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A new method of investigation of a certain class of integral equations describing the dynamics of physical processes

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THE PAPER investigates the properties of a certain class of systems of integral equations in which the non-linearities have the forms of series of multi-linear Volterra-type functionals. The analysis is based on the existence and uniqueness theorems of solutions of a certain operator equation defined in a Banach space, the systems of equations considered being reducible to that form. By this theorem, and using certain definite Banach spaces, the conditions are formulated sufficient for the solutions to possess such properties like continuity and boundedness, or convergence to zero at infinity.

W pracy rozważono własności pewnej klasy układów równań całkowych, w których nieliniowości mają postać szeregów wieloliniowych funkcjonałów typu Volterry. Analizę oparto na twierdzeniach o istnieniu i jednoznaczności rozwiązań pewnego równania operatorowego, określonego w przestrzeni Banacha, do którego sprowadzić można rozważane układy równań. Posługując się tymi twierdzeniami oraz pewną określoną przestrzenią Banacha, sformułowano warunki dostateczne na to, aby otrzymane rozwiązania posiadały takie własności jak ciągłość, ograniczoność lub zbieżność do zera w nieskończoności.

В работе обсуждены свойства некоторого класса систем интегральных уравнений, в которых нелинейности имеют вид рядов многолинейных функционалов типа Вольтерра. Анализ основан на теоремах существования и единственности решений некоторого операторного уравнения определенного в банаховом пространстве, к которому можно свести рассматриваемые системы уравнений. Послуживаясь этими теоремами и некоторым определенным банаховым пространством сформулированы достаточные условия для того, чтобы полученные решения обладали такими свойствами, как непрерывность, ограниченность или сходимость к нулю в бесконечности.

1. Introduction

IN MOST practical cases, the motion of a system of a finite number of material points may be described, under prescribed equations of motion and definite constraints, by the following set of equations

$$\frac{dx_i}{dt} = F_i(t, x_1, \ldots, x_M), \quad i = 1, \ldots, M. \tag{1.1}$$

According to the form of F_i, it may represent a system of differential or integro-differential equations. In particular, when F_i are simply functions, the following system of ordinary differential equations is obtained:

$$\frac{dx_i}{dt} = f_i(t, x_1, \ldots, x_M). \tag{1.2}$$

If we further assume that the functions f_i can be expanded into the McLaurin series with respect to $x_1, \ldots, x_M$, and that $f_i(t, 0, \ldots, 0) = 0$, then the Eqs. (1.2) may be rewritten in the form

$$\frac{dx_i}{dt} = \sum_{j=1}^{M} a_{ij}(t)x_j + \sum_{p=2}^{\infty} \sum_{m_1=1}^{M} \ldots \sum_{m_p=1}^{M} a_{\text{pim}_1 \ldots m_p}(t) x_{m_1} \ldots x_{m_p}. \tag{1.3}$$

If the functions $x_1, \ldots, x_M$ in Eqs. (1.3) are considered as elements of a definite Banach space X ($x_i \in X$ for $i = 1, \ldots, M$), and if certain assumptions are made concerning the functions $a_{ij}(t)$ and $a_{\text{pim}_1 \ldots m_p}(t)$, then the expressions

$$G_{\text{pim}_1 \ldots m_p}(x_{m_1}, \ldots, x_{m_p}) = a_{\text{pim}_1 \ldots m_p}(t) x_{m_1} \ldots x_{m_p} \tag{1.4}$$

may be treated, in turn, as p-linear operators [1] transforming the space considered into itself; this fact will be denoted in our paper by the symbol $G_{\text{pim}_1 \ldots m_p} \in (X^p \to X)$. Introducing moreover the vector Banach space $W(X)$ containing vectors $x = \{x_1, \ldots, x_p\}$ as elements, we may write the following p-linear operator (defined on this space and with values from the space x):

$$G_{pi}(x_1, \ldots, x_p) = \sum_{m_1=1}^{M} \ldots \sum_{m_p=1}^{M} G_{\text{pim}_1 \ldots m_p}(x_{1m_1}, \ldots, x_{pm_p}). \tag{1.5}$$

Here $x_l = \{x_{lm}\}$, $l = 1, \ldots, p$; $m = 1, \ldots, M$. Finally, by defining the operator

$$G_p = \{G_{pi}\}, \quad i = 1, \ldots, M, \tag{1.6}$$

it may easily be demonstrated that $G_p \in \big([W(X)]^p \to W(X)\big)$. Thus the system of Eqs. (1.3) is now written in the form of an operator equation

$$\frac{dx}{dt} = G_1 x + \sum_{p=2}^{\infty} G_p x^p. \tag{1.7}$$

where $G_p x^p \overset{\text{def}}{=} G_p(x, \ldots, x)$. Under the initial condition $x(0) = x_0$, the En. (1.7) is transformed to the form:

$$x = z + A \sum_{p=2}^{\infty} G_p x^p. \tag{1.8}$$

Here, $z = K(t)x_0$, $K(t) = \{k_{ij}(t)\}$ is the solution of the linear part of the Eq. (1.7), while the operator A is defined as:

$$[Ax](t) = \int_0^t K(t) K^{-1}(\tau) x(\tau) d\tau. \tag{1.9}$$

It is easily observed that the expressions (1.4) may be treated as special cases of the following Volterra-type integral operators:

$$G_{\text{pim}_1 \ldots m_p}(x_{m_1}, \ldots, x_{m_p}) = \int_0^t \ldots \int_0^t k_{\text{pim}_1 \ldots m_p}(t, \tau_1, \ldots, \tau_p) x_{m_1}(\tau_1) \ldots x_{m_p}(\tau_p) d\tau_1 \ldots d\tau_p. \tag{1.10}$$

In order to obtain the form of the Eq. (1.4), it is sufficient to substitute $k_{p1m_1\ldots m_p}(t, \tau_1, \ldots \ldots, \tau_p) = a_{p1m_1\ldots m_p}(t) \cdot \delta(t-\tau_1, \ldots, t-\tau_p)$, δ denoting the Dirac delta. The Eq. (1.8) then becomes (in view of the fact that A is also an integral operator) the simplified notation of a non-linear system of integral equations. If the operators G_p [defined by the Eqs. (1.6) and (1.10)] transform the space $W(X)$ into itself and, moreover, $z \in W(X)$ and $A \in \big(W(X) \to W(X)\big)$, then the investigation of properties of the system of integral equations just formulated reduces to the analysis of the Eq. (1.8) which constitutes an operator equation in the Banach space $W(X)$. The present paper sets out to analyse the solutions of the Eq. (1.8) as referred to an arbitrary Banach space, and then to examine properties of integral operators in several selected types of spaces.

2. Analysis of properties of solutions of the operator equation

Let the Eq. (1.8) be given under the assumptions that z is an element of an arbitrary Banach space X, A — a linear (bounded) operator transforming the space X into itself, and $G_p \in (X^p \to X)$, $p = 2, 3, \ldots$. This equation, depending on the forms of z, A and G_p, may possess one or many solutions within the space X, or may have no solutions at all. The following theorem deals with the condition necessary for the equation considered to have exactly one solution.

THEOREM 2.1. *If the series* $\sum_{p=2}^{\infty} \|G_p\| y^p$ *(y — real number) has a positive radius of convergence, then there exist positive numbers α and β such that for each z satisfying the inequality $\|z\| \leqslant \alpha$, the Eq. (1.8) possesses within a sphere $K(0,\beta) = \{x: \|x\| \leqslant \beta\} \subset X$ a unique solution x^* equal to the limit of successive approximations*

$$x_{n+1} = z + A \sum_{p=2}^{\infty} G_p x_n^p. \tag{2.1}$$

Here the term x_0 may be an arbitrary element of X such that $\|x_0\| \leqslant y^*$, where y^* is the minimum non-negative solution of

$$y - \|A\| \sum_{p=2}^{\infty} \|G_p\| y^p = \|z\|. \tag{2.2}$$

The following estimate holds:

$$\|x^*\| \leqslant y^*. \tag{2.3}$$

The proof is based on the following lemma:

LEMMA 2.1. *In the equation*

$$y - \sum_{p=2}^{\infty} a_p y^p = u, \tag{2.4}$$

u and a_p denote the given real numbers ($a_p \geqslant 0$ for $p = 2, 3, \ldots$). If the series $\sum_{p=2}^{\infty} a_p y^p$ has the radius of convergence $R > 0$, then there exist positive numbers α and β such that for each $u \in \langle 0, \alpha \rangle$ the Eq. (2.4) has within the interval $\langle 0, \beta \rangle$ exactly one solution y^* continuously dependent on u.

Proof of the lemma. From the assumptions it follows that the radius of convergence R' of the series $\sum_{p=2}^{\infty} pa_p y^{p-1}$ satisfies the inequality $0 < R' < R$. It follows that there must exist a number $r \in (0, R'\rangle$ such that the function

$$u = f(y) = y - \sum_{p=2}^{\infty} a_p y^p \tag{2.5}$$

is differentiable within the interval $\langle -r, r\rangle$. Its derivative is equal to

$$f'(y) = 1 - \sum_{p=2}^{\infty} pa_p y^{p-1}. \tag{2.6}$$

From the form of Eq. (2.6) it follows that along the segment $\langle 0, r\rangle$, $f'(r)$ is a decreasing function and $f'(0) = 1$. If then $f'(r) \leqslant 0$, then there exists $\tilde{y} \in (0, r\rangle$ such that $f'(\tilde{y}) = 0$ and for each $y \in \langle 0, \tilde{y}\rangle$ the inequality is fulfilled: $0 \leqslant f'(y) \leqslant 1$. However, if $f'(r) > 0$, then all the more $f'(y) > 0$ for $y \in (0, r\rangle$. Thus if β is an arbitrary number from within the interval $(0, \min(\tilde{y}, r))$, then for each $y \in \langle 0, \beta\rangle$ the inequality $0 < f'(y) \leqslant 1$ holds true. It follows that there exists a continuous function $y = f^{-1}(u)$ inverse to f and defined in the interval $\langle 0, \alpha\rangle$, where $\alpha = f(\beta)$. Thus the Eq. (2.4) has exactly one solution $y^*(u) \in (0, \beta\rangle$ within that interval and it is linearly dependent on u.

Proof of the Theorem 2.1. The substitution $a_p = \|A\| \cdot \|G_p\|$, $u = \|z\|$ is easily observed to transform the Eq. (2.2) to the form (2.4). For arbitrary z such that $\|z\| \leqslant \alpha$ we may consider the operation:

$$F_z(x) = z + A \sum_{p=2}^{\infty} G_p x^p, \tag{2.7}$$

defined within a closed sphere $K_z(0, y^*(z)) \overset{\text{df}}{=} \{x\colon \|x\| \leqslant y^*(z)\}$, where y^* is a solution of the Eq. (2.2). Since

$$\|F_z(x)\| \leqslant \|z\| + \|A\| \sum_{p=2}^{\infty} \|G_p\| \, \|x\|^p \leqslant \|z\| + \|A\| \sum_{p=2}^{\infty} \|G_p\| [y^*(z)]^p = y^*(z), \tag{2.8}$$

for each fixed z, the operation F_z transforms the sphere K_z into itself. Simultaneously, for arbitrary $x_1, x_2 \in K_z$ the inequality

$$\|F_z(x_2) - F_z(x_1)\| \leqslant \|A\| \sum_{p=2}^{\infty} \|G_p x_2^p - G_p x_1^p\| \leqslant \|x_2 - x_1\| \, \|A\| \sum_{p=2}^{\infty} p \|G_p\| [y^*(z)]^{p-1} \tag{2.9}$$

is satisfied.

Lemma 2.1 yields the conclusion that $\|A\| \sum_{p=2}^{\infty} p \|G_p\| [y^*(z)]^{p-1} < 1$ which, together with the Eqs. (2.7) and (2.8) proves that F_z is a contracting operation within the sphere K_z. From the Banach theorem on contracting operations there immediately follows the first part of the theorem, the estimate (2.3) being an obvious conclusion from the inequality (2.8).

The considerations presented yield, moreover, a conclusion concerning the continuous dependence of x^* on z. In particular, for $z = 0$ (zero element of the space X) also $x^* = 0$.

It should be noted that for each particular equation of the form of (1.8), the values of α and β may be evaluated numerically. If $G_p \equiv 0$ for $p \geqslant 3$, then the Eq. (1.6) assumes the form

$$x - AG_2x^2 = z, \tag{2.10}$$

and for each z such that $\|z\| \leqslant (4\|A\|\,\|G_2\|)^{-1}$, it has a unique solution satisfying the inequality:

$$\|x^*\| \leqslant (2\|A\|\,\|G_2\|)^{-1}\left(1 - \sqrt{1 - 4\|A\|\,\|G_2\|\,\|z\|}\right). \tag{2.11}$$

Assuming $x_0 = 0$, it is possible, in accordance with the Eq. (2.1), to represent the solution x^* in the form:

$$x = \lim_{n\to\infty} x_n = z + \sum_{p=2}^{\infty} U_p z^p, \tag{2.12}$$

where $U_p \in (X^p \to X)$ for $p = 2, 3, \ldots$. Operators U_p may be determined by substituting the solution (2.12) into the original equation; this yields

$$z + \sum_{p=2}^{\infty} U_p z^p = z + A \sum_{p=2}^{\infty} G_p\Big[z + \sum_{k=2}^{\infty} U_k z^k\Big]^p,$$

or, in the explicit form,

$$U_2 z^2 + U_3 z^3 + \ldots = AG_2(z + U_2 z^2 + \ldots)^2 + AG_3(z + U_2 z^2 + \ldots)^3 + \ldots$$
$$= AG_2 z^2 + AG_2[z(U_2 z^2) + (U_2 z^2)z] + AG_3 z^3 + \ldots.$$

Equating the terms containing equal "powers", we obtain:

$$U_2 z^2 = AG_2 z^2,$$
$$U_3 z^3 = AG_2[z(U_2 z^2) + (U_2 z^2)z] + AG_3 z^3,$$

etc., every consecutive operator U_p depending exclusively on the operators U_k, $k = 2, 3, \ldots, p-1$.

3. Integral operators

In the case in which X is a space of functions (real or complex) of a variable t, and the operators A and G_p have the forms of integrals, the Eq. (1.6) assumes the form of an integral equation (or set of equations). From the Theorem 2.1 it follows that investigation of properties of its solutions (including the existence and uniqueness theorems) is reduced to the determination of conditions under which the operators transform the space X into itself.

The following Banach spaces will be utilized in this paper:

(1) Euclidean M-dimensional space l_M^2 consisting of the sequences $x = \{\xi_k\}$ $k = 1, \ldots, M$, with the norm

$$\|x\|_2 = \Big\{\sum_{k=1}^{M} |\xi_k|^2\Big\}^{1/2};$$

(2) Space C with elements being continuous and bounded functions in the interval $\langle 0, \infty)$, with the norm $\|x\|_C = \sup\limits_{t \geq 0} |x(t)|$;

(3) Space C_0 consisting of functions continuous in the interval $(-\infty, +\infty)$, equal to zero for $t \leq 0$ and bounded for $t > 0$. The norm is defined as before: $\|x\|_{C_0} = \sup\limits_{t \geq 0} |x(t)|$;

(4) The quotient space $K \overset{\mathrm{df}}{=} C/N$, N being a subspace of the space C and consisting of functions converging to zero at infinity. Elements of the space K are classes $\tilde{x}$ of elements of C differing by the element belonging to N. The norm is defined as $\|\tilde{x}\|_K = \lim\limits_{T \to \infty} \sup\limits_{t \geq T} |x(t)|$; x is an arbitrary element of the class x.

Each element $x \in C$ belongs to exactly one class $\tilde{x} \in K$ and hence the class $\tilde{x}$ may be replaced by its arbitrary representative x. In particular, we may write $\|x\|_K = \lim\limits_{T \to \infty} \sup\limits_{t \geq T} |x(t)|$;

(5) The quotient space $K_0 \overset{\mathrm{df}}{=} C_0/N_0$ defined in a manner quite analogous to the preceding case;

(6) Space V. Function $h(t)$ is an element of the space V if $h(t) = 0$ for $t \leq 0$ and $\operatorname*{Var}\limits_{\langle 0\, \infty)} h < \infty$, where

$$\operatorname*{Var}_{\langle 0, \infty)} h = \lim_{T \to \infty} \operatorname*{Var}_{\langle 0,\, T\rangle} h = \lim_{T \to \infty} \sup_{n=1,2,\dots} \sup_{t_i \in \langle 0,\, T\rangle} \sum_{i=1}^{n} |h(t_{i+1}) - h(t_i)|, \quad (i = 1, \dots, n).$$

The norm is expressed as $\|h\|_V = \operatorname*{Var}\limits_{\langle 0,\, \infty)} h$;

(7) Space $C(l_M^2) \overset{\mathrm{df}}{=} C^{M2}$. This is the space of M-dimensional sequences $x = \{x_i\}$, $i = 1, \dots, M$, the elements of which are functions continuous and bounded in the interval $\langle 0, \infty)$, $(x_i \in C)$, with the norm $\|x\|_{2,C} = \sup\limits_{t > 0} \left\{ \sum\limits_{i=1}^{M} |x_i(t)|^2 \right\}^{1/2}$;

(8, 9, 10) Spaces $C_0(l_M^2)$, $K(l_M^2)$ and $K_0(l_M^2)$ defined as in (7).

The examples presented concern real spaces. They may easily be generalized to spaces with complex elements; symbol $|x|$ must then be treated as the modulus of a complex number.

The subject of further investigation will be the properties of the integral operators defined in one of the spaces (2)–(5) or (7)–(10); first, for the sake of simplicity, we shall prove the lemma yielding the conclusion that investigation of properties of operators in vector spaces may be reduced to a similar investigation in scalar spaces.

LEMMA 3.1. *The following operators are given*:

$$G_p(x_1, \dots, x_p)\,(p = 1, 2, \dots), \quad x_l = \{x_{lm}\} \begin{pmatrix} l = 1, \dots, p \\ (m = 1, \dots, M \end{pmatrix}, \tag{3.1}$$

where

$$G_p = \{G_{pi}\} \quad (i = 1, \dots, M), \tag{3.2}$$

$$G_{pi}(x_1, \dots, x_p) = \sum_{m_1=1}^{M} \dots \sum_{m_p=1}^{M} G_{pim_1 \dots m_p}(x_{1m_1}, \dots, x_{pm_p}). \tag{3.3}$$

For each $p = 1, 2, \dots$ *the identity*

$$\{G_p \in ([X(l_M^2)]^p \to X(l_M^2))\} \equiv \{G_{pim_1 \dots m_p} \in (X^p \to X), \quad i, m_1, \dots, m_p = 1, \dots, M\}, \tag{3.4}$$

X denoting an arbitrary Banach space.

Proof. If $G_{pim_1\ldots m_p} \in (X^p \to X)$ for $i, m_1, \ldots, m_p = 1, \ldots, M$ then (as follows from the definition) $G_{pi} \in ([X(l_M^2)]^p \to X)$. Hence, if $y_i = G_{pi}(x_1, \ldots, x_p)$, then $y_i \in X$ whence $y = \{y_i\}_{i=1\ldots M} = G_p(x_1, \ldots, x_p) \in X(l_M^2)$, which means that $G_p(x_1, \ldots x_p) \in \in ([X(l_M^2)]^p \to X(l_M^2))$. However, if such indices $i_0, m_{10}, \ldots, m_{p0}$ exist that $G_{pi\,m_{10}\ldots m_{p0}} \notin \notin (X^p \to X)$, then also a sequence $\{x_{1m_{10}}, \ldots, x_{pm_{p0}}\}$ must exist such that $G_{pi_0m_{10}\ldots m_{p0}}(x_{1m_{10}}, \ldots, x_{pm_{p0}}) \notin X$. It follows that if

$$x_{10} = \{0, \ldots, 0, x_{1m_{10}}, 0, \ldots, 0\},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x_{p0} = \{0, \ldots, 0, x_{pm_{p0}}, 0, \ldots, 0\},$$

then $y_{i_0} = G_{pi_0}(x_{10} \ldots x_{p0}) \notin X$ — that is, $y = \{y_1, \ldots, y_{i_0}, \ldots, y_M\} \notin X(l_M^2)$, which concludes the proof.

In the particular case of $p = 1$, G_1 is a linear matrix operator:

$$G_1 = A = \{A_{ij}\}_{(i,j=1,\ldots,M)}, \tag{3.5}$$

where

$$Ax = \Big\{\sum_{j=1}^{M} A_{ij}x_j\Big\}_{(i=1,\ldots,M)}. \tag{3.6}$$

3.1. Operators in scalar spaces

Let us now analyze the properties of the following integral operators:

$$G_p(x_1, \ldots, x_p) = \int_0^t \ldots \int_0^t k(t, \tau_1, \ldots, \tau_p)x_1(\tau_1) \ldots x_p(\tau_p)d\tau_1 \ldots d\tau_p, \tag{3.7}$$

$$G_p(x_1, \ldots, x_p) = \prod_{i=1}^{p} \int_0^t x_i(t-\tau)dh_i(\tau), \tag{3.8}$$

$$G_p(x_1, \ldots, x_p) = \int_0^t \ldots \int_0^t x_1(t-\tau_1) \ldots x_p(t-\tau_p)k(\tau_1, \ldots, \tau_p)d\tau_1 \ldots d\tau_p \tag{3.9}$$

with $p = 1, 2, \ldots$, the integrals in (3.8) being defined in the sense of Stieltjes.

3.1.1. Operators in spaces C and C_0.

Let us formulate and prove the theorems determining the conditions sufficient for the operators (3.7)–(3.9) to transform C^p into C (or C_0^p into C_0) for $p = 1, 2, \ldots$.

THEOREM 3.1. *If $k(t, \tau_1, \ldots, \tau_p)$ is a continuous function of $p+1$ variables within the region $\{\langle 0, \infty); \ldots \langle 0, \infty)\}$ and $\sup\limits_{t\geq 0} \int_0^t \ldots \int_0^t |k(t, \tau_1, \ldots, \tau_p)|d\tau_1 \ldots d\tau_p < \infty$, then the operator (3.7) transforms C^p into C and its norm satisfies the inequality*

$$\|G_p\|_C \leq \sup_{t\geq 0} \int_0^t \ldots \int_0^t |k(t, \tau_1, \ldots, \tau_p)|d\tau_1 \ldots d\tau_p, \tag{3.10}$$

which for $p = 1$ is transformed to an equality (for $p \geqslant 2$ the equality holds only for particular forms of the kernel k).

Proof. If $x_i \in X$ for $i = 1, \ldots, p$ and

$$y(t) = \int\limits_0^t \ldots \int\limits_0^t k(t, \tau_1, \ldots, \tau_p) x_1(\tau_1) \ldots x_p(\tau_p) d\tau_1 \ldots d\tau_p,$$

then y is a continuous function for each value of t. In addition, the inequality

$$\sup_{t \geqslant 0} |y(t)| \leqslant \prod_{i=1}^{p} [\sup_{t \geqslant 0} |x_i(t)|] \sup_{t \geqslant 0} \int\limits_0^t \ldots \int\limits_0^t |k(t, \tau_1, \ldots, \tau_p)| d\tau_1 \ldots d\tau_p,$$

holds: from which it follows that $y \in C$ and

$$\|y\|_C \leqslant \sup_{t \geqslant 0} \int\limits_0^t \ldots \int\limits_0^t |k(t, \tau_1, \ldots, \tau_p)| d\tau_1 \ldots d\tau_p \prod_{i=1}^{p} \|x_i\|_C,$$

which confirms the inequality (3.10). In order to prove the equality for $p = 1$, it is sufficient to observe that for each number $\varepsilon > 0$ t_0 can be found such that $\sup\limits_{t \geqslant 0} \int\limits_0^{t_0} |k(t, \tau)| d\tau - \int\limits_0^{t_0} |k(t_0, \tau)| d\tau < \varepsilon/2$ and a function $x_0 \in C(\|x_0\|_C = 1)$ can be established such that

$$\int\limits_0^{t_0} |x_0(\tau) - \operatorname{sgn} k(t_0, \tau)| d\tau < \varepsilon/2 \sup_{0 \leqslant \tau \leqslant t_0} |k(t_0, \tau)|.$$

The possibility of determination of such x_0 results from the fact that the set of continuous functions is dense within the space $L(0, t_0)$. The following estimates may then be made:

$$\sup_{t \geqslant 0} \int\limits_0^t |k(t, \tau)| d\tau - \int\limits_0^{t_0} k(t_0, \tau) x_0(\tau) d\tau = \sup_{t \geqslant 0} \int\limits_0^t |k(t, \tau)| d\tau - \int\limits_0^{t_0} |k(t_0, \tau)| d\tau$$

$$+ \int\limits_0^{t_0} |k(t_0, \tau)| d\tau - \int\limits_0^{t_0} k(t_0, \tau) x_0(\tau) d\tau \leqslant \sup_{t \geqslant 0} \int\limits_0^t |k(t, \tau)| d\tau - \int\limits_0^{t_0} |k(t_0, \tau)| d\tau$$

$$+ \sup_{0 \leqslant \tau \leqslant t_0} |k(t_0, \tau)| \, |x_0(\tau) - \operatorname{sgn} k(t_0, \tau)| d\tau < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which yield $\|G_1\|_C = \sup\limits_{t \geqslant 0} \int\limits_0^t |k(t, \tau)| d\tau$.

THEOREM 3.2. *If* $\operatorname*{Var}\limits_{<0,\infty)} h_i < \infty$ $(i = 1, \ldots, p)$, *then the operator* (3.8) *transforms* C_0^p *into* C *and its norm is equal to*

$$\|G_p\|_{C_0} = \prod_{i=1}^{p} \operatorname*{Var}_{<0,\infty)} h_i. \tag{3.11}$$

The corresponding proof in the case of $p = 1$ may be found in [2]. Its validity for all remaining values of p is obtained by means of a direct generalization.

THEOREM 3.3. *If* $\int\limits_0^\infty \ldots \int\limits_0^\infty |k(\tau_1, \ldots, \tau_p)| d\tau_1 \ldots d\tau_p < \infty$ *(the integral being defined in the sense of Lebesgue), then the operator* (3.9) *transforms* C_0^p *into* C_0, *and*

$$\|G_p\|_{C_0} \leqslant \int\limits_0^\infty \ldots \int\limits_0^\infty |k(\tau_1, \ldots, \tau_p)| d\tau_1 \ldots d\tau_p. \tag{3.12}$$

With $k(\tau_1, \ldots, \tau_p) = \prod\limits_{i=1}^{p} k_i(\tau_i)$, *the inequality is transformed to equality.*

P r o o f. If $x \in C_0$ for $i = 1, \ldots, p$, and

$$y(t) = \int\limits_0^t \ldots \int\limits_0^t x_1(t-\tau_1) \ldots x_p(t-\tau_p) k(\tau_1, \ldots, \tau_p) d\tau_1 \ldots d\tau_p,$$

then obviously $y(t) = 0$ for each $t \leqslant 0$. If, in addition, $t_1 > t_2$, the following estimate holds true

$$|y(t_1)-y(t_2)| = \Big|\int\limits_0^{t_1} \ldots \int\limits_0^{t_1} x_1(t_1-\tau_1) \ldots x_p(t_1-\tau_p) k(\tau_1, \ldots, \tau_p) d\tau_1 \ldots d\tau_p$$

$$- \int\limits_0^{t_2} \ldots \int\limits_0^{t_2} x_1(t_2-\tau_1) \ldots x_p(t_2-\tau_p) k(\tau_1, \ldots, \tau_p) d\tau_1 \ldots d\tau_p\Big|$$

$$= \Big|\int\limits_0^{t_1} \ldots \int\limits_0^{t_1} [x_1(t_1-\tau_1) \ldots x_p(t_1-\tau_p) - x_1(t_2-\tau_1) \ldots x_p(t_2-\tau_p)] k(\tau_1, \ldots, \tau_p) d\tau_1 \ldots d\tau_p\Big|$$

$$= \Big|\int\limits_0^{t_1} \ldots \int\limits_0^{t_1} \Big\{\sum_{i=1}^{p} [x_i(t_1-\tau)-x_i(t_2-\tau_i)]\Big\} \prod_{j>i} x_j(t_2-\tau_j) \prod_{l>i} x_l(t_1-\tau_l) \times$$

$$\times k(\tau_1, \ldots, \tau_p) d\tau_1 \ldots d\tau_p\Big| \leqslant \sup_{i=1,\ldots,p} \sup_{0 \leqslant \tau \leqslant t_1} |x_i(t_1-\tau)-x_i(t_2-\tau)| \cdot p \times$$

$$\times \Big\{\sup_{i=1,\ldots,p} \sup_{0 \leqslant \tau \leqslant t_1} |x_i(t_1-\tau)|\Big\}^{p-1} \int\limits_0^{t_1} \ldots \int\limits_0^{t_1} |k(\tau_1, \ldots, \tau_p)| d\tau_1 \ldots d\tau_p$$

$$\leqslant M(T) \cdot \sup_{i=1,\ldots,p} \sup_{0 \leqslant \tau \leqslant T} |x_i(t_1-\tau)-x_i(t_2-\tau)|$$

for each $T > t_1$, which implies the function $y(t)$ to be continuous in the interval $(-\infty, +\infty)$. If it is additionally observed that

$$\sup_{t \geqslant 0} |y(t)| \leqslant \int\limits_0^\infty \ldots \int\limits_0^\infty |k(\tau_1, \ldots, \tau_p)| d\tau_1 \ldots d\tau_p \prod_{i=1}^{p} \sup_{t \geqslant 0} |x_i(t)|,$$

then the inequality (3.12) immediately follows, and concludes the proof.

In the case in which $k(\tau_1, \ldots, \tau_p) = \prod\limits_{i=1}^{p} k_i(\tau_i)$, operator (3.9) is a particular case of (3.8), which makes it possible to replace the inequality (3.12) with an equality. If, however, $k(\tau_1, \ldots, \tau_p)$ is a continuous function of all its variables for $\tau_i \geqslant 0$, $i = 1, \ldots, p$, then the operator (3.9) becomes a special case of (3.7), thus transforming C^p into C.

3.1.2. Operators in spaces K and K_0

The theorems exposed in the preceding section also hold (under certain additional conditions) with respect to the space K or K_0.

THEOREM 3.4. *If the assumptions of Theorem* 3.1. *are fulfilled and if for each fixed $t_0 > 0$ the condition*

$$\lim_{t\to\infty}\int_0^{t_0}\dots\int_0^{t_0}|k(t,\tau_1,\dots,\tau_p)|d\tau_1\dots d\tau_p = 0 \tag{3.13}$$

is satisfied, then the operator (3.7) *transforms K^p into K ($p = 1, 2, \dots$) and*

$$\|G_p\|_K \leqslant \sup_{t\geqslant 0}\int_0^t\dots\int_0^t|k(t,\tau_1,\dots,\tau_p)|d\tau_1\dots d\tau_p. \tag{3.14}$$

Proof. From the assumption it follows that for each $t_0 > 0$ and $\varepsilon > 0$ we can select such T_0 that for each $t > T_0$ and for arbitrary $x_i \in C$ ($x_i(t) \not\equiv 0$ for $i = 1, \dots, p$) there occurs the inequality:

$$\int_0^t\dots\int_0^t|k(t,\tau_1,\dots,\tau_p)|d\tau_1\dots d\tau_p < \frac{\varepsilon}{2\prod\limits_{i=1}^{p}\sup\limits_{t\geqslant 0}|x_i(t)|}.$$

Hence, if $\lim\limits_{t\to\infty}x_i(t) = 0$, there exist such numbers T_i ($i = 1, \dots, p$) that

$$\sup_{t\geqslant T_i}|x_i(t)| < \frac{\varepsilon}{2p\prod\limits_{j\neq l}\sup\limits_{t\geqslant 0}|x_j(t)|\sup\limits_{t\geqslant 0}\int_0^t\dots\int_0^t|k(t,\tau_1,\dots,\tau_p)|d\tau_1\dots d\tau_p}.$$

If now $y(t) = \int_0^t\dots\int_0^t k(t,\tau_1,\dots,\tau_p)x_1(\tau_1)\dots x_p(\tau_p)d\tau_1\dots d\tau_p$, then for each $t > T = T_0 + \sum\limits_i T_i$ we can estimate:

$$\begin{aligned}|y(t)| = \Big|&\int_0^T\int_0^T k(t,\tau_1,\dots,\tau_p)x_1(\tau_1)\dots x_p(\tau_p)d\tau_1\dots d\tau_p\\ &+\sum_{i=1}^{p}\int_0^T\dots\int_0^T\int_T^t\int_0^t\dots\int_0^t k(t,\tau_1,\dots,\tau_p)x_1(\tau_1)\dots x_p(\tau_p)d\tau_1\dots d\tau_p\Big|\\ &\leqslant\prod_{i=1}^{p}[\sup_{t\geqslant 0}|x_i(t)|]\int_0^T\dots\int_0^T|k(t,\tau_1,\dots,\tau_p)|d\tau_1\dots d\tau_p\\ &+\sum_{i=1}^{p}\Big\{\sup_{t\geqslant T}|x_i(t)|\prod_{j=i}^{p}[\sup_{t\geqslant 0}|x_i(t)|]\sup_{t\geqslant 0}\int_0^t\dots\int_0^t|k(t,\tau_1,\dots,\tau_p)|d\tau_1\dots d\tau_p\Big\}\\ &<\frac{\varepsilon}{2}+p\frac{\varepsilon}{2p}=\varepsilon,\end{aligned}$$

which means that $\lim\limits_{t\to\infty}y(t) = 0$.

Let $x_i = x_{i0}+u_i$ $(i = 1, ..., p)$, where $x_{i0} \in C$, $u_i \in N$. Then

$$G_p(x_1, ..., x_p) = G_p(x_{01}+u_1, ..., x_{0p}+u_p) = G_p(x_{01}, ..., x_{0p})$$

$$+\sum_{i=1}^{p} G_p(x_{01}, ..., x_{0(i-1)}, u_i, x_{0(i+1)}+u_{i+1}, ..., x_{0p}+u_p) = G_p(x_{01}, ..., x_{0p})+w.$$

Here $w \in N$. It follows that $G_p \in (K^p \to K)$. If it is additionally assumed that $\|x_{0i}\|_C = 1$,

$$\|G_p\|_K = \sup_{||x_i||K=1} \|G_p(x_1, ..., x_p)\|_K = \sup_{||x_{0i}||C=1} \|G_p(x_{01}, ..., x_{0p})\|_K$$

$$\leqslant \sup_{||x_{0i}||C=1} \|G_p(x_{01}, ..., x_{0p})\|_C = \|G_p\|_C,$$

whence, by means of the Theorem 3.1, Eq. (3.14) is derived.

THEOREM 3.5. *If the assumptions of Theorem* 3.2 *are satisfied, the operator* (3.8) *is found to transform* K_0^p *into* K_0 *and its norm is equal to*

$$\|G_p\|_{K_0} = \prod_{i=1}^{p} \operatorname*{Var}_{\langle 0,\infty)} h_i. \tag{3.15}$$

The corresponding proof for $p = 1$ may be found in [2]. Using the Theorem 3.2 it may easily be generalized to the remaining values of p.

THEOREM 3.6. *If the assumptions of Theorem* 3.3 *are satisfied, the operator* (3.9) *transforms* K_0^p *into* K_0 *and*

$$\|G_p\|_{K_0} \leqslant \int_0^\infty ... \int_0^\infty |k(\tau_1, ..., \tau_p)| d\tau_1 ... d\tau_p. \tag{3.16}$$

(If $k(\tau_1, ..., \tau_p) = \prod_{i=1}^{p} k_i(\tau_i)$, *inequality is changed to equality*).

The proof is analogous to that of Theorem 3.4.

3.2. Operators in vector spaces

Using the results obtained for operators defined in scalar spaces let us now pass to the operators defined in vector spaces (described by the Eqs. (3.1)–(3.4)) and obtain the corresponding norm estimates.

3.2.1. Linear operators

THEOREM 3.7. *The operator*

$$[Ax](t) = \int_0^t k(t, \tau)x(\tau)d\tau, \tag{3.17}$$

is given with $k(t, \tau) = \{k_{ij}(t, \tau)\}_{(i,j=1,...,M)}$. *If each of the operators* A_{ij} *(here* $[A_{ij}x_j](t) = \int_0^t k_{ij}(t, \tau)x_j(\tau)d\tau$*) transforms the space* C *(or* K*) into itself, then the operator* (3.17) *transforms the space C^{M2} (or K^{M2}) into itself, and*

$$\|A\|_{2,K} \leqslant \|A\|_{2,C} \leqslant \sup_{t\geqslant 0} \int_0^t \|k(t,\tau)\|_2 d\tau. \tag{3.18}$$

Proof. Lemma 3.1 being taken into account, it proves sufficient to establish the validity of estimates of the norm of A. From the expression

$$\|Ax\|_{2,C} = \sup_{t\geqslant 0} \| \int_0^t k(t,\tau)x(\tau)d\tau\|_2 \leqslant \sup_{t\geqslant 0} \int_0^t \|k(t,\tau)\|_2 \|x(\tau)\|_2 d\tau$$

$$\leqslant \sup_{t\geqslant 0} \|x(t)\|_2 \sup_{t\geqslant 0} \int_0^t \|k(t,\tau)\|_2 d\tau = \sup_{t\geqslant 0} \int_0^t \|k(t,\tau)\|_2 d\tau \cdot \|x\|_{2,C}$$

it immediately follows that $\|A\|_{2,C} \leqslant \sup_{t\geqslant 0} \int_0^t \|k(t,\tau)\|_2 d\tau$. With $A \in (K^{M2} \to K^{M2})$, and proceeding as in the Theorem 3.4, we obtain the left-hand side inequality from Eq. (3.18), which concludes the proof.

The next theorem concerns the following operator

$$Ax = \Big\{ \sum_{l=1}^{M} A_{im} x_m \Big\}_{(M=1,\dots,M)}. \tag{3.19}$$

Here, $[A_{im}x_m](t) = \int_0^t x_m(t-\tau)dh_{im}(\tau)$, $\underset{\langle 0,\infty)}{\mathrm{Var}}\, h_{im} < \infty$.

THEOREM 3.8. *Operator* (3.19) *transforms C_0^{M2} (as also K_0^{M2}) into itself, and*

$$\|A\|_{2,K_0} \leqslant \|A\|_{2,C_0} \leqslant \|\{\underset{\langle 0,\infty)}{\mathrm{Var}}\, h_{im}\}_{(i,m=1,\dots,M)}\|_2. \tag{3.20}$$

Proof. The first part of that theorem follows directly from the Lemma 3.1. Norm estimates in C_0^{M2} result from the inequalities:

$$\|Ax\|_{2,C_0} = \sup_{t\geqslant 0} \Big\{ \sum_{i=1}^{M} \Big[\sum_{m=1}^{M} \int_0^t x_m(t-\tau)dh_{im}(\tau)\Big]^2\Big\}^{1/2} \leqslant \Big\{ \sum_{i=1}^{M} \Big[\sum_{m=1}^{M} \sup_{t\geqslant 0} |x_m(t)| \underset{\langle 0,\infty)}{\mathrm{Var}}\, h_{im}\Big]^2\Big\}^{1/2},$$

$$\|A\|_{2,C_0} = \sup_{\|x\|_{2,C_0}=1} \|Ax\|_{2,C_0} \leqslant \sup_{\sum_{m=1}^{M} [\sup_{t\geqslant 0}|x_m(t)|]^2=1} \Big\{ \sum_{i=1}^{M} \Big[\sum_{m=1}^{M} \sup_{t\geqslant 0} |x_m(t)| \underset{\langle 0,\infty)}{\mathrm{Var}}\, h_{im}\Big]^2\Big\}^{1/2}$$

$$= \|\{\underset{\langle 0,\infty)}{\mathrm{Var}}\, h_{im}\}_{(i,m=1,\dots,M)}\|_2.$$

Estimates in the space K_0^{M2} may be obtained in the same manner as before.

In particular, if $h_{im}(t) = \int_0^t k_{im}(\tau)d\tau$ for $i, m = 1, \dots, M$, then a stronger estimate may be obtained:

$$\|A\|_{2,K_0} \leqslant \|A\|_{2,C_0} \leqslant \int_0^\infty \|k(\tau)\|_2 d\tau,$$

Here, $k(\tau) = \{k_{im}(\tau)\}_{(i,m=1,\dots,M,)}$.

3.2.2. Multilinear operators

Let us present the properties of the multi-linear operator (3.1) under the assumption that operators $G_{pim_1\ldots m_p}$ assume one of the following forms:

$$(3.21)\qquad G_{pim_1\ldots m_p}(x_{1m_1},\ldots,x_{pm_p}) = \int_0^t\ldots\int_0^t k_{pim_1\ldots m_p}(t,\tau_1,\ldots,\tau_p)x_{1m_1}(\tau_1)\ldots x_{pm_p}(\tau_p)d\tau_1\ldots d\tau_p,$$

$$(3.22)\qquad G_{pim_1\ldots m_p}(x_{1m_1},\ldots,x_{pm_p}) = \prod_{l=1}^{p}\int_0^t x_{lm_l}(t-\tau)dh_{pim_l}(\tau),$$

$$(3.23)\qquad G_{pim_1\ldots m_p}(x_{1m_1},\ldots,x_{pm_p}) = \int_0^t\ldots\int_0^t k_{pim_1\ldots m_p}(\tau_1,\ldots,\tau_p)x_{1m_1}(t-\tau_1)\ldots x_{pm_p}(t-\tau_p)d\tau_1\ldots d\tau_p.$$

THEOREM 3.9. *If the operators* (3.21) *transform C^p into C (or K^p into K), then the operator G_p transforms $(C^{M2})^p$ into C^{M2} (or $(K^{M2})^p$ into K^{M2}), and*

$$(3.24)\qquad \|G_p\|_{2,K}\leqslant\|G_p\|_{2,C}\leqslant\Big\{\sum_{i=1}^{M}\sum_{m_1=1}^{M}\ldots\sum_{m_p=1}^{M}\Big[\sup_{t\geqslant0}\int_0^t\ldots\int_0^t|k_{pim_1\ldots m_p}(t,\tau_1,\ldots,\tau_p)|\times d\tau_1\ldots d\tau_p\Big]^2\Big\}^{1/2}.$$

Proof. Since

$$\|G_{pi}(x_1,\ldots,x_p)\|_C = \sup_{t\geqslant0}\Big|\sum_{m_1=1}^{M}\ldots\sum_{m_p=1}^{M}\int_0^t\ldots\int_0^t k_{pim_1\ldots m_p}(t,\tau_1,\ldots,\tau_p)\times x_{1m_1}(\tau_1)\ldots x_{pm_p}(\tau_p)d\tau_1\ldots d\tau_p\Big|\leqslant\sum_{m_1=1}^{M}\ldots\sum_{m_p=1}^{M}\Big\{\prod_{l=1}^{p}[\sup_{t\geqslant0}|x_{lm_l}(t)|]\times\sup_{t\geqslant0}\int_0^t\ldots\int_0^t|k_{pim_1\ldots m_p}(t,\tau_1,\ldots,\tau_p)|d\tau_1\ldots d\tau_p$$

$$=\sum_{m_1=1}^{M}\ldots\sum_{m_p=1}^{M}\prod_{l=1}^{p}\|x_{lm_l}\|_C\sup_{t\geqslant0}\int_0^t\ldots\int_0^t|k_{pim_1\ldots m_p}(t,\tau_1,\ldots,\tau_p)|d\tau_1\ldots d\tau_p,$$

we obtain:

$$\|G_p(x_1,\ldots,x_p)\|_{2,C}^2=\sum_{i=1}^{M}\|G_{pi}(x_1,\ldots,x_p)\|_C^2$$

$$\leqslant\prod_{l=1}^{p}\Big[\sum_{m_l=1}^{M}\|x_{lml}\|_C^2\Big]\sum_{i=1}^{M}\sum_{m_1=1}^{M}\ldots\sum_{m_p=1}^{M}\Big[\sup_{t\geqslant0}\int_0^t\ldots\int_0^t k_{pim_1\ldots m_p}(t,\tau_1,\ldots,\tau_p)|d\tau_1\ldots d\tau_p\Big]^2.$$

Now taking into account Lemma 3.1 and Theorem 3.4, the theorem is proved. For $p = 2$ (bilinear operator) a more accurate estimate is found: namely, if $k_{2i} \overset{\mathrm{df}}{=} \{k_{2im_1m_2}\}_{(m_1,m_2=1,\dots,M)}$, then

$$|[G_{2i}(x_1, x_2)](t)| \leqslant \int_0^t \int_0^t \Big| \sum_{m_1=1}^{M} \sum_{m_2=1}^{M} k_{2im_1m_2}(t, \tau_1, \tau_2) x_{1m_1}(\tau_1) x_{2m_2}(\tau_2) \Big| d\tau_1 d\tau_2$$

$$\leqslant \int_0^t \int_0^t \Big\{ \sum_{m_1=1}^{M} \Big| \sum_{m_2=1}^{M} k_{2im_1m_2}(t, \tau_1, \tau_2) x_{2m_2}(\tau_2) \Big|^2 \Big\}^{1/2} \Big\{ \sum_{m_1=1}^{M} x_{1m_1}(\tau_1)|^2 \Big\}^{1/2} d\tau_1 d\tau_2$$

$$= \int_0^t \int_0^t \|k_{2i}(t, \tau_1, \tau_2) x_2(\tau_2)\|_2 \|x_1(\tau_1)\|_2 d\tau_1 d\tau_2$$

$$\leqslant \int_0^t \int_0^t \|k_{2i}(t, \tau_1, \tau_2)\|_2 \|x_1(\tau_1)\|_2 \|x_2(\tau_2)\|_2 d\tau_1 d\tau_2,$$

which means that

$$\|G_2(x_1, x_2)\|_{2,C}^2 \leqslant \Big\{ \sum_{i=1}^{M} \Big[\sup_{t \geqslant 0} \int_0^t \int_0^t \|k_{2i}(t, \tau_1, \tau_2)\|_2 d\tau_1 d\tau_2 \Big]^2 \Big\} \|x_1\|_{2,C}^2 \|x_2\|_{2,C}^2,$$

or, finally,

$$\|G_2\|_{2,C} \leqslant \Big\{ \sum_{i=1}^{M} \Big[\sup_{t \geqslant 0} \int_0^t \int_0^t \|k_{2i}(t, \tau_1, \tau_2)\|_2 d\tau_1 d\tau_2 \Big]^2 \Big\}^{1/2}.$$

In particular, if $k_{2i}(t, \tau_1, \tau_2) = k_{2i}(t-\tau_1, t-\tau_2)$, then we obtain:

$$\|G_2\|_{2,C} \leqslant \Big\{ \sum_{i=1}^{M} \Big[\int_0^\infty \int_0^\infty [\|k_{2i}(\tau_1, \tau_2)\|_2 d\tau_1 d\tau_2 \Big]^2 \Big\}^{1/2}.$$

THEOREM 3.10. *If the operators* (3.22) *transform C_0^p into C_0 (and K_0^p into K_0), then the operator* (3.1) *transforms $[C_0^{M2}]^p$ into C_0^{M2}, as also $[K_0^{M2}]^p$ into K_0^{M2}, and*

$$\|G_p\|_{2,K_0} \leqslant \|G_p\|_{2,C_0} \leqslant \Big\{ \sum_{i=1}^{M} \Big[\sum_{m=1}^{M} (\operatorname*{Var}_{\langle 0,\infty)} h_{pim})^2 \Big] \Big\}^{1/2}. \tag{3.25}$$

The proof is analogous to the previous one.

The theorems concerning the operators (3.23) are quite similar and it is not necessary to formulate them independently. It should be noted, however, that in that case the following norm estimates hold true:

$$\|G_p\|_{2,K_0} \leqslant \|G_p\|_{2,C_0} \leqslant \Big\{ \sum_{i=1}^{M} \sum_{m_1=1}^{M} \dots \sum_{m_p=1}^{M} \Big[\int_0^\infty \dots \int_0^\infty |k_{pim_1\dots m_p}(\tau_1, \dots, \tau_p)| d\tau_1 \dots d\tau_p \Big]^2 \Big\}^{1/2}.$$

In the case of bilinear operators, they may be replaced with stronger estimates

$$\|G_2\|_{2,K_0} \leqslant \|G_2\|_{2,C_0} \leqslant \Big\{ \sum_{i=1}^{M} \Big[\int_0^\infty \int_0^\infty \|k_{2i}(\tau_1, \tau_2)\|_2 d\tau_1 d\tau_2 \Big]^2 \Big\}^{1/2}.$$

Here $k_{2i} = \{k_{2im_1m_2}\}_{(m_1,m_2=1,\dots,M)}$.

4. Conclusions

In the cases in which the operators discussed above constitute elements of the Eq. (1.8), the latter becomes a simplified notation of a system of integral equations (or, as was shown at the beginning, a system of integro-differential or differential equations). The theorems derived here make it possible to determine the sufficient conditions for uniqueness and existence of solutions of that system within the framework of one of the Banach spaces discussed. According to the space assumed, we may investigate such properties of the solutions as their continuity, boundedness and convergence to zero at infinity.

As already indicated, for each system of that type it is possible to determine numerically the range of existence and uniqueness of solutions, as also to estimate their norms.

Note also that the method of ruccessive approximations discussed in Sec. 2 enables us to write the solution in the form of a series representing a generalization of the Volterra functional series applied to non-linear scalar equations.

Particular attention should be paid to the possibilities of application of the analysis presented in this paper to mechanical systems described by the differential equations of the (1.3) type. It is easily observed that if, for an arbitrary initial point t_0 and a certain set of initial values $\|x(t_0)\|_2 \leqslant a$, there exists exactly one solution continuous and bounded in the interval t_0, ∞, then the zero solution must be stable (equilibrium point of the system) in the Lapunov sense. If, in addition, each solution converges, under arbitrary initial conditions, to zero, then the stability is globally asymptotic. Since from the previous considerations (Theorem 2.1) it follows that the set of admissible initial conditions can always be effectively determined, we may state that the amount of information on the solutions so obtained is greater than that resulting from the application of other methods (Lapunov methods in particular).

Similar conclusions may also be drawn with respect to the systems described by integro-differential equations or by differential equations with shifted arguments.

The problem of nuclear reactors [3] may be quoted as an example of practical, technical application of the method; it makes it possible to investigate more complex (as compared with the Lapunov methods used so far) models, and to derive more general stability criteria much more effectively.

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