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Acoustical theory of turbulence ([1])

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Fluid fluctuations are expanded in a complete set of orthogonal wave-type solutions of the linearized Navier-Stokes system. The motion of characteristic waves is governed by Hamiltonian equations of ray acoustics. Wave amplitudes are determined by non-linear interaction terms. The distribution functions (squares of amplitudes) satisfy Boltzmann-type equations and describe completely the averaged properties of turbulence.

Fluktuacje cieczy rozwinięto w szereg funkcji stanowiących pełny i ortogonalny układ rozwiązań falowych zlinearyzowanego układu Naviera-Stokesa. Ruch fal charakterystycznych opisany jest równaniami Hamiltona akustyki falowej. Amplitudy fal określono za pomocą nieliniowych członów oddziaływania wzajemnego. Funkcje rozkładu (kwadraty amplitud) spełniają równania typu Boltzmanna i opisują w sposób pełny uśrednione własności turbulencji.

Флуктуации жидкости разложены в ряд функций составляющих полную и ортогональную систему волновых решений линеаризованной системы Навье-Стокса. Движение характеристических волн описано уравнениями Гамильтона волновой акустики. Амплитуды волн определены при помощи нелинейных членов взаимодействия. Функции распределения (квадраты амплитуд) удовлетворяют уравнениям типа Больцмана и описывают полным образом усредненные свойства турбулении.

1. Introduction

Recent years witnessed numerous efforts to base theories of turbulence on fundamental principles of physics. Thus, to mention only a few, we have seen non-linear acoustics applied by Lighthill (1952, 1962) to the problem of aerodynamically produced sound and random "pseudo-sound," statistical mechanics applied to pseudo-turbulence of suspensions by Buyevich (1970, 1971), weakly non-linear interaction theory of continuum and particle physics applied to plasma turbulence by Kadomtsev (1965), Vedenov (1968), Sagdeev & Galeev (1969), and by Davidson (1967), the theory of generalized Brownian motion used by Chung (1969, 1970) to derive a set of Fokker-Planck equations for description of turbulence, the theory of microfluids, developed by Eringen (1964), and applied by him (1970) to a micropolar description of turbulence, and quantum and wave mechanics used by Krzywoblocki (1971) to formulate the theory of turbulence in terms of solutions of the Schroedinger's equation.

The present paper was inspired by and uses many of the ideas developed by the above mentioned authors. The primary objective of this work was the determination of the coupling, interaction, and separation of the effects of the various modes of fluid oscilla-

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tion as they may exist in turbulence and may have a bearing on noise generation and absorption. Such modes were identified by Kovasznay (1953) by factorization of the linearized fluid dynamical differential operators as the vorticity, entropy, and acoustic modes. In this study, however, we shall identify the wave-type solutions of the flow equations, corresponding to vortical, entropy, and acoustic modes of propagation, with characteristic energy states of the continuum, the latter treated as a vibrating system of infinitely many degrees of freedom, and apply quantum statistical and gas-kinetic methods to arrive at a description of turbulence in terms of averages over the probability distribution. Examples of transport equations for turbulent quantities will be derived to show the special role played by the acoustic mode.

2. Formulation of the theory

2.1. Time-dependent perturbation

We shall consider a compressible, viscous gas with constant material properties. If μ_1 and μ_2 are the first and second coefficients of viscosity, then we may introduce the viscosity number $V = 2+\mu_2/\mu_1$, so that the bulk viscosity becomes $\frac{2}{3}\mu_1+\mu_2 = \mu_1\left(V-\frac{4}{3}\right) \geqslant 0$, and Stokes' assumption amounts to setting $V = 4/3$. For such a gas the governing equations of conservation of mass, momentum, and energy are

$$\varrho_{,t}+u_i\varrho_{,i}+\varrho u_{i,i} = 0,$$

$$u_{i,t}+u_ju_{i,j}+\frac{RT}{\varrho}\varrho_{,i}+RT_{,i}-\frac{\mu_1}{\varrho}[u_{i,jj}+(V-1)u_{j,ij}] = 0, \tag{2.1}$$

$$T_{,t}+u_jT_{,j}+(\gamma-1)Tu_{j,j}-\frac{\gamma\mu_1}{Pr}T_{,jj}-\frac{\mu_1}{\varrho}\Phi = 0,$$

where ϱ — density, u_i — velocity, T — temperature, γ — ratio of specific heats, Φ — dissipation function. Pressure has been eliminated using equation of state, $p = \varrho RT$. Repeated indices denote summation, and differentiation is indicated by a comma.

Perturbing the flow by setting $\varrho = \varrho_0+\varrho'$, $u_i = u_{0_i}+u'_i$, $T = T_0+T'$, symmetrizing the system by introduction of non-dimensional ratios ϱ'/ϱ_0, u'_i/c, $T'/(\sqrt{\gamma-1}T_0)$, where $c^2 = RT_0$, we may put the system (2.1) in the form

$$\frac{\partial\psi_i}{\partial t} = \mathsf{L}\psi_i+\mathsf{B}(\psi_i,\psi_i)+\mathsf{T}(\psi_i,\psi_i,\psi_i), \tag{2.2}$$

where ψ_i is a column vector of the dependent variables, L, B, and T are, respectively, a linear, bilinear, and trilinear spatial differential operators uniquely determined by the perturbation of (2.1). For brevity we shall omit giving the particular expressions for the operators. The "weakly non-linear" system (2.2) has been a subject of study by many researchers as reported, e.g., in the books by Kadomtsev (1965), Sagdeev and Galeev (1969), and Vedenov (1968). We shall depart somewhat from the standard procedures used in plasma physics. Our objectives are not a solution of the initial value problem and time-evolution of turbulence, but a description of instantaneous local averages of random, wave-type oscillations of a flowing gas.

If $\psi_{\alpha\beta} = C_\alpha \phi_{\alpha\beta} \exp[\Gamma_\alpha t + i(x_m k_m - \omega_\alpha t)]$ is taken as any one member of the five ($\alpha = 1, \ldots, 5$) linearly independent, infinite families of solutions of the linear, homogeneous equation

$$\frac{\partial \psi_i}{\partial t} = L\psi_i, \tag{2.3}$$

then solubility of (2.3) determines five sets of real functions $\Gamma_\alpha(k_m)$ and $\omega_\alpha(k_m)$. Due to the fact that the amplitudes may be determined only up to a common factor, one may solve (2.3) for five complex eigenvectors $\phi_{\alpha_i}(k_m)$, linearly independent and normalized, so that

$$\phi_{\alpha_i}\phi_{\beta_i} = 0 \text{ if } \alpha \neq \beta, \quad \text{and} \quad \phi_{\alpha_i}\phi^{*}_{\beta_i} = 1 \text{ if } \alpha = \beta, \tag{2.4}$$

where the asterisk denotes a complex conjugate. This leaves the five scalar (in general complex) amplitude factors, C_α, to be determined by the non-linear terms of (2.2).

Adopting $\phi_{\alpha_i}\exp[\Gamma_\alpha t + i(x_m k_m - \omega_\alpha t)]$ as the basis vectors (state vectors) representing small amplitude plane waves as possible states of the fluid, we may form a formal expansion of the solution vector ψ_i in the differential neighbourhood of an arbitrary point $x_m = t = 0$

$$\psi_i = \sum_\alpha \sum_k C_\alpha(k, t)\phi_{\alpha_i}(k)\exp[\Gamma_\alpha t + i(x_m k_m - \omega_\alpha t)].$$

Substitution into (2.2) gives

$$\sum_\alpha \phi_{\alpha_i}\frac{\partial C_\alpha}{\partial t} = \sum_{\alpha', \alpha'', k', k''} \cdots \sum C'_{\alpha'} C''_{\alpha''} \mathrm{B}(\phi_{\alpha' i}, \phi_{\alpha'' i})\exp[(\Gamma' + \Gamma'' - \Gamma)t]\,\delta(k - k' - k'') \times \times \delta(\omega - \omega' - \omega'') + \ldots, \tag{2.5}$$

where the trilinear terms are indicated by ..., and where $\delta(x) = 1$ when $x = 0$, $\delta(x) = 0$ when $x \neq 0$. The double sum over k' and k'' is thus reduced to a single sum over all possible values such that

$$k'_m + k''_m = k_m \quad \text{and} \quad \omega' + \omega'' = \omega. \tag{2.6}$$

Dispersion relations, $\omega = \omega(k_m)$ to be given later, determine the selection rules for possible three-wave resonances. The trilinear term leads to conditions and selection rules for four-wave resonances.

We multiply the Eq. (2.5) by $\phi_{\beta j} C^*_\beta$ and the conjugate of (2.5) by $\phi^*_{\beta j} C_\beta$ and add the results making use of orthogonality relations (2.4). The result is a formal, but formidable expression for

$$\frac{\partial C_\beta C^*_\beta}{\partial t} = \frac{\partial_e f_\beta(k_m, t)}{\partial t}, \quad f_\beta = C_\beta C^*_\beta = |C_\beta|^2. \tag{2.7}$$

It is customary to pass to the limit of infinitely many moving waves with continuous distribution of the square of the amplitude over the wave-number space. The derivative (2.7) becomes then equal to a sum of double and triple integrals over k_m-space, and it represents the time rate of change of the square of the amplitude of fluid waves having wavenumber k_m at position x_m and time t and associated with the β-mode of propagation. Since, for infinitesimal amplitude plane waves satisfying Eq. (2.3), the amplitudes remain rigorously constant, the symbol $\partial_e f_\beta/\partial t$ in Eq. (2.7) will be used to denote effects of wave interactions (encounters or "collisions") which alter amplitudes, and will be interpreted as a rate of change of the number of wave packets (quasi-particles) in analogy to kinetic theory expressions for chemical source functions in multi-component reacting gases.

In the next Section we shall concentrate on the determination of the plane wave solutions of Eq. (2.3).

2.2. Characteristic states

Substituting a state vector ψ_{α_i} into Eq. (2.3) amounts to replacing time and space differential operators in (2.3) by multiplication operators $(\Gamma-i\omega)$ and ik_m, respectively. Thus (2.3) becomes a 5×5 linear, homogeneous, algebraic system for the complex amplitudes ϕ_{α_i}. The condition, that such a system admit a non-trivial solution, is the vanishing of the characteristic determinant. The determinant has the following form, $(\nu = \mu_1/\varrho)$:

$$\begin{vmatrix} -\lambda & ick_1 & ick_2 & ick_3 & 0 \\ ick_1 & -\nu[k^2+(V-1)k_1^2]-\lambda & -\nu(V-1)k_1k_2 & -\nu(V-1)k_1k_3 & ick_1\sqrt{\gamma-1} \\ ick_2 & -\nu(V-1)k_1k_2 & -\nu[k^2+(V-1)k_2^2]-\lambda & -\nu(V-1)k_2k_3 & ick_2\sqrt{\gamma-1} \\ ick_3 & -\nu(V-1)k_1k_3 & -\nu(V-1)k_2k_3 & -\nu[k^2+(V-1)k_3^2]-\lambda & ick_3\sqrt{\gamma-1} \\ 0 & ick_1\sqrt{\gamma-1} & ick_2\sqrt{\gamma-1} & ick_3\sqrt{\gamma-1} & \dfrac{\gamma\nu k^2}{Pr}-\lambda \end{vmatrix},$$

where $\lambda = \Gamma+i(u_mk_m-\omega)$. Thus, if the small perturbations are to exist in a fluid, they have to satisfy the characteristic condition with all its consequences. We observe that the matrix A representing the linear operator L is symmetric but not Hermitian. The transformation $k_m = ik'_m$ renders A Hermitian.

After a substantial amount of labor, the algebra being simplified due to symmetrization of the original system (2.1), the expression for the characteristic condition takes the form:

$$(\lambda+\nu k^2)^2\left[\lambda\left(\lambda+\frac{\gamma\nu k}{Pr}\right)(\lambda+V\nu k^2)+a^2k^2\left(\lambda+\frac{\nu k^2}{Pr}\right)\right] = 0.$$

The cubic factor in square brackets may be factored out under the assumption that $V\cdot Pr = 1$. This simplifies the analysis that follows. Consequently, we shall take as the characteristic condition:

$$(\lambda+\nu k^2)^2(\lambda+\nu k^2/Pr)[\lambda(\lambda+\gamma\nu k^2/Pr)+a^2k^2] = 0, \tag{2.8}$$

where $a^2 = \gamma RT$. We note a formal similarity of (2.8) to the characteristic determinant of the theory of characteristics to which (2.8) is reduced if we set $\mu = \Gamma = 0$, and identify k_m with characteristic normal. Each factor of (2.8), separately equated to zero, gives rise to a mode of wave propagation. To the first factor correspond two vorticity modes, to the second factor corresponds an entropy mode, and the quadratic factor corresponds to the acoustic mode. With the subscript α denoting the mode or the root of the characteristic relation (2.8), we have

$$\begin{aligned} &\alpha = 1, 2: && \Gamma_\alpha = -\nu k^2, && \omega_\alpha = u_m k_m, && \text{vorticity modes,} \\ (2.9)\quad &\alpha = 3: && \Gamma_\alpha = -\nu k^2/Pr, && \omega_\alpha = u_m k_m, && \text{entropy mode,} \\ &\alpha = 4, 5: && \Gamma_\alpha = -\frac{\gamma\nu k^2}{2Pr}, && \omega_\alpha = u_m k_m \pm ak\left[1-\left(\frac{\gamma\nu k}{2aPr}\right)^2\right]^{1/2}, && \text{acoustic modes.} \end{aligned}$$

It will be convenient to write the dispersion relation common to all modes as $\omega_\alpha = u_m k_m + c_\alpha ak(1-K^2)^{1/2}$, where $c_\alpha = 0$ for $\alpha = 1, 2, 3$ and $c_4 = 1$, $c_5 = -1$, and where $K = \gamma\nu k/2aPr$ = Knudsen number based on the mean wavelength, $2\pi/k$. The modes loose their identity when $K \to 1$, or $k \to 0(10^5)$ in units of cm^{-1}.

With Γ_α and ω_α determined uniquely from (2.8), the system (2.3) defines a set of eigenvectors satisfying the orthogonality and normalization conditions (2.4) provided $K < 1$. Let the second subscript β denote a component of the column vector $\{\varrho, u, v, w, T\}$. Then the most general solution of (2.3) is a linear combination of the eigenvectors

$$\begin{aligned} &\phi_{1\beta} = \{0, k_2, -k_1, 0, 0\}/\{k_1^2+k_2^2\}^{1/2}, \\ &\phi_{2\beta} = \{0, -k_1k_3, -k_2k_3, k_1^2+k_2^2, 0\}/\{k(k_1^2+k_2^2)^{1/2}\}, \\ (2.10)\quad &\phi_{3\beta} = \{1, -2ik_1K/k\sqrt{\gamma}, \ldots, \ldots, -1/\sqrt{\gamma-1}\}\Big/\left\{\frac{\gamma}{\gamma-1}+\frac{4K^2}{\gamma}\right\}^{1/2}, \\ &\phi_{4\beta} = \{-g, -ik_1\gamma c, \ldots, \ldots, g^*\sqrt{\gamma-1}\}/\{2\gamma a^2k^2\}^{1/2}, \\ &\phi_{5\beta} = \{-g^*, -ik_1\gamma c, \ldots, \ldots, g\sqrt{\gamma-1}\}/\{2\gamma a^2k^2\}^{1/2}, \end{aligned}$$

where $g = -ak[K+i(1-K^2)^{1/2}]$. Due to symmetry of the matrix A, $\phi^*_{\alpha\beta}$, the complex conjugates of $\phi_{\alpha\beta}$ are the solutions of the adjoint of Eq. (2.3).

The state vectors, $\psi_{\alpha\beta} = \phi_{\alpha\beta}\exp[\Gamma_\alpha t + i(x_m k_m - \omega_\alpha t)]$, represent solutions of the Navier-Stokes equations under the assumption of infinitesimal disturbances of a locally uniform flow. Further, the state vectors represent plane waves whose extent in space is infinite. Localized disturbances, obtained by superposition of plane waves ("wave packets"), are described by the theory of group velocity, cf., e.g., LIGHTHILL (1965). According to that theory, a wave packet at (x_m, k_m, t) moves, so that

$$(2.11)\qquad U_j = \frac{dx_j}{dt} = \frac{\partial\omega}{\partial k_j}, \quad F_j = \frac{dk_j}{dt} = -\frac{\partial\omega}{\partial x}.$$

Thus, for the α-th mode, we have

$$(2.12)\qquad U_{\alpha j} = u_j + c_\alpha a(k_j/k)(1-2K^2)(1-K^2)^{-1/2},$$

$$(2.13)\qquad F_{\alpha j} = -\left[k_m\frac{\partial u_m}{\partial x_j} + c_\alpha k(1-K^2)^{-1/2}\frac{\partial a}{\partial x_j}\right],$$

where $U_{\alpha j}$ and $F_{\alpha j}$ are the velocity of and a "force" acting on a wave packet. The force $F_{\alpha j}$ vanishes in uniform flow.

2.3. Association with quantum mechanics

Equations (2.11) are of the Hamiltonian form and permit association of fluid waves with particle motion. With the substitutions

$$\hbar\omega = H, \quad \hbar k_j = p_j, \quad x_j = q_j, \quad \hbar = h/2\pi, \tag{2.14}$$

where H — Hamiltonian, p_j — momentum, q_j — position coordinate, h — Planck's constant, Eqs. (2.11) take the familiar canonical form. Differentiating the state vector ψ once with respect to time and twice with respect to space, eliminating ψ itself and using (2.14), one obtains in the non-dissipative case, $\Gamma = 0$, the wave equation of Schroedinger, the starting point of the wave-mechanical theory of turbulence developed by KRZYWOBLOCKI (1971). We note that the quantum-mechanical problem for the Schroedinger equation, generalized to include dissipation ($\Gamma \neq 0$), namely the determination of all the characteristic energy states, has been solved in the preceding Section by giving the expressions for the frequencies (energies) of the five modes of wave propagation. In the sequel we shall use the corpuscular point of view, justified by the Hamiltonian form of Eq. (2.11), in order to apply gas-kinetic methods to the dynamics of wave packet motion.

Let $f(x_m, k_m, t)dx_1dx_2dx_3dk_1dk_2dk_3 = f dx^3dk^3$ be the number of particles in the volume dx^3dk^3 of the phase-space having position x_m and wavenumber k_m at time t. Then the Liouville's equation,

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_j}\frac{dx_j}{dt} + \frac{\partial f}{\partial k_j}\frac{dk_j}{dt} = 0,$$

must hold for the wave packets between or in absence of "collisions" as a consequence of the Hamiltonian form of the equations of motion. In the case of wave interactions, the time rate of change of the number density must be set equal to the source term, Eq. (2.7). Specializing to the α-mode, $\alpha = 1, \ldots, 5$, and using (2.11), we obtain the Maxwell-Boltzmann equation:

$$\frac{\partial f_\alpha}{\partial t} + U_{\alpha j}\frac{\partial f_\alpha}{\partial x_j} + F_{\alpha j}\frac{\partial f_\alpha}{\partial k_j} = \frac{\partial f_\alpha}{\partial t} + \frac{\partial \omega_\alpha}{\partial k_j}\frac{\partial f_\alpha}{\partial x_j} - \frac{\partial \omega_\alpha}{\partial x_j}\frac{\partial f_\alpha}{\partial k_j} = \frac{\partial_e f_\alpha}{\partial t}. \tag{2.15}$$

Equation (2.15) implies that a redistribution of any property of the wave packets in the phase-space is brought about only by wave interactions. In particular, energy may flow in the phase-space as a consequence of energy transfer between waves under three-wave and four-wave resonance condition.

In order to justify the use of f_α as number density, we proceed as follows. Let $\varepsilon_\alpha(k)$ be the energy in the α-mode per unit volume of phase space. By (2.14) the energy per particle is $\hbar\omega_\alpha$. Thus the number of particles in unit volume at time t having energy $\hbar\omega_\alpha$ is $f_\alpha = \varepsilon_\alpha(k)/\hbar\omega_\alpha$. The total content of energy in the α-mode is $\int \varepsilon_\alpha(k)dk^3 = \hbar\int \omega_\alpha f_\alpha dk^3$ per unit volume. If we require that energy be proportional to the square of amplitude then $\omega_\alpha f_\alpha$ must be proportional to $\omega_\alpha|C_\alpha|^2$. We may, therefore, normalize C_α, so that

$$\int f_\alpha dk^3 = \int |C_\alpha|^2 dk^3 = n_\alpha, \tag{2.16}$$

where n_α — number density of α-type quasi-particles in the x-space. Thus all five number densities must be known as functions of space and time in order to normalize the distribution functions, the latter being needed to form statistical averages. The subject of the choice of definitions for the statistical averages will be discussed next.

3. Statistical properties

3.1. Statistical averages

In what follows all integrals are assumed to converge, and integration is extended to the limit of validity of continuum concepts, $k \leqslant k_{max}$, for which value of k the collective wave motion ceases to be distinguishable from the molecular random motion. The choice of k_{max} will be made later.

Let m_α be the (variable) mass of a quasi-particle. Then the partial density of species α, ϱ_α, and total density of the gas, ϱ, are

$$\varrho_\alpha = \int m_\alpha f_\alpha dk^3, \quad \varrho = \sum_\alpha \varrho_\alpha.$$

Let the momentum of a quasi-particle relative to the fluid be

$$p_j = \hbar,\ k_j = m_\alpha(U_{\alpha j} - u_j).$$

Summing over all particles and requiring that the total momentum in laboratory frame be ϱu_j, and that the average of the relative momentum vanish, we have, using (2.12),

$$\hbar \sum_\alpha \int k_j f_\alpha dk^3 = \sum_\alpha \int m_\alpha a c_\alpha \frac{1-2K^2}{\sqrt{1-K^2}} \frac{k_j}{k} f_\alpha dk^3 = 0. \tag{3.1}$$

Thus f_α must satisfy the condition

$$\sum_\alpha \int k_j f_\alpha dk^3 = 0 \tag{3.2}$$

and, for consistency of the quasi-particle model, Eq. (3.1) requires that

$$m_\alpha = \frac{\hbar k \sqrt{1-K^2}}{a(1-2K^2)}, \quad \alpha = 4, 5. \tag{3.3}$$

Observe that the mass m_α, $\alpha = 4, 5$, equals 0 or ∞ when $k = 0$ or $k = \sqrt{2} aPr/\gamma\nu = k_{max}$, respectively. Thus the limits of integration, k_{max}, are chosen so that beyond k_{max} there are no quasi particles since f_α must tend to zero sufficiently fast as $k \to k_{max}$, so that all partial densities, ϱ_α, are finite. Further, our choice of m_α renders mass equal to zero when the particle moves with the signal velocity, $|U_{\alpha j} - u_j| = a$, and equal to infinity when at rest relative to the fluid.

When $\alpha = 1, 2$, or 3, the group velocity relative to the fluid vanishes for all k, and so does the relative momentum. Thus m_α is indeterminate for vorticity and entropy modes, but it must remain finite in order that the partial densities be bounded.

We may now interpret $\hbar(\omega - \omega_0) = p^2/2m = \frac{1}{2} m|U_j - u_j|^2$ as kinetic energy of the acoustic mode relative to fluid. Vorticity and entropy modes contribute nothing to quasi-kinetic energy. Let e = specific kinetic energy of quasi-particles relative to fluid. Then $\varrho e = \hbar \sum_\alpha \int \omega_\alpha f_\alpha dk^3$. Note that $\omega_0 = u_m k_m$, the Doppler frequency shift, contributes nothing to the kinetic energy due to condition (3.2).

Other averages, having the form of statistical correlations and being of major interest in the study of turbulence, may be obtained as follows. Recalling that the components of the state vector give, except for a common factor, the components of the vector of the nondimensionalized perturbations, $\{\varrho'/\varrho_0, u'/c, v'/c, w'/c, T'/(T_0\sqrt{\gamma-1})\}$, we may take, e.g., $\beta = 1$ and write

$$\frac{\varrho'_\alpha}{\varrho_0} = \int C_\alpha \phi_{\alpha 1} \exp[\Gamma_\alpha t + i(x_m k_m - \omega_\alpha t)] dk^3$$

as the perturbation of density carried by the α-mode. Thus the total perturbation of density is given by $\varrho' = \sum_\alpha \varrho'_\alpha$. Only the real part of ϱ' has a physical meaning and, due to an oscillatory nature of this representation, the time or space average of ϱ' vanishes. Setting $\beta = 2, 3, 4, 5$, we obtain, in turn, corresponding expressions for perturbations in u, v, w, T. The averages of squares of perturbations are, e.g.,

$$\overline{\left(\frac{\varrho'}{\varrho_0}\right)^2} = \sum_\alpha \int |C_\alpha|^2 |\phi_{\alpha 1}|^2 \exp(2\Gamma_\alpha t) dk^3 = \sum_\alpha \int |\phi_{\alpha 1}|^2 \exp(2\Gamma_\alpha t) f_\alpha dk^3,$$

and a typical double correlation, $\overline{(u', v')}$, is given by

$$\frac{\overline{(u'v')}}{RT_0} = \sum_{\alpha'} \sum_{\alpha''} \int\int C_{\alpha'} C_{\alpha''} \phi_{\alpha' 2} \phi_{\alpha'' 3} \exp\{(\Gamma_{\alpha'} + \Gamma_{\alpha''})t + i[x_m(k'_m + k''_m) + i(\omega_{\alpha'} + \omega_{\alpha''})t]\} dk'^3 dk''^3$$

The main contribution to this average will come from those wavenumbers for which $k'_m = -k''_m$ and $\omega_{\alpha'}(k'_m) = -\omega_{\alpha''}(-k''_m)$. Because the dispersion relation for the acoustic waves does not allow for both of these conditions to be satisfied simultaneously, the acoustic modes will have a negligible effect on double correlations between different components of the state vector. In the random phase approximation one could write for the Reynolds stress tensor, $m, n = 1, 2, 3$,

$$\frac{\overline{(u'_m u'_n)}}{RT_0} = \sum_{\alpha=1}^{\alpha=3} \int \phi_{\alpha, m+1} \phi^*_{\alpha, n+1} f_\alpha \exp(2\Gamma_\alpha t) dk^3,$$

where the $\phi_{\alpha\beta}$ are the components of the normalized eigenvectors, Eq. (2.10).

Having defined mass, momentum, and energy of quasi-particles in motion relative to fluid, we turn now to the evaluation of transport of mass, momentum, and energy.

3.2. Equation of change

Let ψ_α be any function of x_m, k_m, t, a property of α-type particle, per unit mass. Multiplying the kinetic equation (2.15) by $m_\alpha \psi_\alpha$, integrating over the k-space, and summing over all the modes, gives the equation of change

$$(3.4) \qquad \frac{\partial}{\partial t}\varrho\langle\psi\rangle + \frac{\partial}{\partial x_m}\varrho\langle U_m \psi\rangle = n\left[\overline{\left(\frac{\partial m\psi}{\partial t}\right)} + \overline{\left(U_m \frac{\partial}{\partial x_m} m\psi\right)} + \overline{\left(F_m \frac{\partial}{\partial k_m} m\psi\right)} + \Delta_e(m\psi),\right.$$

where $\langle\psi\rangle = \frac{1}{\varrho}\sum_{\alpha}\int m_\alpha \psi_\alpha f_\alpha dk^3$ — mass weighted average,

$(\bar{\psi}) = \frac{1}{n}\sum_{\alpha}\int \psi_\alpha f_\alpha dk^3$ — particle weighted average,

$\Delta_e(m\psi) = \sum_{\alpha}\int m_\alpha \psi_\alpha \frac{\partial_e f_\alpha}{\partial t} dk^3$ — moment of the collision integral.

Without summation over modes, Eq. (3.4) becomes a partial equation of change. Equation (3.4) has a formal similarity to the Chapman-Enskog equation of change for gas mixtures, CHAPMAN & COWLING (1960), with the major difference being the retention of variable mass under integral signs in (3.4).

It should be pointed out that, if ψ is not a function of k_m, then (3.4) yields the trivial result, namely that the mass-weighted average of ψ is carried by the fluid, i.e., Eq. (3.4) reduces to the product of $\langle\psi\rangle$ times the mass continuity equation.

3.3. Transport equations

Allowing for an implicit dependence of frequency ω, Eq. (2.9), on space and time through the mean values of $u_j(x_m, t)$ and $a(x_m, t)$, we have the following special cases of the equation of change.

Particle conservation. Let $\psi_\alpha = 1/m_\alpha$, so that $m_\alpha\psi_\alpha = 1$. Recalling the definition of number density, Eq. (2.16), and using derivatives of (2.9), we have

$$\frac{\partial n_\alpha}{\partial t} + \frac{\partial}{\partial x_j}[n_\alpha(u_j + ad_{\alpha j})] = \Delta_{e_\alpha}(1), \tag{3.5}$$

where $d_{\alpha j}$, the acoustic particle diffusion vector, is defined as

$$d_{\alpha j} = \int \frac{1-2K^2}{\sqrt{1-K^2}}\frac{k_j}{k}\left[\frac{f_4}{n_4} - \frac{f_5}{n_5}\right]dk^3.$$

Equations (3.5), with the source term, $\Delta_{e_\alpha}(1)$, evaluated for f_α, obtained as solutions of the kinetic equation (2.15), must be integrated subject to appropriate boundary conditions, so that the distribution functions, f_α, may be normalized. If one would define $f'_\alpha = f_\alpha/n_\alpha$, then the collision term in (3.5) could be written as a double sum of terms containing factors such as $n_\alpha n_\beta$, $n_\alpha n_\beta n_\gamma$, the summation extended over β and γ.

Conservation of mass. Let $\psi_\alpha = 1$. Then Eq. (3.4) becomes

$$\frac{\partial\varrho}{\partial t} + \frac{\partial}{\partial x_j}\varrho u_j = \sum_{\alpha}\int f_\alpha\left[\frac{\partial m_\alpha}{\partial t} + \frac{\partial\omega_\alpha}{\partial k_j}\frac{\partial m_\alpha}{\partial x_j} - \frac{\partial\omega_\alpha}{\partial x_j}\frac{\partial m_\alpha}{\partial k_j}\right]dk^3 + \Delta_e(m).$$

The sum of integrals on the right-hand side represents the time rate of change of mass per unit volume, due to the dependence of m_α on x_j, k_j and t when the quasi-particles follow their Hamiltonian trajectories between collisions, and $\Delta_e(m)$ gives the rate of change due to collisions. Without additional assumptions on the dependence of m_α for vorticity and entropy modes, the integrals on the right-hand side cannot be evaluated. We shall assume, however, that the net effect, summed over all particles and all collisions, is to conserve mass of the gas, and write

$$\frac{\partial \varrho}{\partial t} + \frac{\partial}{\partial x_j} \varrho u_j = 0.$$

Conservation of momentum. With $m_\alpha \psi_\alpha = \hbar k_j$, where $\psi_\alpha = U_{\alpha j} - u_j$, quasi-momentum per unit mass relative to fluid, imposing conditions (3.1) and (3.2) and using (2.12) and (2.13), one obtains from the equation of change

$$\frac{\partial}{\partial x_j} p a_{ij} + P \frac{\partial p}{\partial x_j} = \frac{1}{\gamma} \Delta_e(\hbar k_j),$$

where p — thermodynamic pressure — $\varrho RT = \varrho a^2/\gamma$, and where

$$P = \frac{\hbar}{\varrho a} \int \frac{k}{\sqrt{1-K^2}} (f_4 - f_5) dk^3 \text{ — acoustic pressure ratio,}$$

$$a_{ij} = \frac{\hbar}{\varrho a} \int \frac{1-2K^2}{\sqrt{1-K^2}} \frac{k_i k_j}{k} (f_4 - f_5) dk^3 \text{ — acoustic stress tensor.}$$

We note that only the acoustic modes contribute explicitly to turbulent momentum transport relative to fluid, and that the dependence on the remaining modes is implicit through the wave interaction term $\Delta_e(\hbar k_j)$. Further, due to condition (3.2), the Eulerian derivative, $\partial/\partial t + u_j \partial/\partial x_j$, does not appear in the equation of conservation of quasi-momentum relative to fluid.

Conservation of energy. With $m_\alpha \psi_\alpha = \hbar \omega_{\tilde{\alpha}}$ one obtains

$$(1 - {}^1/_2 P) \frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} + \frac{1}{\varrho} \frac{\partial}{\partial x_j} [\varrho T(u_j + u_i a_{ij} + a a_j)] = \frac{1}{c_p(\gamma-1)} \Delta_e(\hbar\omega),$$

where T is local thermodynamic temperature — $a^2/c_p(\gamma-1)$, and where

$$a_j = \frac{\hbar}{\varrho a} \int \frac{1-2K^2}{\sqrt{1-K^2}} k_j (f_4 + f_5) dk^3 \text{ — acoustic energy diffusion vector.}$$

In order to display other uses of the equation of change, we shall consider the transport of the entropy functional.

Entropy generation. Let $\psi_\alpha = -k^* \log f_\alpha$, where k* is Boltzmann constant, and take the ψ_α moment of the kinetic equation. Defining specific entropy as

$$s = -\frac{k^*}{\varrho} \sum_\alpha \int f_\alpha \log f_\alpha dk^3,$$

one obtains from the equation of change

$$\frac{\partial}{\partial t} \varrho s + \frac{\partial}{\partial x_j} [\varrho(s u_j + a s_j)] = -k^* \sum_\alpha \left\{ \frac{\partial n_\alpha}{\partial t} + \frac{\partial}{\partial x_j} [n_\alpha (u_j + a d_{\alpha j})] \right\} + \Delta_e(-k^* \log f).$$

The first term on the right-hand side represents entropy of mixing, and the second term the entropy source due to wave interactions. The entropy diffusion vector, s_j, is defined as

$$s_j = -\frac{k^*}{\varrho}\int \frac{1-2K^2}{\sqrt{1-K^2}}\frac{k_j}{k}(f_4\log f_4 - f_5\log f_5)dk^3.$$

4. Discussion

Several comments are in order. Since energy and momentum (ω and k_j) are conserved on microscopic scale in resonant wave interactions, it is reasonable to expect that $\Delta_e(\hbar\omega)$ and $\Delta_e(\hbar k_j)$, but not $\Delta_e(m)$, $\Delta_{e_\alpha}(1)$, and $\Delta_e(-k^*\log f)$, vanish identically. Such results have been demonstrated rigorously making use of certain symmetries, cf., e.g. p. 355, DAVIDSON (1967) and LITVAK (1960). Further, Chapman-Enskog perturbation procedure may be used to arrive at an approximate solution of the kinetic equation. It is, therefore, reasonable to expect that one may be able to give expressions for the scalar P, the vectors $d_{\alpha j}$, a_j and s_j, and the tensor $a_{\bar{\alpha}j}$ in terms of gradients of average fluid properties with transport coefficients defined in terms of fluid properties and "collision cross-sections" defined in terms of the distribution functions f_α.

In several special cases, e.g. DAVIDSON (1967) and Appendix C of VEDENOV (1968), the interaction integral arising from the bilinear terms contains factors of the form

$$f_2f_3 - f_1f_2 - f_1f_3 = f_1f_2f_3(f_1^{-1} - f_2^{-1} - f_3^{-1})$$

in present notation. Consequently, for equilibrium to exist, $\partial f/\partial t = 0$, it is sufficient that f^{-1} be a linear combination of the collisional invariants, k_j and ω, corresponding to quasi-particle momentum and energy, respectively. Trilinear terms lead to the same conclusion. With a, b, and c_j constant, we have for the equilibrium distribution function

$$f_\alpha = (a + b\omega_\alpha + c_jk_j)^{-1},$$

which is a distribution of the Rayleigh-Jeans type.

The present model of turbulence offers an independent justification of KRZYWOBLOCKI'S (1971) characterization of turbulence as "a self-excitable flutter (vibration) phenomenon," (p. 384) which "becomes observable above critical Reynolds number but exists everywhere above $T = 0$," (p. 368). With a five-fold infinity of energy containing states of the fluid admitted by the present theory, turbulence "exists" even when energy has an equilibrium distribution and the total amount of energy is small. The generally non-linear "flutter phenomenon" would appear naturally as a consequence of a balance between the tendency of each wave to serve as an almost infinite capacity energy storage and the tendency to exchange this energy and to return to equipartition. The balance between rates of gaining and loosing energy is governed by the non-linear interaction terms. No other external agency is needed to cause self-excitation of fluid waves safe for a right combination of the physical quantities which enter into the coefficients of the interaction terms. Evidently, that combination of the physical quantities which renders the Reynolds number sufficiently high is the experimentally observed "sufficient condition" for the "flutter phenomenon" to occur.

We have not touched in this paper upon the question of boundary conditions for the distribution function $f(x_m, k_m, t)$ which solves the kinetic equation (2.15). It is expected that boundary conditions may lead to a discrete spectrum, possibly superimposed on a continuous spectrum, as a consequence of a possible existence of a discrete set of eigenfrequencies. It is quite possible that the intermittency of turbulent boundary layer edges and Clear Air Turbulence (CAT) in stratified atmosphere may belong to a class of eigenvalue problems for the kinetic equation (2.15).

It is the sincere hope of the author that the readers will examine critically the premises on which this preliminary outline of the theory of turbulence is based, and that improvements, refinements, and generalizations will follow resulting, eventually, in applications to technical problems. Ultimately, the test of the theory lies in the correctness of its predictions and in an agreement with observations.

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