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Bifurcation of a T-periodic flow towards an nT-periodic flow and their non-linear stabilities

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We consider a basic T-periodic flow as a solution of the Navier-Stokes equations. Let us suppose that this flow becomes unstable when a parameter passes through a critical value, then we give sufficient conditions to obtain a new periodic solution with a multiple period nT, and we study its stability.

Rozpatrujemy podstawowy przepływ T-okresowy stanowiący rozwiązanie równań Naviera-Stokesa. Zakładając, że przepływ taki staje się niestateczny po osiągnięciu przez pewien parametr wartości krytycznej, podaje się warunki dostateczne do uzyskania nowych rozwiązań okresowych o zwielokrotnionym okresie nT oraz rozważa się zagadnienie stateczności tych rozwiązań.

Рассматривается основное T-периодическое течение составляющее решение уравнений Навье-Стокса. Предполагая, что такое течение становится неустойчивым, когда некоторый параметр достигает критическое значение, приводятся достаточные условия периодом nT, а также рассматривается проблема устойчивости этих решений.

1. Introduction

In 1972, C. S. Yih and C. H. Li [1] studied the convection phenomenon of a viscous incompressible fluid between two plane horizontal plates, submitted to different temperatures *periodic in time* (upper plate: $\theta = \theta_1 + \theta_2 \cos\omega t$, lower plate: $\theta = \theta_0 - \theta_2 \cos\omega t$). They observed numerically that, for a fixed value of $\theta_2/\theta_0 - \theta_1$, if they increase the Rayleigh number (proportional to $\theta_0 - \theta_1$), then, for a critical value of this parameter, the periodic known basic flow (the rest for speed, plus a periodic distribution for temperature and pressure) becomes unstable, and there appears a *new periodic flow*, either with the same period, or with a *double period*, according to the value of $\theta_2/\theta_0 - \theta_1$.

On the other hand, G. S. Markman [2] has mathematically shown for a similar problem how there may occur a bifurcation towards a new periodic solution with the same period as the basic one.

In [3], the same author formed that this is related to the fact that an eigenvalue of the monodromy operator, noted hereinafter by $S_\lambda(T)$, escapes from the unit disc, passing by 1 when $\lambda = \lambda_0$, which is the critical value of the parameter λ of the problem (such as the Rayleigh number).

Finally, D. D. Joseph [4] has given a formal method for building, for Navier-Stokes equations, a bifurcated quasi-periodic solution (with two fundamental periods) which appears when two conjugated simple eigenvalues of the monodromy operator escape from the unit disc, passing through two points, *not roots of unity*, on the unit circle. But, when the two conjugated eigenvalues cross the unit circle passing by roots of unity, his method fails.

In this paper, our method gives in general the solution in this last case, and enables us to justify mathematically the first cited work [1]—i.e., the apparition of a new periodic flow with a multiple period nT, n being determined by the eigenvalues of the monodromy operator at the neutral stability. This case is obviously very different from the case studied in [4] which is in fact not mathematically justified, where the quasi-periodic bifurcated solution is in fact periodic for certain values of the parameter λ, with period kT, but a non-fixed k.

2. General formulation

Generally the flow is characterized by (V, p) satisfying

$$(2.1) \qquad \begin{cases} \left.\begin{aligned} &\frac{\partial V}{\partial t} + V\cdot\nabla V + \nabla p = \nu \Delta V + f(t) \\ &\nabla\cdot V = 0 \end{aligned}\right\} \text{in a regular bounded domain } \Omega, \\ V|_{\partial\Omega} = a(t), \end{cases}$$

where ν is the inverse of the Reynolds number, f and a are given T-periodic vector functions (obviously not necessarily constant in space). In fact, when we have thermal phenomena, there is a coupling between this system and the energy equation, with the occurrence of temperature, but this does not change the structure of the system (2.1) (see [5]). On the other hand, in the case in which the domain of the flow is unbounded, we suppose that the flow has a spatially periodic structure such that we can always take a bounded Ω.

We assume that there exists a basic flow (V_0, p_0) solution of (2.1), which is T-periodic in time, and we note by (U, ω) the perturbation of this solution. Then we introduce some classical functional spaces to facilitate study of the evolution in time of U, the initial condition for (U, ω) being given.

Let us note

$$H = \{U \in \{L^2(\Omega)\}^3; \nabla\cdot U = 0,\ U\cdot n|_{\partial\Omega} = 0\}(^1),$$
$$\mathscr{D} = \{U \in \{H^2(\Omega)\}^3; \nabla\cdot U = 0,\ U|_{\partial\Omega} = 0\},$$
$$K = \{U \in \{H^1(\Omega)\}^3; \nabla\cdot U = 0,\ U\cdot n|_{\partial\Omega} = 0\},$$

where $H^m(\Omega)$ is the classical Sobolev space, and n the exterior normal of the domain Ω. We have the following compact imbeddings: $\mathscr{D} \subsetneq K \subsetneq H$, where we have put the usual Hilbertian structure on the spaces. Now, we project the first equation of (2.1) orthogonally on H, in $\{L^2(\Omega)\}^3$, as in LADYZHENSKAYA's [7], and we obtain, for the perturbation U, an evolution equation of the following form

$$(2.2) \qquad \frac{dU}{dt} = \mathscr{A}_\lambda(t)\,U + M(U).$$

(1) For the complete justification of this, see J. L. LIONS [6].

We seek solutions U of (2.2) in $C^0(0, \infty; \mathscr{D}) \cap C^1(0, \infty; H)$, the space of continuous and bounded functions in $\mathscr{D}$, with a continuous and bounded derivative in H. We assume the initial condition $U(0) = U_0$ belonging to $\mathscr{D}$. The characteristic parameter λ of the problem occurs by V_0 or (and) ν. We assume that λ belongs to some real (for physical meaning) compact interval I_0.

2.1. Properties of the linear operator $A_\lambda(t)$

Let us give the different properties of the operators in (2.2). First, we have a decomposition:

$$\mathscr{A}_\lambda(t) = A_\lambda + B_\lambda(t), \tag{2.3}$$

where A_λ and $B_\lambda(t)$ are linear and correspond to the parts not depending or depending on time in the term

$$\nu \Delta U - [V_0(t) \cdot \nabla U + U \cdot \nabla V_0(t)]$$

projected on H. Moreover, we can define a scalar product on $\mathscr{D}$, of the form:

$$(U, V)_{\mathscr{D}} = (A_{\lambda_0} U, A_{\lambda_0} V)_H + (U, V)_H,$$

all norms being equivalent to any λ_0 in I_0. In fact, there exists a complex domain $D_0 \supset I_0$ such that $\{A_\lambda\}_{\lambda \in D_0}$ can be extended in a holomorphic family of (A) type in H, with domain $\mathscr{D}$ (see [8] for the definition).

A classical result is that A_λ is the infinitesimal generator of a holomorphic semi-group $\{e^{A_\lambda t}\}$ in H, and is with compact resolvent in H (see [5]). Finally, this semi-group satisfies the important following estimate [9]:

$$\|e^{A_\lambda t}\|_{\mathscr{L}(K; \mathscr{D})} \leqslant c t^{-\alpha}, \quad \alpha = 3/4, \quad t \in]0, T], \tag{2.4}$$

where c is a constant and $\mathscr{L}(K; \mathscr{D})$ denotes the space of linear bounded operators from K to $\mathscr{D}$.

For the family $B_\lambda(t)$, it can easily be shown that $t \to B_\lambda(t)$ is T-periodic, continuous and bounded in $\mathscr{L}(\mathscr{D}; K)$ and holomorphic in λ. This results from the fact that for $U \in \mathscr{D}$, we have

$$V_0 \cdot \nabla U + U \cdot \nabla V_0 \in \{H^1(\Omega)\}^3.$$

2.2. Properties of the non-linear operator M

The function $U \to M(U)$ is quadratic, continuous from $\mathscr{D}$ to K, and we have a constant γ such that

$$\|M(U)\|_K \leqslant \gamma \|U\|_{\mathscr{D}}^2. \tag{2.5}$$

This is obtained immediately from the Sobolev imbedding theorems in dimension 2 or 3.

2.3. Resolution of the linearized evolution problem

Let us consider the following problem:

$$\left\{ \frac{dV}{dt} = \mathscr{A}_\lambda(t) V, \quad V(0) = V_0 \in \mathscr{D}, \right. \tag{2.6}$$

where we seek a function $t \to V(t)$ continuous in $\mathscr{D}$ for $t \geqslant 0$, with a continuous derivative in H for $t > 0$.

Knowledge of the solution of (2.6) is necessary to give us a "good formulation" of our problem — i.e., investigation of a bifurcated non trivial periodic solution of (2.2).

To solve (2.6), we consider the term $B_\lambda(t)V$ in the second member as a perturbation of $A_\lambda V$, due to the properties cited in 2.1. Indeed, we can write (2.6) as

$$(2.7)\qquad \begin{cases} V(t) = e^{A_\lambda t}V_0 + \int\limits_0^t e^{A_\lambda(t-\tau)}B_\lambda(\tau)V(\tau)d\tau, \quad V_0 \in \mathscr{D}, \\ t \to V(t) \text{ continuous in } \mathscr{D} \text{ for } t \geqslant 0, \end{cases}$$

and there exists $T > 0$ such that (2.7) can be written

$$V(t) = e^{A_\lambda t}V_0 + \mathscr{L}_t V$$

in $C^0(0, T; \mathscr{D})$, where $\|\mathscr{L}\| < 1$ (we take the norm of uniform convergence in $\mathscr{D}$ for $C^0(0, T; \mathscr{D})$). Hence $(1-\mathscr{L})^{-1}$ is bounded in $C^0(0, T; \mathscr{D})$, and we have

$$(2.8)\qquad V(t) = S_\lambda(t)V_0,\ t \in [0, T] \text{ (definition of } S_\lambda(t)).$$

In what follows, we shall have to consider the similar problem

$$\frac{dV}{dt} = \mathscr{A}_\lambda(t+\delta)V$$

with the same other conditions for V. Then we arrive at

$$(2.9)\qquad V(t) = S_\lambda(t, \delta)V_0 \text{ (definition of } S_\lambda(t, \delta)),$$

and we have the identity (resulting from the definition):

$$(2.10)\qquad S_\lambda(t-\tau, \tau)\cdot S(\tau-\eta, \eta) = S_\lambda(t-\eta, \eta), \quad t \geqslant \tau \geqslant \eta,$$

which enables us to determine $S_\lambda(t, \delta)$, $\forall t \in (0, \infty)$. The following properties of the family $S_\lambda(t, \delta)$ are not very difficult to prove:

i) $S_\lambda(\cdot, \delta)$ is strongly continuous at 0^+ in $\mathscr{D}$; $S_\lambda(0, \delta) = 1$.

ii) $S_\lambda(\cdot, \delta)$ is continuous in $\mathscr{L}(\mathscr{D})$, for $t > 0$.

iii) $S_\cdot(t, \delta)$ is analytic in D_0, with values in $\mathscr{L}(\mathscr{D})$.

iv) $S_\lambda(t, \delta)$ is compact in $\mathscr{D}$ for $t > 0$.

v) $\dfrac{\partial}{\partial t}S_\lambda(t-\tau, \tau) = \mathscr{A}_\lambda(t)\cdot S_\lambda(t-\tau, \tau) \in \mathscr{L}(\mathscr{D}; H)$.

vi) $\|S_\lambda(t, \delta)\|_{\mathscr{L}(K;\mathscr{D})} \leqslant ct^{-\alpha}$, $\alpha = 3/4$, $t \in]0, T]$.

The properties i), ii), iii), vi) can be shown from the formulation (2.7), and v) results from the construction. The property iv) results from (2.7) and the compactness of $e^{A_\lambda t}$ for $t > 0$ (see [5]); this has also been proved by G. S. Markman in [3].

Another class of very useful properties, of the family $S_\lambda(t, \delta)$, is obtained by making use of the T-periodicity of $\mathscr{A}_\lambda$. Then we obtain:

vii) $S_\lambda(t, \delta+T) = S_\lambda(t, \delta)$,

viii) $S_\lambda(t+T, \delta) = S_\lambda(t, \delta)\cdot S_\lambda(T, \delta)$, this latter being a particular case of (2.10) with vii). The identity viii) is very important for study of the behaviour of the solution $V(t)$ of (2.6) when $t \to \infty$. This fact was first noted by V. I. IUDOVICH in [10]. If the spectral radius of $S_\lambda(T, \delta)$ is less than 1, then $V(t) \to 0$ exponentially in $\mathscr{D}$, when $t \to \infty$. If the spectral radius of $S_\lambda(T, \delta)$ ("the monodromy operator") is greater than 1, then there exists V_0 in $\mathscr{D}$ such that $\|V(t)\|_{\mathscr{D}}$ is unbounded when $t \to \infty$. Note that the spectrum of $S_\lambda(T, \delta)$ is independent of δ (see [5] for the demonstration).

2.4. A good formulation for the non-linear evolution problem

First, let us consider the following nonhomogeneous problem:

$$\frac{dV}{dt} = \mathscr{A}_\lambda(t)V + f(t), \qquad V(0) = 0, \tag{2.11}$$

where $f \in C^0(0, \infty; K)$, with the same conditions on V as for (2.6). The unique solution of (2.11) is given by:

$$V(t) = S_\lambda(t)V_0 + \int_0^t S_\lambda(t-\tau, \tau)f(\tau)d\tau. \tag{2.12}$$

The demonstration of the required properties of V is analogous to that made in [11, Ch. 7].

Now, we consider the complete evolution problem:

$$\frac{dU}{dt} = \mathscr{A}_\lambda(t)U + M(U), \qquad U \in C^0(0, \infty; \mathscr{D}) \cap C^1(0, \infty; H), \qquad U(0) = U_0 \in \mathscr{D}. \tag{2.13}$$

Then, due to the resolution of (2.11), we have the following equivalent formulation:

$$U(t) = S_\lambda(t)U_0 + \int_0^t S_\lambda(t-\tau, \tau)M[U(\tau)]d\tau, \qquad U_0 \in \mathscr{D}, \qquad U \in C^0(0, \infty; \mathscr{D}). \tag{2.14}$$

Now, it is easy to show that:

• If $\operatorname{spr} S_\lambda(T) < 1$ (spectral radius), $\exists \delta > 0$, such that $\|U_0\|_{\mathscr{D}} \leqslant \delta$ induces the existence of a unique solution U of (2.13), depending analytically on U_0 and which tends exponentially towards 0 when $t \to \infty$.

• If $\operatorname{spr} S_\lambda(T) > 1$, $\exists U_0 \neq 0$ with an arbitrary fixed small norm, such that the solution U of (2.13) in $C^0(0, T_1; \mathscr{D})$ $(T_1 < \infty)$, leaves a fixed neighbourhood of 0 for $t > t_0$.

3. The necessary condition for bifurcation

If we want $U(t)$ not to tend towards 0 when $t \to \infty$, we have to suppose that the spectral radius $\operatorname{spr} S_\lambda(T)$ is at least 1. Physically this means that when the basic flow loses its stability, the parameter λ passes through a critical value λ_0 such that $\operatorname{spr} S_{\lambda_0}(T) = 1$. Now, we have to improve this point — i.e., to study how the eigenvalues of greatest moduli leave the unit disc, to yield a bifurcation of a new periodic solution.

In fact, we seek a solution of (2.13) small in norm, U_0 being unknown, such that $U(t) = U(t+nT)$, $\forall t \in \mathbf{R}$, where n is also to be determined. Now, it is not difficult to show that the following formulation is equivalent to the preceding one (here we have only expressed $U(0) = U(nT)$):

$$U(t) = S_\lambda(t)U_0 + \int_0^{t_-} S_\lambda(t-\tau, \tau)M[U(\tau)]d\tau,$$

(3.1)
$$[1 - S_\lambda(nT)]U_0 = \int_0^{nT} S_\lambda(nT-\tau, \tau)M[U(\tau)]d\tau,$$

$$U_0 \in \mathcal{D}, \quad U \in C^0(0, nT; \mathcal{D}),$$

where we seek a solution $U(t)$ remaining of small norm in $\mathcal{D}$, and where we have to determine U_0 and n.

But, for $\|U_0\|_{\mathcal{D}}$ sufficiently small, we can solve $(3.1)_1$ by using the implicit function theorem, with respect to U, on $[0, nT]$. Then we obtain $U(t) = \mathcal{U}(U_0, \lambda, t)$, $t \in [0, nT]$, with an analytic $\mathcal{U}$ in (U_0, λ) in the neighbourhood of $(0, \lambda_0)$ $\forall \lambda_0 \in D_0$. In fact, we have

(3.2)
$$\mathcal{U}(U_0, \lambda, t) = S_\lambda(t)U_0 + 0(\|U_0\|_{\mathcal{D}}^2).$$

Putting $\mathcal{U}$ in $(3.1)_2$, we obtain now in $\mathcal{D}$

(3.3)
$$[1 - S_\lambda(nT)]U_0 = \int_0^{nT} S_\lambda(nT-\tau, \tau)M[\mathcal{U}(U_0, \lambda, \tau)]d\tau,$$

which is of the type

(3.4)
$$(1 - K_\lambda)U_0 = B_\lambda(U_0, U_0) + C(U_0, \lambda),$$

with $\|C(U_0, \lambda)\|_{\mathcal{D}} \leqslant c_1\|U_0\|_{\mathcal{D}}^3$ for $\|U_0\|_{\mathcal{D}} \leqslant \delta$, and

$$B_\lambda(U_0, U_0) = \int_0^{nT} S_\lambda(nT-\tau, \tau)M[S_\lambda(\tau)U_0]d\tau \qquad \text{(quadratic in } U_0).$$

The study of the existence of a solution $U_0 \neq 0$ of (3.4) is a classical problem, because K_λ is compact in $\mathcal{D}$ and depends analytically on λ, whereas the second member is analytic in (U_0, λ) in the neighbourhood of $(0, \lambda_0)$ and begins with at least a quadratic term in U_0. We have immediately (see [12]):

THEOREM 1. *It can appear a bifurcation of the trivial solution of* (3.1), *towards an nT-periodic one, only from a* λ_0 *such that the spectrum of* $S_{\lambda_0}(T)$ *contains at least a* ζ_0 *satisfying* $\zeta_0^n = 1$.

Indeed, 1 is an eigenvalue of $S_{\lambda_0}(nT) = [S_{\lambda_0}(T)]^n$ following the property viii.

4. Calculus of bifurcated solutions

4.1. Precise hypotheses

To calculate explicitly the new solution, we have to make further precise assumptions.

H1. $\exists \lambda_0$ such that spr $[S_{\lambda_0}(T)] = 1$, $\frac{d}{d\lambda}[\operatorname{spr} S_\lambda(T)]_{\lambda=\lambda_0} > 0$. Physically, this means that the basic flow (the null solution) is stable for $\lambda < \lambda_0$ and unstable for $\lambda > \lambda_0$. On the other hand, for $\lambda = \lambda_0$, there exists a finite number of eigenvalues of $S_{\lambda_0}(T)$, of modulus 1, the other eigenvalues being of moduli strictly less than 1. We shall make now *one of the two* following assumptions:

H2a. $S_{\lambda_0}(T)$ has only $\zeta_0 = 1$ or -1 as an eigenvalue of modulus one, and ζ_0 is simple.

H2b. $S_{\lambda_0}(T)$ has only ζ_0 and $\bar{\zeta}_0$ as eigenvalues of moduli one, with $\zeta_0^n = \bar{\zeta}_0^n = 1$, and these eigenvalues are simple(²).

Note that we do not consider the case when ζ_0 and $\bar{\zeta}_0$, *not roots of unity*, are the eigenvalues of moduli 1 of $S_{\lambda_0}(T)$. We have already observed that this case has been formally treated in [4], and corresponds to a bifurcation towards a quasi-periodic solution. Now, by the perturbation theory (see [8]), we know that for $\lambda \in \mathscr{V}(\lambda_0)$ (real neighbourhood of λ_0),

1) if H2a is verified, $\exists$ a simple *real* eigenvalue $\zeta_1(\lambda)$ of $S_\lambda(T)$, satisfying

$$\zeta_1(\lambda) = \zeta_0[1+(\lambda-\lambda_0)\zeta^{(1)}+0(\lambda-\lambda_0)^2], \tag{4.1}$$

where $\zeta_0 = 1$ or -1 and $\zeta^{(1)} > 0$ by H1.

2) If H2b is verified, $\exists$ two simple conjugated eigenvalues $\zeta_1(\lambda)$ and $\bar{\zeta}_1(\lambda)$ of $S_\lambda(T)$, satisfying (4.1) with $\mathrm{Re}\,\zeta^{(1)} > 0$ by H1. Noting E_λ the invariant projection operator associated with $\zeta_1(\lambda)$, we know that ζ_1 and E are analytic functions in $\mathscr{V}(\lambda_0)$. In the case of the assumption H2b, we note $E(\lambda) = E_\lambda + \bar{E}_\lambda$, whereas in the case of H2a $E(\lambda) = E_\lambda$.

To solve (3.3), we use the Liapunov-Schmidt technique. First, we split the equation in the following manner:

$$U_0 = X+V, \quad \text{with} \quad X = E(\lambda_0)U_0, \quad V = [\mathbf{1}-E(\lambda_0)]U_0,$$

and we use the development

$$\mathbf{1}-S_\lambda(nT) = \mathbf{1}-S_{\lambda_0}(nT)-\sum_{k=1}^{\infty}(\lambda-\lambda_0)^k S_n^{(k)},$$

where $\mathbf{1}-S_{\lambda_0}(nT)$ has a bounded inverse Q in $[\mathbf{1}-E(\lambda_0)]\mathscr{D}$. Solving first with respect to V the Eq. (3.3) projected on $[\mathbf{1}-E(\lambda_0)]\mathscr{D}$, by using the implicit function theorem, we find $V = \mathscr{V}(X, \lambda)$ with some good estimations (not made explicit here). Then reporting in the equation projected on $E(\lambda_0)\mathscr{D}$, we arrive at the "bifurcation equation":

$$(\lambda-\lambda_0)E(\lambda_0)S_n^{(1)}E(\lambda_0)X+E(\lambda_0)B_{\lambda_0}(X, X)+G(X, \lambda) = 0, \tag{4.2}$$

with

$$\|G(X, \lambda)\|_{\mathscr{D}} \leqslant c_2\|X\|_{\mathscr{D}}\{|\lambda-\lambda_0|^2+\|X\|_{\mathscr{D}}\cdot|\lambda-\lambda_0|+\|X\|_{\mathscr{D}}^2\}.$$

In the case of H2a, the Eq. (4.2) is a scalar equation, whereas in the case of H2b, (4.2) has two dimensions. Moreover we have after some easy calculations:

case H2a: $E(\lambda_0)S_n^{(1)}E(\lambda_0) = n\zeta^{(1)}E_{\lambda_0}$, with $n = 1$ or 2,

case H2b: $E(\lambda_0)S_n^{(1)}E(\lambda_0) = n\zeta^{(1)}E_{\lambda_0}+n\bar{\zeta}^{(1)}\bar{E}_{\lambda_0}$.

Then we separate the two cases:

(²) Note n the smallest integer satisfying the identity.

4.2. Case of the assumption H2a

We note $U^{(0)}$ the eigenvector of $S_{\lambda_0}(T)$ for ζ_0, and $E(\lambda_0)B_{\lambda_0}(U^{(0)}, U^{(0)}) = \alpha U^{(0)}$.
H3a. We suppose $\alpha \neq 0$.

Then it is easy to see that (4.2) has one non trivial solution which is analytic at the neighbourhood of λ_0. In fact

$$X = -(\lambda-\lambda_0)\frac{n\zeta^{(1)}}{\alpha}U^{(0)}+0(\lambda-\lambda_0)^2.$$

Let us now express the theorem:

THEOREM 2. *Let the hypotheses* H1, H2a, H3a *be verified, then there exists a neighbourhood of* λ_0, $\mathscr{V}(\lambda_0)$ *such that if* $\lambda \in \mathscr{V}(\lambda_0)$, *there exists one and only one bifurcated non trivial nT-periodic solution of* (2.13), *analytic in* λ. *The principal part of this solution is*:

$$U(t) = -(\lambda-\lambda_0)\frac{n\zeta^{(1)}}{\alpha}S_{\lambda_0}(t)\,U^{(0)}+0(\lambda-\lambda_0)^2,$$

where $n = 1$ or 2 according as $\zeta_0 = 1$ or -1.

4.3. Case of the assumption H2b

We note $U^{(0)}$ and $\overline{U}^{(0)}$ the eigenvectors of $S_{\lambda_0}(T)$ for the eigenvalues ζ_0 and $\overline{\zeta}_0$, and

$$E(\lambda_0)B_{\lambda_0}(U^{(0)}, U^{(0)}) = \alpha U^{(0)}+\overline{\beta}\,\overline{U}^{(0)},$$

$$2E(\lambda_0)B_{\lambda_0}(U^{(0)}, \overline{U}^{(0)}) = \gamma U^{(0)}+\overline{\gamma}\overline{U}^{(0)}.$$

Then, if we assume

H3b. $$|\alpha\overline{\gamma}-\overline{\beta}\gamma| \neq \big||\alpha|^2-|\beta|^2\big|,$$

$$2|(\alpha\overline{\zeta}^{(1)}-\overline{\gamma}\zeta^{(1)})^2+3\overline{\beta}\zeta^{(1)}(\gamma\overline{\zeta}^{(1)}-\overline{\alpha}\zeta^{(1)})| \neq |9|\beta\overline{\zeta}^{(1)}|^2-|\alpha\overline{\zeta}^{(1)}-\overline{\gamma}\zeta^{(1)}|^2|$$

to be verified, there always exists at least one non trivial solution of (4.2), having the following form:

$$X(\lambda) = (\lambda-\lambda_0)Y_0+\sum_{k=2}^{\infty}(\lambda-\lambda_0)^k X_k,$$

where $Y_0 \in E(\lambda_0)$ satisfies

$$[n\zeta^{(1)}E_{\lambda_0}+n\overline{\zeta}^{(1)}\overline{E}_{\lambda_0}]\,Y_0+E(\lambda_0)B_{\lambda_0}(Y_0, Y_0) = 0. \tag{4.3}$$

THEOREM 3. *Let the hypotheses* H1, H2b, H3b *be verified, then there exists a neighbourhood of* $\lambda_0, \mathscr{V}(\lambda_0)$, *such that if* $\lambda \in \mathscr{V}(\lambda_0)$, *there exists at least one bifurcated non trivial nT-periodic solution of* (2.13), *analytic in* λ. *The principal part of these solutions is of the form*:

$$U(t) = (\lambda-\lambda_0)S_{\lambda_0}(t)\,Y_0+0(\lambda-\lambda_0)^2, \tag{4.4}$$

where Y_0 *is a non trivial real solution of* (4.3) *in* $E(\lambda_0)\mathscr{D}$.

5. Stability of a bifurcated solution

5.1. General remarks

Let us pose $\mathscr{U}(t, \lambda)$ the bifurcated solution (4.4), and note $U = \mathscr{U}+V$; then the equation satisfied by V is:

$$\frac{dV}{dt} = \mathscr{A}'_\lambda(t)V+M(V), \tag{5.1}$$

where

$$\mathscr{A}'_\lambda(t) = \mathscr{A}_\lambda(t)+DM[\mathscr{U}(t, \lambda)],$$

DM denoting the derivative of M, which takes its values in $\mathscr{L}(\mathscr{D}; K)$. We remark that $\mathscr{A}'_\lambda(t)$ has the same structure as previously $\mathscr{A}_\lambda(t)$. In an analogous way we can define the operator $S'_\lambda(t)$ by:

$$S'_\lambda(t)V_0 = S_\lambda(t)V_0+\int_0^t S_\lambda(t-\tau, \tau)DM[\mathscr{U}(\tau, \lambda)]\cdot S'_\lambda(\tau)V_0\,d\tau. \tag{5.2}$$

Then we arrive at the monodromy operator $S'_\lambda(nT)$, *the spectral radius of which determines the stability of the solution* $V \equiv 0$ of (5.1). Now, using (5.2), we have

$$S'_\lambda(nT) = S_{\lambda_0}(nT)+(\lambda-\lambda_0)S'^{(1)}+0(\lambda-\lambda_0)^2. \tag{5.3}$$

We can obviously verify that for $\lambda = \lambda_0$, $S'_{\lambda_0}(nT) \equiv S_{\lambda_0}(nT)$ because of $\mathscr{A}'_{\lambda_0}(t) \equiv \mathscr{A}_{\lambda_0}(t)$. Moreover, we can calculate $S'^{(1)}$:

$$S'^{(1)} = S_n^{(1)}+\int_0^{nT} S_{\lambda_0}(nT-\tau, \tau)DM[S_{\lambda_0}(\tau)Y_0]\cdot S_{\lambda_0}(\tau)[\cdot]d\tau,$$

due to (4.4) and (5.2). But, using the definition of $B_{\lambda_0}^{nT}$ (§ 3), this can be written

$$S'^{(1)} = S_n^{(1)}+2B_{\lambda_0}(Y_0, \cdot). \tag{5.4}$$

Now, by (5.3), we know that the spectrum of $S'_\lambda(nT)$ is obtained by a perturbation from that of $S_{\lambda_0}(nT)$, where 1 is a semi-simple eigenvalue, other eigenvalues being of smaller moduli. By the perturbation theory (see [8]) we know that an eigenvalue of $S'_\lambda(nT)$ near 1 has the form:

$$\zeta(\lambda) = 1+(\lambda-\lambda_0)\zeta^{(1)}+o(\lambda-\lambda_0),$$

where $\zeta^{(1)}$ is an eigenvalue of $E(\lambda_0)S'^{(1)}E(\lambda_0)$.

5.2. Case of the assumptions H2a, H3a

We can easily calculate the operator $E(\lambda_0)S'^{(1)}E(\lambda_0)$ which operates in a one-dimensional space; we find

$$E(\lambda_0)S'^{(1)}E(\lambda_0) = -n\zeta^{(1)}E_{\lambda_0}.$$

Now, $\zeta^{(1)}$ is positive, hence we can conclude with the important

THEOREM 2′. *Let the assumptions of Theorem 2 be verified, then for $\lambda > \lambda_0$ the bifurcated solution is stable, the null solution being unstable, whereas for $\lambda < \lambda_0$ the bifurcated solution is unstable, the null solution being stable.*

In fact, the eigenvalue of $S'_\lambda(nT)$ which is of modulus near 1 has the form:

$$\zeta(\lambda) = 1 - n\zeta^{(1)}(\lambda - \lambda_0) + 0(\lambda - \lambda_0)^2,$$

where $n = 1$ or 2 according as $\zeta_0 = 1$ or -1.

Remark. Here we have a phenomenon analogous to "the exchange of stability" for the bifurcation of stationary solutions of Navier-Stokes equations (see [11]).

5.3. Case of the assumptions H2b, H3b

Taking in the space $E(\lambda_0)\mathscr{D}$ the basis $\{U^{(0)}, \overline{U^{(0)}}\}$, and denoting $Y_0 = a_0 U^{(0)} + \bar{a}_0 \overline{U^{(0)}}$, the matrix of the operator $E(\lambda_0) S'^{(1)} E(\lambda_0)$ is:

$$\begin{bmatrix} n\zeta^{(1)} + 2a_0\alpha + \bar{a}_0\gamma & a_0\gamma + 2\bar{a}_0\beta \\ 2a_0\bar{\beta} + \bar{a}_0\bar{\gamma} & n\bar{\zeta}^{(1)} + a_0\bar{\gamma} + 2\bar{a}_0\bar{\alpha} \end{bmatrix} \tag{5.5}$$

and H3b ensures its invertibility. Unfortunately, we can prove that the eigenvalues of (5.5) are not necessarily of the same sign. Hence we cannot conclude on the stability or instability of the bifurcated solution, in the general case.

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