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Stationary heat exchange in a system of two spheres in uniform rectilinear motion through a free-molecule medium

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THE PRESENT paper is concerned with a particular problem of stationary interaction of a system of two spheres moving in a free-molecule medium. This is a problem of heat exchange at equal temperatures of the spheres. It is demonstrated that the solution of the continuity equations (impermeability of the wall), which constitute a set of Fredholm integral equations of the second kind can be avoided in this case, in which the heat exchange can be expressed by energy fluxes and numbers of incident particles coming only from the ambient medium and inot from the other sphere, which is the general case). The expression for heat exchange is obtaned in an effective manner in the case of hypersonic motion. As an auxiliary problem the problem of screening of a sphere by the other sphere is solved in detail. The quintuple quadratures of interaction representing heat exchange are expressed in terms of elementary functions.

Praca dotyczy szczególnego zagadnienia stacjonarnego oddziaływania układu dwu kul, poruszającego się w ośrodku swobodno-molekularnym, mianowicie wymiany ciepła przy równych temperaturach kul. Udowodniono, że w przypadku tym można uniknąć rozwiązywania równań ciągłości (nieprzenikalności ścianki), które reprezentują sobą układ równań całkowych Fredholma II rodzaju; wymiana ciepła w takim przypadku daje się wyrazić przez strumienie energii i ilości cząstek, padających tylko z otoczenia (a nie również z drugiej kuli, jak jest w przypadku ogólnym). Wyrażenie na wymianę ciepła zostało efektywnie uzyskane dla przypadku ruchu hipersonicznego w ośrodku. Jako zagadnienie pomocnicze rozwiązano szczegółowo ekranowanie jednej kuli przez drugą. Pięciokrotne kwadratury oddziaływania, reprezentujące wymianę ciepła, wyrażają się przez funkcje elementarne.

Работа касается особенной проблемы стационарного взаимодействия системы двух шаров, движущейся в свободно-молекулярной среде, именно теплообмена при равных температурах шаров. Доказано, что в этом случае можно обойти решение уравнений непроницаемости стенки, которые представляют собой систему интегральных уравнений Фредгольма II рода — теплообмен в этом случае дается выразить через потоки энергии и количества частиц, падающих только из среды (но нет тоже из второго шара, как это есть в общем случае). Формула для теплообмена была эффективно получена для случая гиперзвукового движения в среде. В характере вспомагательной задачи решено детально экранирование одной сферы — другою. 5-кратные интегралы взаимодействия, представляющие теплообмен, выражаются через элементарные функции.

Introduction

THE SOLUTION of the problem of stationary heat exchange gives a (quantitative) answer to the question as to how the bodies belonging to the system considered should be cooled or heated, so that their temperatures may remain constant during motion through the ambient medium. In the general form — that is, for any temperatures of the bodies, any distances between them and any velocity of the system — this problem cannot be tackled even numerically. Thus, in the simplest case of spheres at rest, the solution can be reduced to ordinary integrals which can be found by numerical means only. It emerges, however, that in a certain particular case the problem of heat exchange can be accurately solved by analytic means (this fact does not concern the problem of drag). This is the case of equal temperatures of the spheres (the temperature of the ambient gas being different, of course) and the diffusion model of reflection of gas from the surfaces of the bodies.

This solution is not effective, however, since it is expressed in quadratures which cannot be performed by analytic means. An effective analytic solution can be obtained accurately if the motion of the system is hypersonic.

The problem of heat exchange for a system moving at a hypersonic velocity is important for the prediction of the thermal behaviour of satellite systems moving through the space, and also for the control of their thermal characteristics.

In the non-stationary case, this problem is coupled with that of drag, because the temperatures of the bodies resulting from the heat exchange between the bodies and the ambient medium have a direct influence on the value of the drag. Examples of solutions of the problem of heat exchange for convex bodies may be found in the monographs [1, 2] and for a simple system of two bodies (two parallel plates) in Ref. [3].

1. General expression of the heat exchange and its derivation in an explicit form for spheres at equal temperatures

Let us consider two spherical bodies K_1, K_2 of radii R_1, R_2, and temperatures $T_1 = T_2 = T$, moving in a free-molecule medium the temperature of which is $T^0 \neq T$, at velocity $\mathbf{q}_1 = \mathbf{q}_2 = \mathbf{q}$ (Fig. 1).

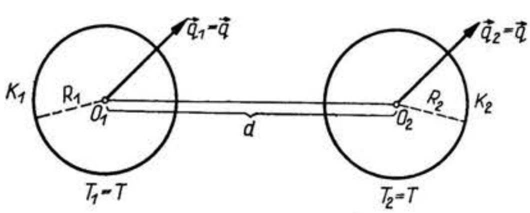


FIG. 1.

It is assumed that

1) $\lambda \gg d$ — that is, the mean free path λ of molecules is considerably longer than the distance d between the bodies and their dimensions;

2) the temperature fields of the bodies are homogeneous, steady and equal (they may be maintained in an artificial manner);

3) if there are no bodies in the medium, it is in a state of global thermodynamic equilibrium, the distribution function being a Maxwell-Boltzmann function;

4) the interaction between the surface of a body and the gas is assumed to be of the diffusion reflection type.

The energy flux E_1 furnished to the sphere K_1 is a difference between the incident in flux $E_1^{(i)}$ and the reflected flux $E_1^{(r)}$

$$E_1 = E_1^{(i)} - E_1^{(r)}. \tag{1.1}$$

The energy flux of incident particles $E_1^{(i)}$ is composed of the fluxes $E_{1(0)}^{(i)}$ and $E_{1in}^{(i)K_2}$ originating from the ambient medium and the sphere K_2, respectively,

$$E_1^{(i)} = E_{1(0)}^{(i)} + E_{1(in)}^{(i)K2}. \tag{1.2}$$

This may also be expressed in the form of a sum of fluxes $E_{10}^{(i)*}$ and $E_{1(in)}^{(i)}$, the former originating from the ambient medium in the absence of the sphere K_2, and the latter connected with the perturbing action of the sphere K_2:

$$E_1^{(i)} = E_{10}^{(i)*} + E_{1(in)}^{(i)}, \tag{1.3}$$

where

$$E_{10}^{(i)*} = E_{10}^{(i)} + E_{1(in)}^{(i)(0)}, \qquad E_{1(in)}^{(i)} = E_{1(in)}^{(i)K2} - E_{1(in)}^{(i)(0)} \tag{1.3'}$$

and $E_{1(in)}^{(i)(0)}$ is the energy flux originating from the ambient medium in the space region screened by the sphere K_2.

The energy fluxes $E_{1(0)}^{(i)*}$ and $E_{1(in)}^{(i)}$ are expressed thus:

$$E_{1(0)}^{(i)*} = \frac{1}{2} m \int_{\Sigma_1} \Big[\int_{\Omega_{1/2}^c} c_{01}^2 (-\mathbf{c}_{01} \cdot \mathbf{n}_1) f_z^{(i)(1)} d^3\mathbf{c}_{01} \Big] d\Sigma_1, \tag{1.4}$$

$$E_{1(in)}^{(i)K2} = \frac{1}{2} m \int_{\Sigma_{w1}} \Big[\int_{\Omega_{K2(P1)}^c} c_{21}^2 (-\mathbf{c}_{21} \cdot \mathbf{n}_1) f_{K2}^{(i)(1)} d^3\mathbf{c}_{21} \Big] d\Sigma_1, \tag{1.5}$$

$$E_{1(in)}^{(i)(0)} = \frac{1}{2} m \int_{\Sigma_{w1}} \Big[\int_{\Omega_{K2(P1)}^c} c_{01}^2 (-\mathbf{c}_{01} \cdot \mathbf{n}_1) f_z^{(i)(1)} d^3\mathbf{c}_{01} \Big] d\Sigma_1, \tag{1.6}$$

where

- m mass of a gas particle,
- $\mathbf{c}_{01}$ velocity of a particle of the ambient medium with reference to the sphere K_1,
- $\mathbf{n}_1$ external normal to the sphere K_1 at an arbitrary point P_1,
- $d^3\mathbf{c}$ volume element of the velocity space,
- Σ_1 total surface of the sphere K_1,
- $d\Sigma$ surface element,
- $\Omega_{1/2}^c$ velocity semi-space connected with the normal $\mathbf{n}_1$,
- $\mathbf{c}_{21}$ velocity of a particle emitted from K_2 as referred to K_1; $\mathbf{c}_{21} = \mathbf{c}_{22} - (\mathbf{q}_1 - \mathbf{q}_2) \equiv \mathbf{c}_{22}$, since $\mathbf{q}_1 = \mathbf{q}_2$,
- Σ_{w1} internal surface of the sphere K_1 (that is, the surface viewed from the sphere K_2),
- $\Omega K_\alpha(P_\beta)$ solid angle of view of K_α from the point P_β; $\alpha, \beta = 1, 2$; $\alpha \neq \beta$,
- $f_z^{(i)(1)}$ velocity distribution function of particles of the ambient medium in the system connected with the sphere K_1; $f_z^{(i)(1)} = A_z^{(i)} e^{-B_z^{(i)}(\mathbf{c}_{01}+\mathbf{q}_1)^2}$,
- $f_{K2}^{(i)(1)}$ velocity distribution function of particles emitted from K_2 in the system connected with the sphere K_1; $f_{K2}^{(i)(1)} = A_{K2}^{(i)} e^{-B_2(\mathbf{c}_{21}+\mathbf{q}_1-\mathbf{q}_2)^2}$,
- P_1, P_2 arbitrary points on the surfaces of the spheres K_1 and K_2, respectively,
- $\Omega_{K_\alpha(P_\beta)}^c$ region in the velocity space, corresponding to $\Omega_{K_\alpha(P_\beta)}$.

The energy flux of the reflected particles $E_1^{(r)}$ will be represented as a sum of the flux $E_{1z}^{(r)}$ reflected from the outer part of the surface and the flux $E_{1w}^{(r)}$ reflected from the inner part of the surface. (By the term "inner part" of the surface Σ_{1w} of the sphere K_1 we understand that part of its surface which is viewed from the sphere K_2 and by the outer part of the surface Σ_{21} of the same sphere that part of its surface which is not viewed from K_2)

$$E_1^{(r)} = E_{1z}^{(r)} + E_{1w}^{(r)}. \tag{1.7}$$

The fluxes $E_{1z}^{(r)}$ and $E_{1w}^{(r)}$ are expressed thus:

$$E_{1z}^{(r)} = \frac{1}{2} m \oint_{\Sigma_{z1}} \Big[\int_{\Omega_{1/2}^c} c_{11}^2 (\mathbf{c}_{11} \cdot \mathbf{n}_1) f_{1z}^{(r)} d^3 \mathbf{c}_{11} \Big] d\Sigma_1, \tag{1.8}$$

$$E_{1w}^{(r)} = \frac{1}{2} m \int_{\Sigma_{w1}} \Big[\int_{\Omega_{1/2}^c} c_{11}^2 (\mathbf{c}_{11}^2 \cdot \mathbf{n}_1) f_{1w}^{(r)} d^3 \mathbf{c}_{11} \Big] d\Sigma_{1w}, \tag{1.9}$$

where

$\mathbf{c}_{11}$ velocity of a particle reflected from the sphere K_1 in the system connected with K_1,

$f_{1z}^{(r)}$ distribution function of particles reflected from the outer part of the surface of K_1; $f_{1z}^{(r)} = A_{1z}^{(r)} e^{-B_1 c_{11}^2}$,

$f_{1w}^{(r)}$ distribution function of particles reflected from the inner part of the surface Σ_{w1}; $f_{1w}^{(r)} = A_{1w}^{(r)} e^{-B_1 c_{11}^2}$.

The interaction flux $E_{1(in)}^{(i)K_2}$ originating from the sphere K_2, originally written with reference to the sphere K_1, will now be written with reference to the sphere K_2:

$$E_{1in}^{(i)K2} = \frac{1}{2} m \int_{\Sigma_{w2}} \Big[\int_{\Omega_{K1(P2)}^c} c_{21}^2 (\mathbf{c}_{22} \cdot \mathbf{n}_2) f_{K2}^{(i)(2)} d^3 \mathbf{c}_{22} \Big] d\Sigma_{w2}. \tag{1.10}$$

Since it has been assumed that $f_{K2}^{(i)}$ is independent of time (the motion is stationary), the symbol $A_{K2}^{(i)(2)}(P_2)$ may be written before the sign of integration over the velocity space $\mathbf{c}_{22}$, thus reducing $E_{1in_1}^{(i)K2}$ (owing to the assumption of $\mathbf{q}_1 = \mathbf{q}_2$) to the form:

$$E_{1(in)}^{(i)K2} = \frac{1}{2} m g_{2c} \int_{\Sigma_{w2}} A_{K2}^{(i)(2)}(P_2) g_{2g} d\Sigma_{w2}, \tag{1.11}$$

where

$$g_{2c} = \int_0^\infty c_{22}^5 e^{-B_2 c_{22}^2} dc_{22}, \tag{1.12}$$

$$g_{2g} = \int_{\Omega_{K1(P2)}} (\mathbf{l}_{22} \cdot \mathbf{n}_2) d\Omega_{\mathbf{l}_{22}}, \qquad \mathbf{l}_{22} \overset{\mathrm{df}}{=} \frac{\mathbf{c}_{22}}{c_{22}}, \tag{1.13}$$

$d\Omega_{\mathbf{l}_{22}}$ denoting an element of the solid angle connected with the direction $\mathbf{l}_{22}$. Similarly, $E_{1w}^{(r)}$ can be expressed in the form:

$$E_{1w}^{(r)} = \frac{1}{2} m h_{1c} h_{1g} \int_{\Sigma_{w1}} A_{1w}^{(r)}(P_1) d\Sigma_{1w}, \tag{1.14}$$

where

$$h_{1c} = \int_0^\infty c_{11}^5 e^{-B_1 c_{11}^2} dc_{11}, \tag{1.15}$$

$$h_{1g} = \int_{\Omega_{1/2}} (\mathbf{l}_{11} \cdot \mathbf{n}_1) d\Omega_{\mathbf{l}_{11}}, \quad \mathbf{l}_{11} \overset{\text{df}}{=} \frac{\mathbf{c}_{11}}{c_{11}}. \tag{1.16}$$

The remaining fluxes $E_{10}^{(i)*}$, $E_{1in}^{(i)(0)}$, $E_{z1}^{(r)}$, which are needed for expressing the heat exchange E_1, can be obtained directly by quadratures, all the quantities being known, with the sole exception of $A_{z1}^{(r)}(P_1)$ in (1.8). This can easily be found, however, from the local continuity equation(¹) at the point P_1 of the external surface. (Case with which this may be done is a consequence of the fact that the external surface is not perturbed by the influence of the sphere K_2):

$$N_{z1}^{(i)}(P_1) = N_{z1}^{(r)}(P_1), \tag{1.17}$$

where

$$N_{z1}^{(i)}(P_1) = \int_{\Omega_{1/2}^c} (-\mathbf{c}_{01} \cdot \mathbf{n}_1) f_z^{(i)(1)} d^3\mathbf{c}_{01}, \tag{1.18}$$

$$N_{z1}^{(r)}(P_1) = \int_{\Omega_{1/2}^c} (\mathbf{c}_{11} \cdot \mathbf{n}_1) f_{z1}^{(r)} d^3\mathbf{c}_{11}. \tag{1.19}$$

Thus, we obtain for the heat exchange E_1 the expression:

$$E_1 = E_{10}^{(i)*} - E_{1(in)}^{(i)(0)} + \frac{1}{2} m g_{2c} \int_{\Sigma_{w2}} A_{K2}^{(i)(2)}(P_2) g_{2g} d\Sigma_{w2} - E_{1z}^{(r)} - \frac{1}{2} m h_{1c} h_{1g} \int_{\Sigma_{w1}} A_{1w}^{(r)}(P_1) d\Sigma_{w1}. \tag{1.20}$$

An analogous expression is obtained for the heat furnished to the sphere K_2 (This can be achieved by a formal change of indices $1 \rightleftharpoons 2$):

$$E_2 = E_{20}^{(i)*} - E_{2(in)}^{(i)(0)} + \frac{1}{2} m g_{1c} \int_{\Sigma_{w1}} A_{K1}^{(i)(1)}(P_1) g_{1g} d\Sigma_{w1} - E_{2z}^{(r)} - \frac{1}{2} m h_{2c} h_{2g} \int_{\Sigma_{w2}} A_{2w}^{(r)}(P_2) d\Sigma_{w2}, \tag{1.21}$$

$$g_{1c} = g_{1c}, \quad g_{1g} = \int_{\Omega_{K2(P1)}} (\mathbf{l}_{11} \cdot \mathbf{n}_1) d\Omega_{\mathbf{l}_{11}},$$

$$h_{2c} = g_{2c}, \quad h_{2g} = \int_{\Omega_{1/2}} (\mathbf{l}_{22} \cdot \mathbf{n}_2) d\Omega_{\mathbf{l}_{12}} = h_{1g}.$$

To complete the solution, we must obtain the quantities $A_{1w}^{(r)}(P_1)$ and $A_{2w}^{(r)}(P_2)$ $(A_{K2}^{(i)(2)} \equiv A_{2w}^{(r)}, A_{K1}^{(i)(1)} \equiv A_{1w}^{(r)})$, which are still unknown. We can attempt to find them from the

(¹) The condition of impermeability of the wall.

local continuity equations for the point P_1 of the inner part of the surface of the sphere K_1 and for the point P_2 of the inner part of the surface of the sphere K_2. The situation is not easy, however, because the continuity equations for points of the inner parts of the surface constitute a set of two complicated Fredholm integral equations of the second kind with four variables (the points P_1, P_2 on the surfaces of the spheres being determined in an unequivocal manner by prescribing the relevant directions). In the case of complete symmetry $\mathbf{q}\|\mathbf{0}_1\mathbf{0}_2$, we should have two variables only. This difficulty can, however, be avoided.

Such is the case of equal temperatures of the spheres, $T_1 = T_2 = T \neq T_0$. To demonstrate this, let us write the overall equations for streams of particles. The stream of entering particles is equal to the stream of a body leaving particles.

The overall continuity equations are obtained in the same manner as for the exchange of energy:

$$(1.22)\qquad N_1 = N_{10}^{(i)*} - N_{1(in)}^{(i)(0)} + g_{2c}^{(N)} \int\limits_{\Sigma_{w2}} A_{w2}^{(r)(2)}(P_2) g_{2g}^{(N)} d\Sigma_{w2} - N_{1z}^{(r)} - h_{1c}^{(N)} h_{1g}^{(N)} \int\limits_{\Sigma_{w1}} A_{1w}^{(r)}(P_1) d\Sigma_{w1} = 0,$$

$$(1.23)\qquad N_2 = N_{20}^{(i)*} - N_{2(in)}^{(i)(0)} + g_{1c}^{(N)} \int\limits_{\Sigma_{w1}} A_{w1}^{(r)(1)}(P_1) g_{1g}^{(N)} d\Sigma_{w1} - N_{2z}^{(r)} - h_{2c}^{(N)} h_{2g}^{(N)} \int\limits_{\Sigma_{w2}} A_{2w}^{(r)}(P_2) d\Sigma_{w2} = 0,$$

where

$$g_{2c}^{(N)} = \int\limits_0^\infty c_{22}^3 e^{-B_2 c_{22}^2} dc_{22}, \qquad g_{1c}^{(N)} = \int\limits_0^\infty c_{11}^3 e^{-B_1 c_{11}^2} dc_{11},$$

$$(1.24)\qquad h_{1c}^{(N)} = \int\limits_0^\infty c_{11}^3 e^{-B_1 c_{11}^2} dc_{11} = g_{1c}^{(N)},$$

$$h_{2c}^{(N)} = \int\limits_0^\infty c_{22}^3 e^{-B_2 c_{22}^2} dc_{22} = g_{2c}^{(N)};$$

N_1, N_2 are the total streams of particles supplied to the spheres K_1 and K_2, respectively Since the integrals in the expressions of the exchange of particles and energy differ by the power of the velocity modulus, we have immediately:

$$(1.25)\qquad g_{2g}^{(N)} = g_{2g}, \quad h_{2g}^{(N)} = h_{2g}, \quad h_{1g}^{(N)} = h_{1g}, \quad g_{1g}^{(N)} = g_{1g}.$$

If now we assume that $T_1 = T_2$ — that is, $B_1 = B_2$ — then $g_{2c}^{(N)} = h_{1c}^{(N)}$, and from the. continuity Eqs. (1.22), we find that:

$$(1.26)\qquad \int\limits_{\Sigma_{w2}} A_{w2}^{(r)(2)}(P_2) g_{2g}^{(N)} d\Sigma_{w2} - h_{1g}^{(N)} \int\limits_{\Sigma_{w1}} A_{1w}^{(r)}(P_1) d\Sigma_{w1} = \frac{1}{g_{2c}^{(N)}} (-N_{10}^{(i)*} + N_{1in}^{(i)(0)} + N_{1z}^{(r)}).$$

The same difference will be observed, with the same assumption, in the expression for the exchange of energy ($T_1 = T_2 \to q_{2c} = h_{1c}$). This fact enables us to eliminate this difference in the expression for the exchange of energy (1.20), making use of the continuity equation in the form (1.26). Thus, finally, we obtain the following expression for the exchange of energy:

$$(1.27) \qquad E_1 = E_{1(0)}^{(i)*} - E_{1(in)}^{(i)(0)} - E_{1z}^{(r)} + \frac{1}{2} m \frac{g_{2c}}{g_{2c}^{(N)}} (N_{1z}^{(r)} + N_{1(in)}^{(i)(0)} - N_{1(0)}^{(i)*}).$$

Bearing in mind that

$$(1.28) \qquad E_{1z}^{(r)} = \frac{1}{2} m \frac{g_{2c}}{g_{2c}^{(N)}} N_{1c}^{(r)},$$

the expression for the exchange of energy is seen to be considerably simplified:

$$(1.29) \qquad E_1 = E_{10}^{(i)*} - E_{1(in)}^{(i)(0)} + \frac{1}{2} m \frac{g_{2c}}{g_{2c}^{(N)}} (N_{1(in)}^{(i)(0)} - N_{1(0)}^{(i)*}).$$

It is seen, as a consequence, that the expression of $A_{z1}^{(r)}$ starting out from the local continuity equation at the point P_1 is no longer necessary.

The analogous expression for the energy exchange for the sphere K_2 is:

$$(1.30) \qquad E_2 = E_{2(0)}^{(i)*} - E_{2(in)}^{(i)(0)} + \frac{1}{2} m \frac{g_{2c}}{g_{2c}^{(N)}} (N_{2(in)}^{(i)(0)} - N_{20}^{(i)*}).$$

Thus, the solution of the problem of energy exchange between the ambient medium and the spheres moving at equal velocities is reduced, if their temperatures are equal, to fivefold quadratures. (The equality of temperatures eliminates the necessity of solving a complicated set of integral equations). The quadratures for $E_{(0)}^{(i)*}$, $N_{(0)}^{(i)*}$ do not present major difficulties. They are expressed in terms of erf functions, by contrast with the quadratures for $E_{(in)}^{(i)(0)}$, $N_{(in)}^{(1)(0)}$ which can be reduced to single integrals only even in the case of spheres at rest with reference to the ambient medium. However, these quadratures can be performed, if the system of spheres moves through the ambient medium at hypersonic velocity — that is, if

$$(1.31) \qquad q \Big/ \left(\frac{2kT_0}{m}\right)^{\frac{1}{2}} \gg 1.$$

In such a case the incident stream of gas may be considered to be homogeneous. If we observe that hypersonic velocities are realized in the space adjacent to the Earth, the problem becomes of importance for astronautical practice, since its solution may furnish information on the thermal behaviour of satellite systems.

For the hypersonic problem, the general expression of the heat exchange can be expressed (on the basis of the Eqs. (1.29), (1.2), (1.3) and (1.3′) in the more convenient form:

$$(1.32) \qquad E_1 = E_{1(0)}^{(i)} - \frac{1}{2} m \frac{g_{2c}}{g_{2c}^{(N)}} N_{1(0)}^{(i)}.$$

To evaluate the energy flux $E^{(i)}_{1(0)}$ and the particles flux $N^{(i)}_{1(0)}$ of incident particles for the sphere K_1, we must first, in the general case — that is, for a velocity of the system directed in an arbitrary manner with reference to the line connecting the centres of the two spheres — solve the problem of screening of one sphere by the other.

2. Determination of the screened region

If the two bodies are located in a homogeneous flow, one of them is screened by the other. A body is screened if the direction from the other body to the body considered coincides with the direction of flow. Thus, a certain region of its surface will not be "irradiated" by the flowing particles. The problem of finding the screened region for a system of two spheres K_1 and K_2 is reduced to analysis of the intersection between the cylinder whose generator lines are tangent to one of the spheres and are parallel to the direction of flow and the surface of the other sphere. The screened region is the surface region of the sphere considered, bounded by the intersection line with the cylinder. Let us consider

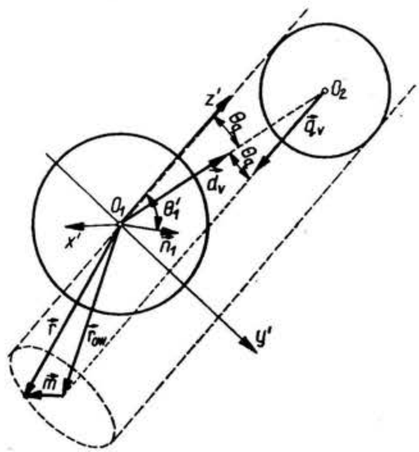


Fig. 2.

a Cartesian system of coordinates with its origin at 0_1 (Fig. 2). The direction of the z'-axis will be parallel and opposite to the direction of flow ($\mathbf{q}$), the y'-axis will lie in the (z', $\mathbf{0}_1\mathbf{0}_2$) plane, and $\mathbf{j}'\cdot\mathbf{d}_v \geqslant 0$ $\left(\mathbf{d}_v \overset{\text{df}}{=} \dfrac{\mathbf{0}_1\,\mathbf{0}_2}{0_1\,0_2}, \mathbf{j}' \text{— unit vector in } y'\text{-axis direction}\right)$. The intersection curve KP of the cylinder W and the sphere K_1 is a solution of the set of equations:

$$KP \equiv \begin{cases} R\cdot W \text{ (the equation of the cylinder)} \\ R\cdot K_1 \text{ (the equation of the sphere } K_1\text{).} \end{cases}$$

In the vector notation, this set of equations is:

$$\mathbf{r} = R_1\mathbf{n}_1, \tag{2.1}$$

$$(\mathbf{m})^2 = R_2^2, \tag{2.2}$$

$$\mathbf{m}\cdot\mathbf{q}_v = 0, \tag{2.3}$$

where

$$\mathbf{m} = \mathbf{r}-\mathbf{r}_{0w}, \quad \mathbf{r}_{0w} = \mathbf{d}+\lambda\mathbf{q}_v, \quad \mathbf{q}_v \stackrel{\mathrm{df}}{=} \frac{\mathbf{q}}{q} \tag{2.3'}$$

and

- $\mathbf{r}$ radius vector of a surface point of the sphere or cylinder,
- $\mathbf{n}_1$ direction normal to K_1,
- $\mathbf{d}$ $= \mathbf{0}_1\mathbf{0}_2$,
- $\mathbf{q}$ velocity of particles,
- $\mathbf{r}_{0w}$ radius vector of the axis of the cylinder with its end determined by the parameter λ.

The solution of the set of Eqs. (2.1), (2.2), (2.3) in polar coordinates φ_1', θ_1' (the polar and azimuthal angle of $\mathbf{n}_1$) is the curve $\theta_1(\boldsymbol{\varphi}_1')$ expressed by the relation:

$$\sin\theta_{1(\pm)}' = \frac{1}{k_1}[\sin\theta_q\sin\varphi_1'\pm(k_2^2-\sin^2\theta_q\cos^2\varphi_1')^{\frac{1}{2}}]. \tag{2.4}$$

θ_q is the angle between the direction z' and the line $\mathbf{0}_1\mathbf{0}_2$ (this angle describes the geometrical characteristic of the problem — that is, the direction of flow with reference to the line connecting the centres of the two spheres), $k_1 = R_1/d$, $k_2 = R_2/d$. The $\pm$ sign in the equation of the curve means that, under certain conditions (with appropriate values of the parameter θ_q and if k_1 and k_2 are considered to be constant for the particular problem under consideration), there may exist two solutions $\theta_{1(+)}'$ and $\theta_{1(-)}'$ for a single value of φ_1. The existence of an intersection curve (2.4) requires the satisfaction of the following three criteria:

$$1) \quad k_2^2-\sin^2\theta_q\cos^2\varphi_1' \geqslant 0 \tag{2.5}$$

(the radicand must be positive)

$$2) \quad \cos\theta_{1(\pm)}' \geqslant 0. \tag{2.6}$$

(Although the region $\cos\theta_{1(\pm)}' \leqslant 0$ is screened against the flowing particles, this screening action is due to the upper surface of the same sphere)

$$3) \quad \sin\theta_{1(\pm)}' \leqslant 1. \tag{2.7}$$

In agreement with the above the variability interval D of φ_1' is a conjunction of three intervals of φ_1' denoted D^{Δ}, D_+, $D_{\leqslant 1}$, for which the respective criteria 1, 2 and 3 are satisfied:

$$D \equiv D_+ \wedge D^{\Delta} \wedge D_{\leqslant 1}, \tag{2.8}$$

$\wedge$ being the conjunction sign.

The conditions for which each particular criterion is satisfied are established and discussed in detail in Ref. [5].

Knowing the region D of existence of the screening curve, we can determine the solid region corresponding to the part of the surface "irradiated" by a homogeneous stream.

The integral may be expressed thus:

$$\int_{\Omega} \dots d\Omega = \int_{\Omega_0} - \int_{\Omega_z}, \tag{2.23}$$

where Ω is the region "irradiated" by flowing particles, Ω_0 — the region "irradiated" under conditions of absence of the other body, and Ω_z — the screened region. In view of the symmetry with reference to the (z', y') — plane and the differentiation of the interval D in the regions $\varphi_1' \in \left(\frac{\pi}{2}, \pi\right)$ and $\varphi_1' \in \left(\pi, \frac{3}{2}\pi\right)$, the integration over the screened region Ω_z can be expressed in the form:

$$\int_{\Omega_z} = 2\Big[\int_{\Omega_1^\varphi} \dots d\Omega + \int_{\Omega_2^\varphi} \dots d\Omega\Big], \tag{2.24}$$

where

$$\Omega_1^\varphi \equiv \varphi_1' \in \left(\frac{\pi}{2}, \pi\right), \quad \Omega_2^\varphi \equiv \varphi_1' \in \left(\pi, \frac{3}{2}\pi\right). \tag{2.25}$$

The region Ω_z is directly connected with the screening curve KP. The latter determines in an unequivocal manner Ω_z, because it determines the surface region on the sphere K_1 corresponding to the solid angle Ω_z. By discussing the existence of a region D depending on the value of θ_q, we can find the region Ω_z — therefore also a method of integration with respect to the variables φ_1', θ_1'. The results are represented in Tables 1, 2 and 3 corresponding to the three cases occurring in the discussion of that region — namely, those of $A1 \equiv k_1 > k_2,\ k_1 < 2k_2,\ A2 \equiv k_1 > k_2,\ k_1 > 2k_2$ and $B \equiv k_1 < k_2$. The variability interval φ_1' is divided in the Tables into two subregions $\sin\varphi_1' \geqslant 0$ and $\sin\varphi_1' \leqslant 0$. The first column contains sketches representing the projections of the system on the (x', y')-plane for the limiting values of θ_q of the interval of θ_q considered. The subsequent column contains the regions D^+, D^- of existence of the intersection curve + and −, and the last — the integration method over the solid region Ω_z.

3. Integration of the expression of the heat exchange

The screened region having been effectively determined, and the integration method in that region being known, we can proceed to perform the quadrature of the expression of the heat exchange. This double quadrature can be performed for each case separately (the number of cases is large because there are several cases in each of the schemes A_1, A_2, B); we shall attempt, however, to express it in a general manner. The heat exchange is characterized in our problem by the incident fluxes only — that is, the flux of energy E_{10}^i and particles N_{10}^i. Starting out from the formulas expressing the fluxes

$$E_{1(0)}^{(i)} = \frac{1}{2}mq^2 \int_{\Sigma_1} (-\mathbf{q}\cdot\mathbf{n}_1) n_0 \, d\Sigma_{w1}, \quad N_{1(0)}^{(i)} = n_0 \int_{\Sigma_1} (-\mathbf{q}\cdot\mathbf{n}_1) d\Sigma_1, \tag{3.1}$$

Table 1. $A_1 \equiv (k_2 < k_1, k_1 < 2k_2)$

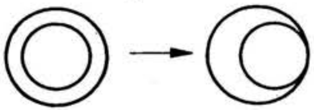
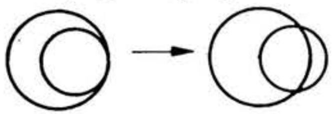
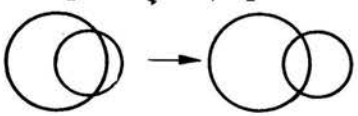
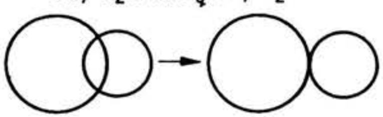
Case and plane diagram	$\sin\varphi_1' \geqslant 0$			$\sin\varphi_1' \leqslant 0$		
	D^+	D^-	Integral form	D^+	D^-	Integral form
$0 \leqslant \sin\theta_q \leqslant k_1 - k_2$ 	$\left(\frac{\pi}{2}, \pi\right)$	0	$\int\limits_{\frac{\pi}{2}}^{\pi} \left[\int\limits_{0}^{\theta^+} \dots d\theta\right] d\varphi_1'$	$\left(\pi, \frac{3}{2}\pi\right)$	0	$\int\limits_{\pi}^{\frac{3}{2}\pi} \left[\int\limits_{0}^{\theta^+} \dots d\theta\right] d\varphi_1'$
$k_1 - k_2 \leqslant \sin\theta_q \leqslant k_2$ 	$(\varphi_1'^{x}, \pi)$	0	$\int\limits_{\varphi_1'^{x}}^{\pi} \left[\int\limits_{0}^{\theta^+} \dots d\theta\right] d\varphi_1' + \int\limits_{\frac{\pi}{2}}^{\varphi_1'^{x}} \left[\int\limits_{0}^{\frac{\pi}{2}} \dots d\theta\right] d\varphi_1'$	$\left(\pi, \frac{3}{2}\pi\right)$	0	$\int\limits_{\pi}^{\frac{3}{2}\pi} \left[\int\limits_{0}^{\theta^+} \dots d\theta\right] d\varphi_1'$
$k_2 \leqslant \sin\theta_q \leqslant \sqrt{k_1^2 + k_2^2}$ 	$(\varphi_1'^{x}, \varphi_1'^{\Delta})$	$\left(\frac{\pi}{2}, \varphi_1'^{\Delta}\right)$	$\int\limits_{\frac{\pi}{2}}^{\varphi_1'^{x}} \left[\int\limits_{\theta^-}^{\frac{\pi}{2}} \dots d\theta\right] d\varphi_1' + \int\limits_{\varphi_1'^{x}}^{\varphi_1'^{\Delta}} \left[\int\limits_{\theta^-}^{\theta^+} \dots d\theta\right] d\varphi_1'$	0	0	$\Omega_z = 0$
$\sqrt{k_1^2 + k_2^2} \leqslant \sin\theta_q \leqslant k_1 + k_2$ 	0	$\left(\frac{\pi}{2}, \varphi^{x}\right)$	$\int\limits_{\frac{\pi}{2}}^{\varphi_1'^{x}} \left[\int\limits_{\theta^-}^{\frac{\pi}{2}} \dots d\theta\right] d\varphi_1'$	0	0	$\Omega_z = 0$

Table 2. $A_2 \equiv (k_2 < k_1, k_1 < 2k_2)$

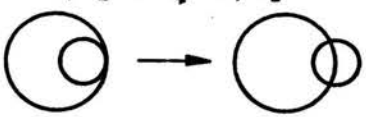
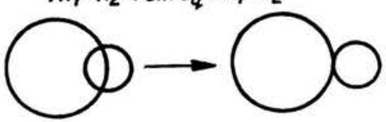
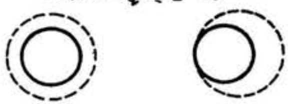
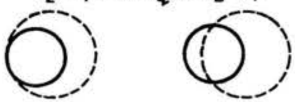
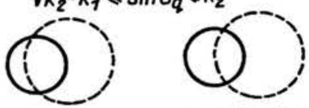
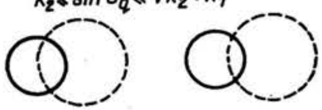
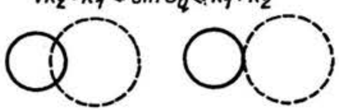
Case and plane diagram	$\sin\varphi_1' \geqslant 0$			$\sin\varphi_1' \leqslant 0$		
	D^+	D^-	Integral form	D^+	D^-	Integral form
$0 \leqslant \sin\theta_q \leqslant k_2$ 	$\left(\frac{\pi}{2}, \pi\right)$	0	$\int\limits_{\frac{\pi}{2}}^{\pi}\Big[\int\limits_{0}^{\theta^+} \ldots\, d\theta\Big]d\varphi_1'$	$\left(\pi, \frac{3}{2}\pi\right)$	0	$\int\limits_{\pi}^{\frac{3}{2}\pi}\Big[\int\limits_{0}^{\theta^+} \ldots\, d\theta\Big]d\varphi_1'$
$k_2 \leqslant \sin\theta_q \leqslant k_1 - k_2$ 	$\left(\frac{\pi}{2}, \varphi_1'^{\Delta}\right)$	$\left(\frac{\pi}{2}, \varphi_1'^{\Delta}\right)$	$\int\limits_{\frac{\pi}{2}}^{\varphi_1'^{\Delta}}\Big[\int\limits_{\theta^-}^{\theta^+} \ldots\, d\theta\Big]d\varphi_1'$	0	0	$\Omega_z = 0$
$k_1 - k_2 \leqslant \sin\theta_q < \sqrt{k_1^2 + k_2^2}$ 	$(\varphi_1'^{x}, \varphi_1'^{\Delta})$	$\left(\frac{\pi}{2}, \varphi_2'^{\Delta}\right)$	$\int\limits_{\frac{\pi}{2}}^{\varphi_1'^{x}}\Big[\int\limits_{\theta^-}^{\frac{\pi}{2}} \ldots\, d\theta\Big]d\varphi_1' + \int\limits_{\varphi_1'^{x}}^{\varphi_1'^{\Delta}}\Big[\int\limits_{\theta^-}^{\theta^+} \ldots\, d\theta\Big]d\varphi_1'$	0	0	$\Omega_z = 0$
$\sqrt{k_1^2 + k_2^2} \leqslant \sin\theta_q \leqslant k_1 + k_2$ 	0	$\left(\frac{\pi}{2}, \varphi_1'^{x}\right)$	$\int\limits_{\frac{\pi}{2}}^{\varphi_1'^{x}}\Big[\int\limits_{\theta^-}^{\frac{\pi}{2}} \ldots\, d\theta\Big]d\varphi_1'$	0	0	$\Omega_z = 0$

Table 3. $B \equiv (k_2 > k_1)$

Case and plane diagram	$\sin\varphi_1' \geqslant 0$			$\sin\varphi_1' \leqslant 0$		
	D^+	D^-	Integral form	D^+	D^-	Integral fomr
$0 \leqslant \sin\theta_q \leqslant k_2-k_1$ 	0	0	$\int\limits_{\frac{\pi}{2}}^{\pi}\left[\int\limits_{0}^{\frac{\pi}{2}}\dots d\theta\right]d\varphi_1'$	0	0	$\int\limits_{\frac{\pi}{2}}^{\pi}\left[\int\limits_{0}^{\frac{\pi}{2}}\dots d\theta\right]d\varphi_1'$
$k_2-k_1 \leqslant \sin\theta_q \leqslant \sqrt{k_2^2-k_1^2}$ 	0	0	$\int\limits_{\frac{\pi}{2}}^{\pi}\left[\int\limits_{0}^{\frac{\pi}{2}}\dots d\theta\right]d\varphi_1'$	$\left(\bar{\varphi}_1'^{x}, \frac{3}{2}\pi\right)$	0	$\int\limits_{\varphi_1'^{x}}^{\frac{3}{2}\pi}\left[\int\limits_{0}^{\theta^+}\dots d\theta\right]d\varphi_1' + \int\limits_{\pi}^{\bar{\varphi}_1'^{x}}\left[\int\limits_{0}^{\frac{\pi}{2}}\dots d\theta\right]d\varphi_1'$
$\sqrt{k_2^2-k_1^2} \leqslant \sin\theta_q \leqslant k_2$ 	$(\varphi_1'^{x}, \pi)$	0	$\int\limits_{\frac{\pi}{2}}^{\varphi_1'^{x}}\left[\int\limits_{0}^{\frac{\pi}{2}}\dots d\theta\right]d\varphi_1' + \int\limits_{\varphi_1'^{x}}^{\pi}\left[\int\limits_{0}^{\theta^+}\dots d\theta\right]d\varphi_1'$	$\left(\pi; \frac{3}{2}\pi\right)$	0	$\int\limits_{\varphi_1'^{x}}^{\frac{3}{2}\pi}\left[\int\limits_{0}^{\theta^+}\dots d\theta\right]d\varphi_1'$
$k_2 \leqslant \sin\theta_q \leqslant \sqrt{k_2^2+k_1^2}$ 	$(\varphi_1'^{x}, \varphi_1'^{\Delta})$	$\left(\frac{\pi}{2}, \varphi_1'^{\Delta}\right)$	$\int\limits_{\frac{\pi}{2}}^{\varphi_1'^{x}}\left[\int\limits_{\theta^-}^{\frac{\pi}{2}}\dots d\theta\right]d\varphi_1' + \int\limits_{\varphi_1'^{x}}^{\varphi_1'^{\Delta}}\left[\int\limits_{\theta^-}^{\theta^+}\dots d\theta\right]d\varphi_1'$	0	0	$\Omega_z = 0$
$\sqrt{k_2^2+k_1^2} \leqslant \sin\theta_q \leqslant k_1+k_2$ 	0	$\left(\frac{\pi}{2}, \varphi_1'^{x}\right)$	$\int\limits_{\frac{\pi}{2}}^{\varphi_1'^{x}}\left[\int\limits_{\theta^-}^{\frac{\pi}{2}}\dots d\theta\right]d\varphi_1'$	0	0	$\Omega_z = 0$

and replacing the integral over the "irradiated" surface by that over the solid angle Ω and the integration region Ω by the difference between the region Ω_0 and the screened region Ω_z, we transform the two fluxes to obtain:

$$E^{(i)}_{1(0)} = \frac{1}{2} mq^3 n_0 R_1^2 \Big[\int_{\Omega_0} \cos\theta_1' d\Omega_1' - \int_{\Omega_z} \cos\theta_1' d\Omega_1',$$

$$N^{(i)}_{1(0)} = qn_0 R_1^2 \Big(\int_{\Omega_0} \cos\theta_1' d\Omega_1' - \int_{\Omega_z} \cos\theta_1' d\Omega_1' \Big), \tag{3.2}$$

$$d\Omega_1' = \sin\theta_1' d\theta_1' d\varphi_1'.$$

The integration over the solid region Ω_0 is very simple:

$$\int_{\Omega_0} \cos\theta_1' d\Omega_1' = \pi. \tag{3.3}$$

The integral over the screened region is split up, in agreement with (2.24), into two:

$$\int_{\Omega_z} \cos\theta_1' d\Omega_1' = 2\Big[\int_{D_{(1)}} \cos\theta_1' d\Omega_1' + \int_{D_{\varphi(2)}} \cos\theta_1' d\Omega_1' \Big], \tag{3.4}$$

where

$$D_{\varphi(1)} \equiv \varphi_1' \in \Big(\frac{\pi}{2}, \pi\Big), \quad D_{\varphi(2)} \equiv \varphi_1' \in \Big(\pi, \frac{3}{2}\pi\Big). \tag{3.5}$$

The integrals over the regions $D_{\varphi(1)}$ and $D_{\varphi(2)}$ may be expressed in the form:

$$I = \int_{\varphi_1}^{\varphi_2} I_\theta d\varphi_1', \tag{3.6}$$

where φ_1 and φ_2 are the limits of φ_1' in the region $D_{\varphi(1)}$ or $D_{\varphi(2)}$, and I_θ is the result of integration with respect to θ_1'

$$I_\theta = \int_{\theta_1'}^{\theta_2'} \cos\theta_1' \sin\theta_1' d\theta_1' = -\frac{1}{2}(\cos^2\theta_2' - \cos^2\theta_1'), \tag{3.7}$$

where θ_2', θ_1' are the corresponding limits for θ_1'.

From the analysis of the integration schemes A_1, A_2, B it is inferred that the limits θ_2', θ_1' are at most of the type:

$$\theta_2' = \theta^+ \vee \frac{\pi}{2}, \quad \theta_1' = \theta^- \vee 0; \tag{3.8}$$

$\vee$ is the alternative sign.

This enables I_θ to be expressed, for any possible combination of the limits θ_2', θ_1', in the form:

$$I_\theta = -\frac{1}{2}(\alpha\cos^2\theta^+ - \beta\cos^2\theta^- - \gamma), \tag{3.9}$$

where α, β and γ are indices characterizing the particular combination of the limits θ_2', θ_1'. Thus, for each combination, we have:

$$\begin{cases} \theta_2' = \dfrac{\pi}{2} \\ \theta_1' = 0 \end{cases} \quad \alpha = 0, \quad \beta = 0, \quad \gamma = 1; \tag{3.10}$$

$$\begin{cases} \theta_2' = \dfrac{\pi}{2} \\ \theta_1' = \theta^- \end{cases} \quad \alpha = 0, \quad \beta = 1, \quad \gamma = 0; \tag{3.11}$$

$$\begin{cases} \theta_2' = \theta^+ \\ \theta_1' = 0 \end{cases} \quad \alpha = 1, \quad \beta = 0, \quad \gamma = 1; \tag{3.12}$$

$$\begin{cases} \theta_2' = \theta^+ \\ \theta_1' = \theta^- \end{cases} \quad \alpha = 1, \quad \beta = 1, \quad \gamma = 0. \tag{3.13}$$

To facilitate the integration of I_θ with respect to φ_1', we express I_θ in a somewhat different form:

$$I_\theta = -\frac{1}{2}(\beta \sin^2\theta^- - \alpha \sin^2\theta^+ + \gamma^*), \quad \gamma^* = \alpha - \beta - \gamma. \tag{3.14}$$

On performing in I integration with respect to φ_1', we obtain, in general (for all the possible cases of the schemes A_1, A_2 and B):

$$I = \left(-\frac{1}{2}\right)\frac{1}{k_1^2}[2(\beta-\alpha)\sin^2\theta_q I_2 + (\beta-\alpha)(k_2^2 - \sin^2\theta_q) I_0 + 2(\beta+\alpha)\sin\theta_q I_1] - \frac{1}{2}\gamma^*(\varphi_2 - \varphi_1), \tag{3.15}$$

where

$$I_0 = \int_{\varphi_1}^{\varphi_2} d\varphi_1' = \varphi_2 - \varphi_1, \tag{3.16}$$

$$I_2 = \int_{\varphi_1}^{\varphi_2} \sin^2\varphi_1' d\varphi_1' = -\frac{1}{2}\sin\varphi_2\cos\varphi_2 + \frac{1}{2}(\varphi_2 - \varphi_1) + \frac{1}{2}\sin\varphi_1\cos\varphi_1, \tag{3.17}$$

$$\begin{aligned} I_1 &= -\int_{\varphi_1}^{\varphi_2} \sin\varphi_1'(k_2^2 - \sin^2\theta_q\cos^2\varphi_1')^{\frac{1}{2}} d\varphi_1' \\ &= \frac{1}{2}\cos\varphi_2(k_2^2 - \sin^2\theta_q\cos^2\varphi_2)^{\frac{1}{2}} + \frac{1}{2}k_2^2\frac{1}{\sin\theta_q}\arcsin\left(\cos\varphi_2\frac{\sin\theta_q}{k_2}\right) \\ &\quad - \frac{1}{2}\cos\varphi_1(k_2^2 - \sin^2\theta_q\cos^2\varphi_1)^{\frac{1}{2}} - \frac{1}{2}k_2^2\frac{1}{\sin\theta_q}\arcsin\left(\cos\varphi_1\frac{\sin\theta_q}{k_2}\right). \end{aligned} \tag{3.18}$$

The fluxes N_{10}^{i} and E_{10}^{i} can therefore be represented thus:

$$N_{10}^{(i)} = 2qn_0 R_1^2\left[\frac{\pi}{2} - I_{D\varphi(1)} - I_{D\varphi(2)}\right], \tag{3.19}$$

$$E_{10}^{(i)} = mq^3 n_0 R_1^2\left[\frac{\pi}{2} - I_{D\varphi(1)} - I_{D\varphi(2)}\right], \tag{3.20}$$

where

$$\begin{aligned} I_{D\varphi(1)} &\equiv I(\varphi_1, \varphi_2), \quad \varphi_1, \varphi_2 \in D_{\varphi(1)}, \\ I_{D\varphi(2)} &\equiv I(\varphi_1^*, \varphi_2^*), \quad \varphi_1^*, \varphi_2^* \in \left(\pi, \frac{3}{2}\pi\right). \end{aligned} \tag{3.21}$$

From the analysis of the integration schemes it follows that the regions $D_{\varphi(1)}$, $D_{\varphi(2)}$ can be divided by a characteristic point φ_1' (such as $\varphi_1'^{+}$, $\varphi_1'^{x}$, $\varphi_1'^{\Delta}$) into two subregions. Bearing this in mind, we can reduce the incident fluxes N_{10}^{i}, E_{10}^{i} of particles and energy to the form:

$$N_{10}^{i} = 2qn_0 R_1^2\left[\frac{\pi}{2} - \sum (I+I^*)\right], \tag{3.22}$$

$$E_{10}^{(i)} = mq^3 n_0 R_1^2\left[\frac{\pi}{2} - \sum (I+I^*)\right], \tag{3.23}$$

where

$$\begin{aligned} I &= I(\varphi_1, \varphi_2), \quad \varphi_1, \varphi_2 \in \left(\frac{\pi}{2}, \pi\right), \\ I^* &= I(\varphi_1^*, \varphi_2^*), \quad \varphi_1^*, \varphi_2^* \in \left(\pi, \frac{3}{2}\pi\right); \end{aligned} \tag{3.24}$$

Σ is in agreement with the subdivision of the regions $D_{\varphi(1)}$, $D_{\varphi(2)}$ into two subregions. The final expression of the heat exchange is:

$$E_1 = mqn_0 R_1^2\left[\frac{\pi}{2} - \sum (I+I^*)\right]\left(q^2 - \frac{q_{2c}}{q_{2c}^{(N)}}\right), \tag{3.25}$$

or, after finding q_{2c}, $q_{2c}^{(N)}$:

$$E_1 = mqn_0 R_1^2\left(q^2 - \frac{4kT}{m}\right)\left[\frac{\pi}{2} - I_g\right], \quad I_g \stackrel{\mathrm{df}}{=} \sum (I+I^*). \tag{3.26}$$

The expression in square brackets concerns the geometrical characteristics. Bearing in mind that the factor $\pi/2$ corresponds to the case in which the sphere K_1 is not screened, it is seen that the screening effect is proportional to $\sum (I+I^*)$. The expression (3.26) for the total energy flux imparted to the sphere $K1$ is of a general character. The form of E_1 in each particular case is obtained within the schemes A_1, A_2 and B by substituting in I and I^* the limits for φ_1' and θ_1' — that is, the indices α, β, γ and the limits φ_1, φ_2.

Below, making use of the general formula (3.26), we shall determine E_1 in a few cases of screening selected for simplicity from geometrical and physical interpretation owing to the structure of the Eq. (3.26) in which the geometrical and energy parts are separated.

The aim of this analysis will be not only to find E_1 in some particular cases but, principally, to verify the functionality of the Eq. (3.26).

4. Particular cases

1. A_1, $\sin\theta_q = 0$

The projections on the (x', y') and (z', y')-plane are as follows (Fig. 3). From the scheme A_1, in the case of $\sin\theta_q = 0$, we have:

$$(4.1)\quad \sum (I+I^*) = \int_{\pi/2}^{\pi} \Big[\int_0^{\theta^+} \dots d\theta_1'\Big] d\varphi_1' + \int_{\pi}^{\frac{3}{2}\pi} \Big[\int_0^{\theta^+} \dots d\theta_1'\Big] d\varphi_1' = I(\varphi_1, \varphi_2) + I(\varphi_1^*, \varphi_2^*).$$

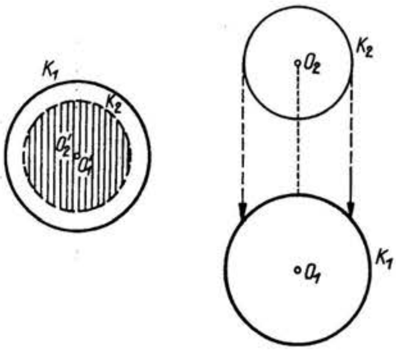


FIG. 3.

In this case, the sum reduces to a single term (the intervals $D_{\varphi(1)}$, $D_{\varphi(2)}$ are complete and contain no characteristic values φ_1'). The indices α, β, γ, γ^* and the limits φ_1, φ_2 are as follows:

$$\varphi_1 = \frac{\pi}{2}, \quad \varphi_2 = \pi, \quad \alpha = 1, \quad \beta = 0, \quad \gamma = 1, \quad \gamma^* = 0$$

for the first integral and

$$\varphi_1^* = \pi, \quad \varphi_2^* = \frac{3}{2}\pi, \quad \alpha = 1, \quad \beta = 0, \quad \gamma = 1, \quad \gamma^* = 0$$

for the second integral.

On substituting them into the formula for E_1, we find:

$$(4.2)\qquad E_1 = mqn_0 R_1^2 \frac{\pi}{2}\left(1 - \frac{k_2^2}{k_1^2}\right)\left(q^2 - \frac{4kT}{m}\right).$$

The energy flux is less than 1 by the value k_2^2/k_1^2, due to the screening effect. If $k_2 = k_1$ $(R_1 = R_2)$, we have $E_1 = 0$, which corresponds to complete screening. If $k_2 = 0$ $(R_2 = 0)$, we have $E_1 = E_{10}^* = mqn_0 R_1^2 \frac{\pi}{2}\left(q^2 - \frac{4kT}{m}\right)$, which corresponds to the absence of the sphere K_2 (there is no possibility of $k_1 = 0$ in the scheme A_1).

2. A_1, $\sin\theta_q = k_1 + k_2$

The projections on the (x', y') and (z', y')-plane are as follows (Fig. 4). In the case under consideration we have:

$$\sum (I+I^*) = I(\varphi_1, \varphi_2),$$

$$\varphi_1 = \frac{\pi}{2}, \quad \varphi_2 = \varphi_1'^x = \frac{\pi}{2}, \tag{4.3}$$

$$\alpha = 0, \quad \beta = 1, \quad \gamma = 0, \quad \gamma^* = -1.$$

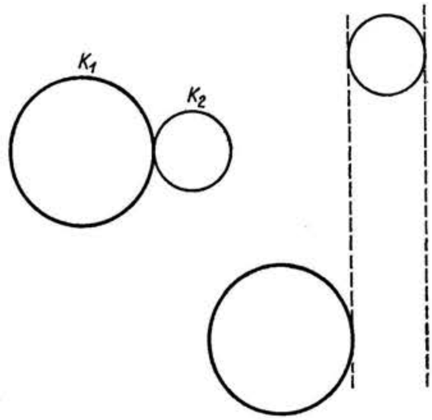


Fig. 4.

From the Eq. (3.26), we obtain:

$$E_1 = mqn_0 R_1^2 \frac{\pi}{2}\left(q^2 - \frac{4kT}{m}\right) \tag{4.4}$$

— that is, the energy flux for a non-screened sphere, which is correct.

3. A_2, $\sin\theta_q = k_1 - k_2$ (Fig. 5)

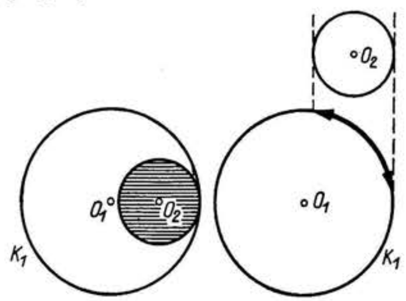


Fig. 5.

The energy flux E_1 can be found by making use of the fact that $\sin\theta_q$ belongs either to the interval $k_2 \leqslant \sin\theta_q \leqslant k_1 - k_2$ or to the neighbouring interval $k_1 - k_2 \leqslant \sin\theta_q \leqslant k_1$

a) $k_2 \leqslant \sin\theta_q \leqslant k_1 - k_2$,

b) $k_1 - k_2 \leqslant \sin\theta_q \leqslant k_1$.

The results obtained in a) and b) are identical. We find:

$$E_1 = mqn_0 R_1^2 \frac{\pi}{2}\left(1 - \frac{k_2^2}{k_1^2}\right)\left(q^2 - \frac{4kT}{m}\right).$$

4. B, $\sin\theta_q = k_2$ (Fig. 6)

a) $\sqrt{k_2^2 - k_1^2} \leqslant \sin\theta_q \leqslant k_2$,

b) $k_2 \leqslant \sin\theta_q \leqslant \sqrt{k_1^2 + k_2^2}$.

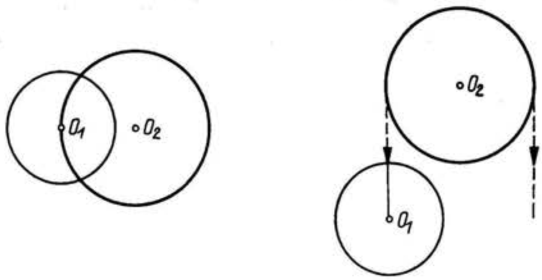


FIG. 6.

The results for a) and b) are identical, namely:

$$E_1 = mqn_0 R_1^2 \left(q^2 - \frac{4kT}{m}\right)\frac{\pi}{2}\, g_1(\eta),$$

where

$$g_1(\eta) = 1 - \frac{2}{\pi}\left\{-\frac{1}{2}\arcsin\left(\frac{1}{2}\eta\right) + \frac{\pi}{4} + \frac{1}{\eta^2}\left[\frac{1}{2}\arcsin\left(\frac{1}{2}\eta\right) - \frac{1}{4}\eta\sqrt{4-\eta^2} + \frac{\pi}{4} - \frac{1}{2}\arcsin\left(\frac{1}{2}\sqrt{4-\eta^2}\right)\right]\right\}.$$

5. B, $\sin\theta_q = \sqrt{k_2^2 - k_1^2}$ (Fig. 7)

a) $k_2 - k_1 \leqslant \sin\theta_q \leqslant \sqrt{k_2^2 - k_1^2}$,

b) $\sqrt{k_2^2 - k_1^2} \leqslant \sin\theta_q \leqslant k_2$.

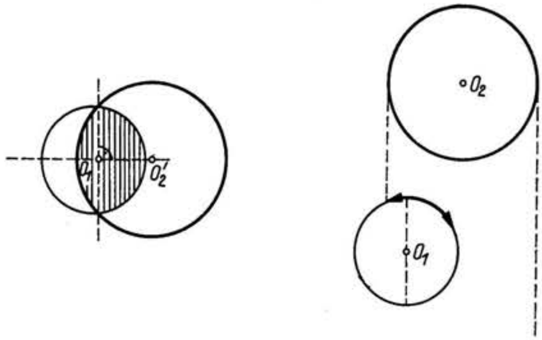


FIG. 7.

The results for a) and b) are identical:

$$E_1 = mqn_0 R_1^2 \left(q^2 - \frac{4kT}{m}\right) \frac{\pi}{2} g_2(\eta),$$

$$g_2(\eta) = 1 - \frac{2}{\pi}\left\{\frac{\pi}{2} - \frac{1}{2\eta^2}\left[-\frac{\pi}{2} + \eta\sqrt{1-\eta^2} + \arcsin\sqrt{1-\eta^2}\right]\right\}.$$

On the basis of computations performed for a few examples it can be stated that the general equation (3.26) enables us to obtain in a simple manner the total energy flux E_1 in all cases with the schemes A_1, A_2, B. In addition, its verification in the cases 1 and 2 has given a positive result. The diagrams of the geometrical factors $g_1(\eta)$ and $g_2(\eta)$ in the cases 4 and 5 (the remaining cases being simple) are represented in Figs. 8 and 9.

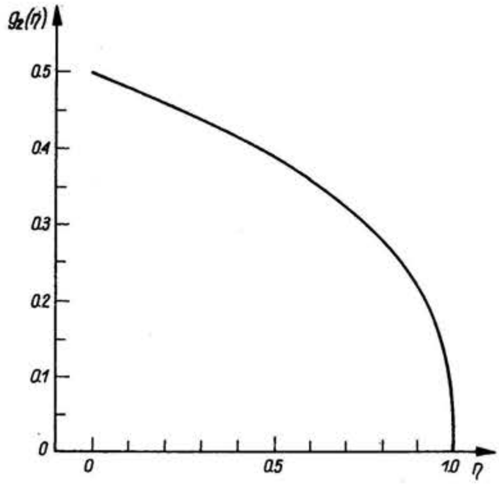


FIG. 8.

Their deviations from 1 characterize the screening effect. From the graphs it is seen that $g_1(\eta)$ and $g_2(\eta) \xrightarrow[\eta \to 0]{} \frac{1}{2}$ and $g_2(\eta) \xrightarrow[\eta \to 1]{} 0$, which is correct.

Thus an accurate analytic solution has been obtained for the problem of heat exchange in a system of two spheres of equal temperatures moving at hypersonic speed in a free-molecule medium using the diffusion model of reflection.

This problem comprises three problems which are independent from the mathematical point of view: 1) the problem of the continuity equation constituting in its general formulation a system of two Fredholm integral equations of the second kind in four variables; 2) the problem of screening of a sphere by another sphere in a homogeneous flow; and 3) the problem of quadratures of the expression for the heat exchange, reducing to fivefold integrals. Despite the necessity of considering over a dozen cases separately (the schemes A_1, A_2, B), we have succeeded in expressing the quadrature in a general manner.

The solution of the problem is a new item in the small group of non-trivial exact analytic solutions in the theory of flows past non-convex bodies or systems of bodies.

The structure of the equation obtained for the heat exchange enables simple (and very accurate) verification of the validity of the interaction model assumed. For example, if we consider the ratio of thermal powers supplied to the screened sphere for two different values of the varying parameters (for example two attack angles of the gas with reference to the system considered, two different ratios of radii or two different distances), it is found from the solution that this ratio depends on neither the density

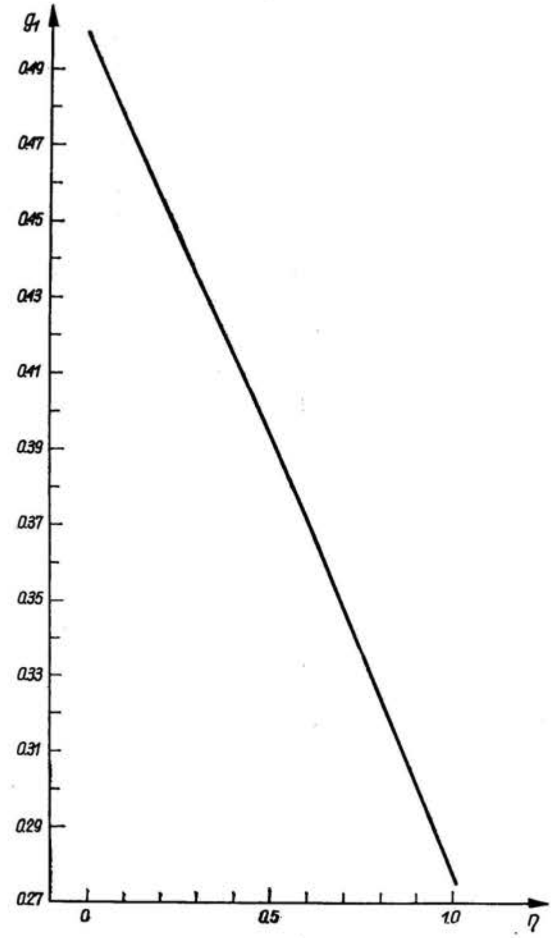


FIG. 9.

of the medium nor the velocity of the system, nor the temperature of the sphere; therefore, it is independent of quantities which may be incorrectly determined or assumed.

Since the ratio of the thermal powers imparted to the sphere can be determined by direct experiment, this fact offers a possibility of simple and accurate verification of the diffusion reflection model assumed (in a very broad sense, because the analysis can be made for several values of various parameters).

Finally, we shall appraise the heat exchange which can be expected in the space adjacent to the Earth, the conditions of which are those of a free-molecule medium. As a heat flux S transmitted to the sphere considered, we shall assume the energy flux of incident particles originating from the ambient medium under conditions of absence of the other sphere. (Thus, we assume for the appraisal the maximum flux, disregarding the screening effects and the energy flow of reflected particles. These, assuming that the temperature of the body is lower than the temperature corresponding to the velocity of the stream, may constitute only a fraction of the maximum flux). On the basis of the principle of calorimetry, the time Δt necessary to heat the body by a definite temperature rise ΔT is:

$$\Delta t = \frac{Mc_{pw.}\Delta T}{S} = \frac{8}{3}\frac{\varrho_m c_{pw.}\Delta T R_1}{\varrho_0 q^3},$$

where M is the mass of the body and c_{pw}, the specific heat of the body. (The above equation is valid for sufficiently small times). Assuming

$$c_{pw} = 0.22\frac{\text{cal}}{g\cdot\text{grade}}, \quad \Delta T = 1°\text{K}, \quad q \approx 7.2\frac{\text{km}}{\text{sek}}$$

$\varrho_m = 3\,\text{g/cm}^3$ (density of the material of the sphere), we find in the zone 50–130km high (above sea level) the time $\Delta t(1°)$ necessary to heat the sphere by 1°K for sphere with different radii (the radius is varied with the altitude in such a manner, however, that the conditions of a free-molecule medium are preserved).

The results are collected in the following Table 1.

Table 4

H [km]	ϱ_0 [g/cm³]	λ [cm]	R_1 [cm]	t [1°]
130	7.6×10^{-12}	1.02×10^3	10^2	2.8×10^3 s
100	5×10^{-10}	1.6×10	1	4.2 s
90	3.1×10^{-9}	2.56	10^{-1}	7×10^{-3} s
50	1.03×10^{-6}	8.3×10^{-3}	10^{-4}	2×10^{-8} s

The values of ϱ_0 and λ have been taken from Ref. [4].

It is seen that the thermal effect is essential. The times of heating $\Delta t_{(1°)}$ for particles of 1μ (micrometeorites) at low altitudes are so small that the particles will be burnt.

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