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A continuum theory for granular media with a critical state

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In this work an attempt is presented to formulate a theory of the mechanical behaviour of granular media for finite deformations. The critical state assumption is fundamental, while no idea of plastic limit condition is presented. No true elastic range is exibited. Dilatancy effects are taken into account. An explicit form of the constitutive equations has been obtained and stress-strain relations are presented both for loading and unloading processes.

W pracy przedstawiono próbę sformułowania teorii mechanicznego zachowania się materiałów ziarnistych przy odkształceniach skończonych. Podstawowym założeniem pracy jest przyjęcie hipotezy stanu krytycznego przy równoczesnym odrzuceniu idei warunku plastyczności. W rozważaniach pominięto zakres sprężysty. Uwzględniono efekty dylatancji. Otrzymano jawną postać równań konstytutywnych, w których związki między naprężeniami i odkształceniami obowiązują zarówno w procesie obciążenia jak i odciążenia.

В работе представлена попытка формулировки теории механического поведения зернистых материалов при конечных деформациях. Основным предположением является принятие гипотезы критического состояния при одновременном непринятие идеи условия пластичности. В рассуждениях пренебрегается упругой областью. Учтены эффекты дилатации. Получен явный вид определяющих уравнений, в которых соотношения между напряжениями и деформациями обязывают так в процессе нагрузки, как и разгрузки.

1. Introduction

THE GRANULAR media this paper will be dealing with are isotropic, one-phase, non viscous, cohesionless solids. They are conceived as the continuum idealization of real materials as e.g. dry sand with uniform grain size. Many soil materials as well as some organic or artificial granular materials can show, at least under convenient assumptions, similar mechanical properties. In this context, the aim of the present paper is to develop, by a phenomenological approach, a theoretical model for the mechanical behaviour of granular media that can give an appropriate description of some basic properties of these materials. Let us now introduce the subject by some brief considerations that by no means intend to be a review of the overwhelming literature in this field. Only very few specific works will be listed, as being more strictly connected with the ideas that will be presented in this paper.

Classical soil mechanics theories were conceived to predict failure of soils, rather than to obtain a good description of real deformation processes. Such a point of view is evident in the assumption of a rigid or elastic behaviour up to rupture, and a Coulomb type failure condition. More attention to the deformation process brought about the formulations of new, more sophisticated models. Best developed and most satisfactory are the theories considering granular media as elasto-plastic materials admitting plastic potential with associated flow rule. The first attempts in this direction failed because, e.g. assuming a Coulomb type limit surface as yield surface, too large volume increase during plastic flow was predicted [1]. But these shortcomings are absent in the theories that consider the yield surface as a function of the density, giving account of the softening and hardening due to density variations. Worth mentioning are two main approaches. The Cambridge theory [2] developed the critical state conception in the framework of an elasto-plastic model, obtaining the plastic potential function, i.e. the yield surface, by the assumption of a special simple form for the specific power dissipation during plastic flow, which is supposed to be of "frictional" nature. On a similar line is Mróz theory of density hardening media [3] that can be considered, to a certain extent, a generalization of the previous one. This approach has also more sound foundations in general plasticity theory. The plastic potential is in general a function of the stress invariants and of the density, and can be chosen in the most convenient form. This theory has been generalized to allow for non-linear behaviour in the elastic range [16]. Many serious objections have been raised against classical elasto-plastic models with regard to the description of plastic deformations of metals and much more can be said thinking of granular media. In general, no elastic range is exhibited by these materials, in the sense that also small deformations are partially irreversible. For this reason ROSCOE and BURLAND presented a generalization of Cambridge theory for "wet" clays [4], giving account of irreversible shear deformations that take place beneath the yield surface, confirming so the non-existence of a true elastic region. Moreover, the stress strain relations are very smooth, so that also if irreversible deformations up to a certain amount are disregarded, the definition of conventional yield points is arbitrary. Different definitions give rise to different experimental determination of the initial yield surface. These differences can be important and any way become non-negligible for the subsequent yield surfaces, e.g. during hardening of the material, giving rise to big discrepancies between the deformation histories associated. It must be said that while experimental evidence shows that for granular media the relevant processes involve very large deformations, no one of the previously mentioned theories is properly formulated to take account of finite deformations. So there are good reasons to reject the previous approaches and to try new ones.

T. Y. THOMAS first attempted to develop a theory of the plastic behaviour of non-viscous metals without a yield surface [5]. Assuming a special form of Truesdell's Hypoelastic [6, 7] constitutive relations, he described continuous transition from elastic to perfectly plastic behaviour during loading processes. The form of the constitutive relations range from the incremental version for infinitesimal elasticity (for zero stresses), to a correctly invariant form for finite deformations of the Prandtl-Reuss equations for isochoric perfectly plastic flow (when von Mises condition is satisfied). This is obtained by an appropriate choice of the constitutive functions. It is clear that von Mises condition in this case looses the meaning of yield limit, conceived in the classical sense, but is a limit condition, never definitively satisfied but asymptotically approached when deviatoric deformations increase. This interpretation is in perfect agreement with Truesdell's observation that during simple shear of a particular hypoelastic material of grade 2 (that can be considered a special case of Thomas material for purely deviatoric deformations) von Mises yield is never reached, but only asymptotically approached when shear deformations increase. T. Y. THOMAS developed also a similar theory for von Mises plasticity and with a reference to a generalized von Mises yield condition [8]. The Hypoelastic yield observed by TRUESDELL is of purely "mathematical" nature. A more general approach was presented by A. E. GREEN [9, 10], always in connection with the formulation of a theory for plastic flow of metals. By means of a representation theorem he reduced the general Hypoelastic constitutive relations to a tensorial polynomial form. Then he assumed a definition of loading and unloading processes in terms of the sign of the deviatoric stress power and determined the constitutive coefficients for processes of each kind, by means of some axiomatic assumptions on the material properties. The different determination of the constitutive coefficients for loading and unloading processes gives account of the irreversible deformations. Green requires that constitutive equations for loading and unloading must coincide when the stress power is zero, i.e. for neutral states, to assure a smooth transition from one process to the other. This hypothesis does not seem necessary and is not supported by the experimental results that show discontinuity of the derivatives at the transition points in the stress-strain relations. Green also tried to reconcile this new approach with the classical conception of a yield surface. He assumed that when the yield condition was satisfied, the constitutive coefficients had to be of such a form as to assure the condition to be satisfied further until unloading occurred. In this conception the yield condition has again the classical meaning and for the non-yield states elastic behaviour is hypothesized. We obtain in such a way a true generalization of Prandtl-Reuss theory, now formulated in a correctly invariant form for finite deformations. The yield condition can be any smooth function of the stress invariants, and "elastic" compressibility during plastic flow is taken into account. There is no idea here of continuous, smooth transition from elastic to perfectly plastic states.

All these theories refer to metallic materials, but it is natural to think that Green's general approach that describes irreversible behaviour assuming different constitutive laws for loading and unloading processes, when these are defined in a convenient way, can suggest a procedure to built up a model for the mechanical behaviour of granular media.

2. Choice of the model

Let us list some well established experimental facts about granular materials as constitutive assumptions:

1. Relevant processes involve finite deformations (so that a properly invariant theory is needed).

2. Density variations play a fundamental role and strongly influence the mechanical response (softening and hardening).

3. No elastic range is observed in general.

4. When undergoing increasing deviatoric deformations these materials may tend to reach "critical states" in which they flow as frictional "fluids". In these states some limit condition on stresses must be satisfied. Later on this behaviour will be discussed in detail.

The peculiarities in the mechanical behaviour of granular materials summarized above justify the choice of the model that will be developed in this paper.

The assumed definition for a granular material and the constitutive assumptions 1) and 2) suggest constitutive equations of the form([1])

$$\mathring{T} = H(T, \varrho)\,[D], \tag{2.1}$$

where T is the Cauchy stress tensor, D is the stretching tensor, ϱ the density, $\mathring{T}$ the corotational stress rate and the tensor function H is isotropic in its tensor arguments and linear in D. The linearity of H in D assures time scale independent mechanical properties. By a special case of a general representation theorem of C. C. WANG, we have [14]:

$$\begin{aligned} H(T, \varrho)\,[D] = {} & [\Box_1 \operatorname{tr} D + \Box_2 \operatorname{tr}(TD) + \Box_3 \operatorname{tr}(T^2 D)]1 \\ & + [\Box_4 \operatorname{tr} D + \Box_5 \operatorname{tr}(TD) + \Box_6 \operatorname{tr}(T^2 D)]T \\ & + [\Box_7 \operatorname{tr} D + \Box_8 \operatorname{tr}(TD) + \Box_9 \operatorname{tr}(T^2 D)]T^2 \\ & + \Box_{10} D + \Box_{11}(DT + TD) + \Box_{12}(DT^2 + T^2 D), \end{aligned} \tag{2.2}$$

where the $\Box_i$ $i = 1, \ldots, 12$ are scalar functions of the fundamental stress invariants and of the density ϱ. In this paper we will be dealing with a special simple form of the general representation (2.2) Indeed it will be assumed:

$$\mathring{T} = [\Box_1 \operatorname{tr} D + \Box_2 \operatorname{tr}(TD)]1 + [\Box_4 \operatorname{tr} D + \Box_5 \operatorname{tr}(TD)]T + \Box_{10} D. \tag{2.3}$$

The irreversible behaviour will be described following Green's approach, assuming the same form of the general constitutive equations for loading and unloading processes, but with different choice of the constitutive coefficients, let us say $\Box_i$ and $\Box_i'$, respectively. Loading states are characterized by positive stress power, i.e. $\operatorname{tr}(TD) > 0$, neutral states by zero stress power, i.e. $\operatorname{tr}(TD) = 0$, and unloading states by negative stress power, i.e. $\operatorname{tr}(TD) < 0$. This definition, also due to A .E. GREEN, seems to be the most appropriate for our purposes. We will reject Green's hypothesis that the constitutive coefficients $\Box_i$ and $\Box_i'$ must coincide for neutral states to assure smooth transition from loading to unloading processes and conversely because, as previously stated in the introduction, it seems not at all justified by the experimental results. Stress-strain relations show discontinuity in the derivatives at the transition points. In what follows we will distinguish only between loading states ($\operatorname{tr}(TD) \geqslant 0$) and unloading states ($\operatorname{tr}(TD) < 0$), assuming that the same constitutive equations are valid for neutral and loading processes.

On the basis of convenient constitutive assumptions it is possible to obtain an explicit form of the coefficients $\Box_i \Box_i'$. The procedure that will be followed is general but will be illustrated with reference to a special case, chosen due to the physical reliability of the constitutive assumptions and the clear meaning of the state parameters introduced.

3. The state space

Most of present knowledge about constitutive properties in soil mechanics come from laboratory triaxial tests on cylindrical specimens. It is evident that in these tests only stress states in which two principal stresses are equal are feasible. Then only two independent

([1]) Constitutive equations of the same form are assumed in Noll's theory of hygrosteric materials [11], and have been assumed also in [12 and 13] with reference to granular media.

stress parameters are needed e.g. the axial and the radial principal stresses t_1 and t_2(²) or the mean pressure $p = -(t_1+2t_2)/3$ and the parameter $q' = t_1 - t_2$ that can be considered a "measure" of deviatoric stresses. In such a situation, to obtain an explicit form of the constitutive relations in terms of well established experimental facts, in this paper a point of view has been adopted similar to that of the Cambridge school, assuming only two constitutive stress parameters that can be considered an appropriate generalization of p and q'. Namely, we shall define:

$$(3.1) \qquad p = -\frac{\operatorname{tr} T}{3}, \qquad q = \sqrt{\operatorname{tr}(T^*)^2}, (^3)$$

where $T^* = T + p1$ is the stress deviator $\left(\text{observe that for triaxial tests it is } q = \sqrt{\frac{2}{3}}\,q'\right)$. Such a choice is quite natural because all the constitutive assumptions, if they are to be founded on experimental evidence, at present can be expressed only in terms of these stress parameters. Anyway the procedure that will allow the determination of the constitutive coefficients is completely general. Therefore, as state parameters in what follows will be considered the pressure p, the density ϱ and the non-negative "measure" of the deviatoric stresses q. The constitutive coefficients $\square_1$, $\square_2$, $\square_4$, $\square_5$, $\square_{10}$ in the Eq. (2.3) will be assumed to be functions of p, q, ϱ only. The three-dimensional space with coordi-

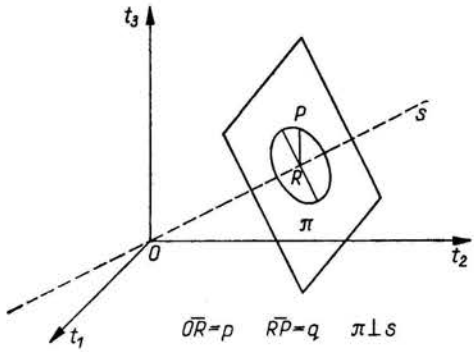

FIG. 1.

nates p, q, ϱ, will be called the state space. For a fixed value of the density, the points representative of the "state" characterized by the pair (p, q) are situated in the Haigh-Westergaard principal stress space, on a circumference of radius q contained in the plane orthogonal to the space diagonal at the point of abscissa p and with center on the space diagonal (Fig. 1).

4. Some experimental results

Let us now recall some basic features of the mechanical behaviour of granular media in the most indicative available experimental tests.

(²) Principal stresses are assumed positive if they correspond to tension, while in soil mechanics usually the opposite convention is adopted.

(³) The non-negative parameter q can be considered as the norm of the stress deviator.

4.1. Purely spherical motion under hydrostatic pressure

It is well known that the set of admissible states (pairs (p, ϱ)) in the p, ϱ plane is bounded by the so called virgin compression line, that represents the set of the "loosest states" of the material. It means that for every value of the pressure p, the point on the virgin compression line corresponds to the least admissible value of the density ϱ and is actually reached only when the material never before experienced greater densities, i.e. when it is yet

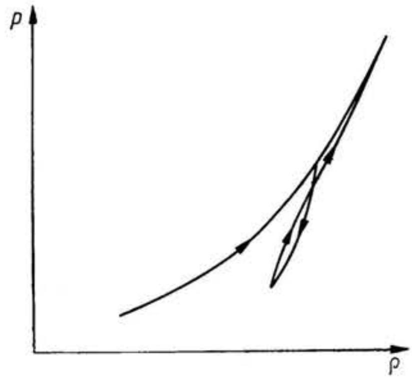

FIG. 2.

"virgin". Figure 2 shows a schematic picture of typical pressure-density paths for cohesionless materials.

4.2. Constant p tests

When deviatoric deformations increase, the density and the stress parameter q tend to reach limit values that depend only on the fixed value of p. If the initial value ϱ_i of the density is greater, equal or less than the limit "critical" one ϱ_c, the responses are of a differ-

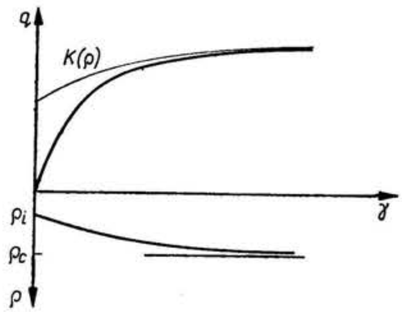

FIG. 3.

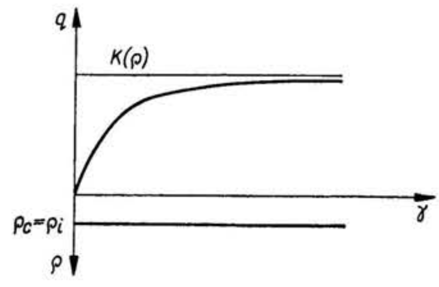

FIG. 4.

ent kind as is shown in Figs. 3, 4 and 5, where q and ϱ are plotted as functions of the parameter γ that indicates a "measure" of the deviatoric deformation.

If $\varrho_i < \varrho_c$ (Fig. 3) (loose states), both ϱ and q tend asymptotically to the final values.

If $\varrho_i = \varrho_c$ (Fig. 4) (critical states), there is no density variation and q behaves as before.

If $\varrho_i > \varrho_c$ (Fig. 5) (dense states), both ϱ and q first increase reaching a maximum, and subsequently decrease tending to reach the critical values.

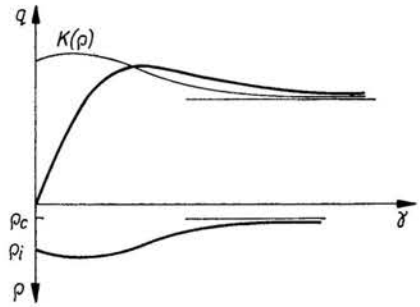

FIG. 5.

4.3. Constant ϱ tests

The deformation process is of course purely deviatoric. When deformation increases, the mean pressure and the stress parameter q tend to reach a limit value that depends only on the fixed value of ϱ. The responses are different if the initial value p_i of the mean pressure is greater, equal or less than the limit "critical" p_c as is shown in Figs. 6, 7 and 8 where p and q are plotted as functions of γ.

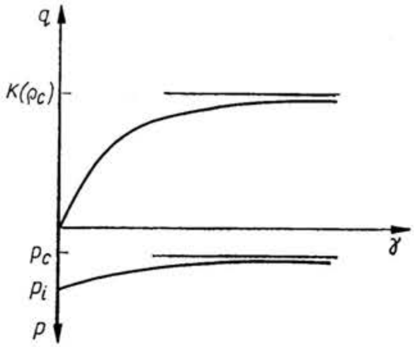

FIG. 6.

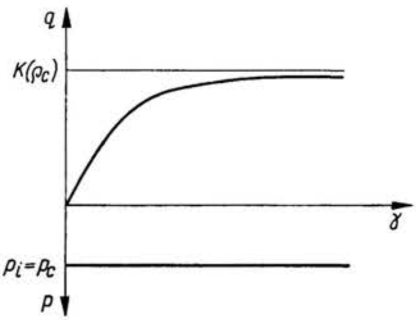

FIG. 7.

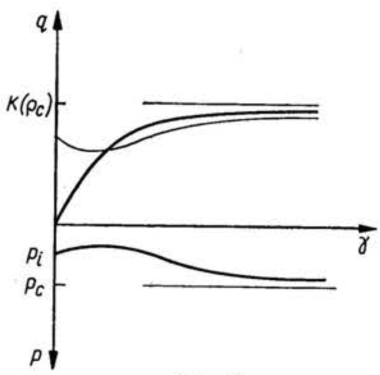

FIG. 8.

If $p_i > p_c$ (Fig. 6) (loose states), the mean pressure decreases monotonically to the critical value, q increases monotonically to the final value.

If $p_i = p_c$ (Fig. 7) (critical states), the mean pressure remains constant while q behaves as before.

If $p_i < p_c$ (Fig. 8) (dense states), the mean pressure reaches a minimum and after increases tending to the critical value, the parameter q behaves like before.

These results will be of great importance in the following formulation of the critical state assumption.

5. The critical state

ROSCOE, SCHOFIELD and WROTH [15] suggested that granular materials when undergoing increasing deviatoric deformations tend to reach a critical state in which they continue to distort without further change of p, q and ϱ. According to the experimental facts previously exposed it will be assumed that in the critical states the following relations must hold:

$$q = k(\varrho), \quad q = \psi p \quad \text{and then} \quad k(\varrho) = \psi p, \tag{5.1}$$

where k is a strictly increasing function of the density ϱ and ψ is a dimensionless positive constant. Relations (5.1) define in the state space two surfaces and a plane that intersect

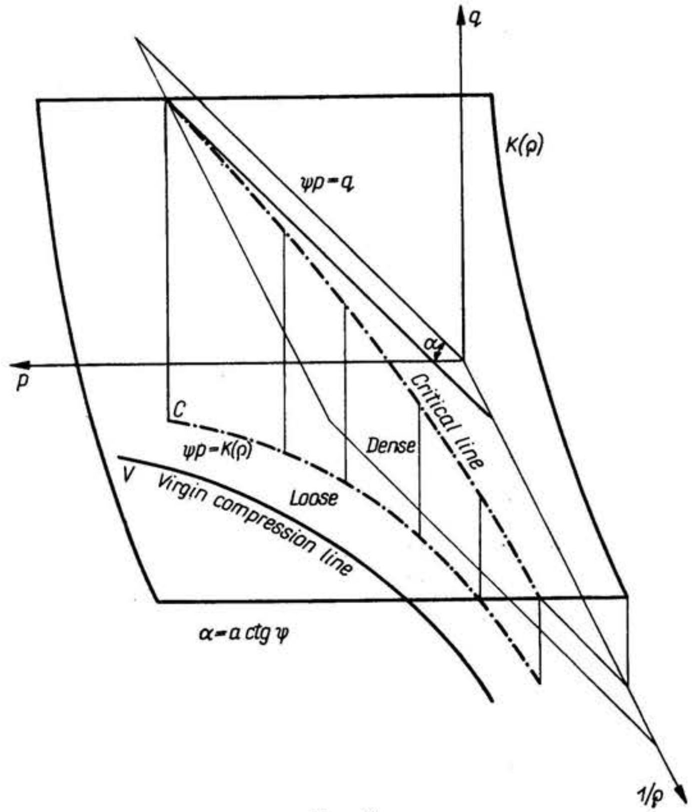

FIG. 9.

in a curve, called the critical state line, that is the set of all the points representative of critical states. In general we will call "dense" these states in which is $k(\varrho) > \psi p$ and "loose" those where $k(\varrho) < \psi p$ (Fig. 9). The critical state assumption can be enounced in the following way:

$$\left.\begin{array}{r} q \to k(\varrho) \\ q \to \psi p \\ \psi p \to k(\varrho) \end{array}\right\} \Rightarrow \dot{p}, \dot{q}, \dot{\varrho} \to 0. \tag{5.2}$$

6. The constitutive equations

To obtain an explicit determination of the constitutive coefficients, it is convenient to split the Eq. (2.3) in two that describe the deviatoric and the spherical stress-responses, respectively. Because $\mathring{T} = \dot{T} - WT + TW = \dot{T}^* - WT^* + T^*W - \dot{p}1 = \mathring{T}^* - \dot{p}1$, where W is the spin tensor, and by the local form of the continuity equation

$$\operatorname{tr} D = -\frac{\dot{\varrho}}{\varrho} \tag{6.1}$$

it can easily be shown that the following system:

$$\begin{aligned} \dot{p} &= \left(\Box_1 + \frac{\Box_{10}}{3} - \Box_4 p - \Box_2 p + \Box_5 p^2\right)\frac{\dot{\varrho}}{\varrho} + (\Box_5 p - \Box_2)\operatorname{tr}(T^*D^*), \\ \mathring{T}^* &= \Box_{10} D^* + \left[(\Box_5 p - \Box_4)\frac{\dot{\varrho}}{\varrho} + \Box_5 \operatorname{tr}(T^*D^*)\right]T^*, \end{aligned} \tag{6.2}$$

is equivalent to the Eq. (2.3).

A new important constitutive assumption is that purely spherical motions does not affect the deviatoric part of the stress response.

The second equation of the system (6.2) for purely spherical motions ($D^* = 0$) reduces to

$$\mathring{T}^* = (\Box_5 p - \Box_4)\frac{\dot{\varrho}}{\varrho} \tag{6.3}$$

and then, by the stated assumption, necessarily must be

$$\Box_5 p - \Box_4 = 0. \tag{6.4}$$

From (6.4) it results $\Box_4 = \Box_5 p$, and substituting in (6.2), we obtain the system

$$\begin{aligned} \dot{p} &= \left(\Box_1 + \frac{\Box_{10}}{3} - \Box_2 p\right)\frac{\dot{\varrho}}{\varrho} + (\Box_5 p - \Box_2)\operatorname{tr}(T^*D^*), \\ \mathring{T}^* &= \Box_{10} D^* - \Box_5 \operatorname{tr}(T^*D^*)T^*, \end{aligned} \tag{6.5}$$

that is equivalent to the unique equation

$$\mathring{T} = [\Box_1 \operatorname{tr} D + \Box_2 \operatorname{tr}(TD)]1 + \Box_5\left[\operatorname{tr}(TD) - \frac{(\operatorname{tr} T)(\operatorname{tr} D)}{3}\right]T + \Box_{10} D. \tag{6.6}$$

The system (6.5) and the Eq. (6.6), by the arbitrariness of the functions $\Box_1$, $\Box_{10}$, $\Box_2$ and $\Box_5$, can, of course, be written in the form:

$$\tag{6.7} \begin{aligned} \dot{p} &= \Gamma(p, q, \varrho)\frac{\dot{\varrho}}{\varrho} + C(p, q, \varrho)\operatorname{tr}(T^*D^*), \\ \mathring{T}^* &= 2\mu(p, q, \varrho)D^* - B(p, q, \varrho)\operatorname{tr}(T^*D^*)T^*, \end{aligned}$$

where

$$\tag{6.8} \begin{aligned} \Gamma &= \frac{\Box_{10}}{3} + \Box_1 - \Box_2 p, \\ C &= \Box_5 p - \Box_2, \\ 2\mu &= \Box_{10}, \\ B &= \Box_5; \end{aligned}$$

Γ and 2μ usually denote the bulk and shear moduli of isotropic linear elastic materials, respectively. The analogy is clear if we consider that the constitutive equations of isotropic infinitesimal elasticity in incremental form is

$$\tag{6.9} \dot{T} = \lambda(\operatorname{tr} T)1 + 2\mu D,$$

where λ and 2μ are the two Lamé constants.

By decomposing (6.9) in the spherical and deviatoric part we obtain

$$\tag{6.10} \dot{p} = \Gamma\frac{\dot{\varrho}}{\varrho}, \qquad \dot{T}^* = 2\mu D^*,$$

where $\Gamma = \lambda + \frac{2}{3}\mu$ is the bulk modulus. Because $\mathring{T}^* = \dot{T}^* - WT^* + T^*W$, for small values of T^* it results $\mathring{T}^* = \dot{T}^*$ and for $T^* = 0$ is $\mathring{T}^* = \dot{T}^*$.

When $T^* \approx 0$, the (6.10) become

$$\tag{6.11} \begin{aligned} \dot{p} &= \Gamma(p, 0, \varrho)\frac{\dot{\varrho}}{\varrho}, \\ \dot{T}^* &= 2\mu(p, 0, \varrho)D^*, \end{aligned}$$

so that the analogy is evident and we conclude that the materials whose mechanical properties are described by the constitutive Eqs. (6.7) behave in the neighbourhoods of states in which the deviatoric stresses are zero as isotropic elastic materials. System (6.7) is equivalent to the single equation

$$\tag{6.12} \mathring{T} = [(\lambda + Bp^2 - Cp)\operatorname{tr} D + (Bp - C)\operatorname{tr}(TD)]1 + B[\operatorname{tr}(TD) + p(\operatorname{tr} D)]T + 2\mu D,$$

which on the basis of the inverses of relations (6.8), namely

$$\tag{6.13} \begin{aligned} \Box_{10} &= 2\mu, \\ \Box_5 &= B, \\ \Box_2 &= Bp - C, \\ \Box_1 &= \Gamma - \frac{2}{3}\mu + Bp^2 - Cp = \lambda + Bp^2 - Cp, \end{aligned}$$

can be written again in the form

$$(6.14)\qquad \overset{\circ}{T} = (\square_1 \operatorname{tr} D + \square_2 \operatorname{tr}(TD))1 + \square_5 \left(\operatorname{tr}(TD) - \frac{(\operatorname{tr} T)(\operatorname{tr} D)}{3}\right) T + \square_{10} D.$$

Let us now look for an explicit form of the constitutive coefficients for loading processes $(\operatorname{tr}(TD) \geqslant 0)$. A differential equation for q can be obtained remembering that $q^2 = \operatorname{tr}(T^*)^2$ and then $\dot{q}q = \operatorname{tr}(\overset{\circ}{T}{}^* T^*)$. Multiplying the second term in (6.10) by T^* and taking the trace, we have

$$(6.15)\qquad \dot{q}q = (2\mu + Bq^2)\operatorname{tr}(T^* D^*).$$

From (6.15) it is clear that the deviatoric stress response in terms of q is described by the constitutive function B. Figure 5 shows that in dense states at least, this response depends more strictly on the density changes than on the mean pressure. A simple interpretation in agreement with the various typical stress results schematically illustrated in Figs. 3–8 comes out naturally in terms of the following constitutive hypothesis:

$$(6.16)\qquad q \to k(\varrho), \quad \dot{q} \to 0.$$

By (6.15) this implies $2\mu + Bq^2 \to 0$ when $q \to k(\varrho)$, so that by continuity it must be $2\mu +$ $+ Bk^2(\varrho) = 0$, from which

$$(6.17)\qquad B = -\frac{2\mu}{k^2(\varrho)}.$$

We can now substitute (6.17) in (6.15) to obtain

$$(6.18)\qquad \dot{q}q = 2\mu\left(1 - \frac{q^2}{k^2(\varrho)}\right)\operatorname{tr}(T^* D^*).$$

From (6.18) it results that for loading processes in which the deviatoric part of the stress power $\operatorname{tr}(T^* D^*) = \operatorname{tr}(TD) - \frac{p\dot{\varrho}}{\varrho}$ is positive, $\dot{q}$ has the same sign of the difference $k(\varrho) - q$. The previous considerations make clear the interpretation of the test results of Figs. 3–8, on the basis of the Eq. (6.18).

Substituting (6.17), the second of (6.7) can be written in the form:

$$(6.19)\qquad \overset{\circ}{T}{}^* = 2\mu\left[D^* - \frac{\operatorname{tr}(T^* D^*)}{k^2(\varrho)} T^*\right].$$

The first of (6.7) for constant p tests assumes the form

$$(6.20)\qquad 0 = \Gamma \frac{\dot{\varrho}}{\varrho} + C\operatorname{tr}(T^* D^*)$$

and for constant ϱ tests

$$(6.21)\qquad \dot{p} = C\operatorname{tr}(T^* D^*).$$

The typical stress results shown in Figs. 3–8 refer to deformation process in which $\operatorname{tr}(T^* D^*) > 0$. Because Γ is always positive, it is possible to deduce some implications concerning the constitutive function C that gives account of the coupling between deviatoric and spherical parts of the stress response. If C is zero, the first of (6.7) could be solved separately to give the spherical stresses. This is the case in [12]. Figs. 3 and 5, when compared with the Eq. (6.20), respectively suggest that

$$k(\varrho) \to \psi p_0, \quad C \to 0,$$

and

$$q \to \psi p_0, \quad C \to 0.$$

Moreover, Fig. 5 shows that ϱ has always the same sign of the difference $\psi p_0 - q$ and that $\psi p_0 = 0 \Rightarrow \dot{\varrho} = 0$.

Because C is a dimensionless function, we propose the following explicit form:

$$C(p, q, \varrho) = \frac{(q - \psi p)|k(\varrho) - \psi p|}{b}, \tag{6.22}$$

where b is a constant with dimension of stress. Of course, the smaller is b, the greater are the dilatancy effects during shear processes of dense materials (Fig. 5).

Expression (6.22) is in agreement with the behaviour illustrated in Fig. 4 and for constant ϱ tests, with that of Figs. 6, 7, 8 when compared with the Eq. (6.21).

We can now summarize the previous results writing (6.7) in the form

$$\begin{aligned} \dot{p} &= \Gamma \frac{\dot{\varrho}}{\varrho} + \frac{(q - \psi p)|k(\varrho) - \psi p|}{b} \operatorname{tr}(T^* D^*), \\ \mathring{T}^* &= 2\mu \left[D^* - \frac{\operatorname{tr}(T^* D^*)}{k^2(\varrho)} T^* \right]. \end{aligned} \tag{6.23}$$

The constitutive functions Γ, μ and k remain now to be given an explicit form. Their determination is really difficult, mainly because of the lack of suitable experimental data. By the same reason, the choice of a suitable form of the constitutive equations for unloading processes is troublesome. Anyway a major step toward the verification of the validity of the proposed model is the evaluation of the reliability, at least from a qualitative point of view, of the solutions obtainable for boundary-value problems that simulate some real processes. With these considerations in mind, the next step will be to conceive an explicit form of the constitutive coefficients that, if strongly simplified, can be realistic.

To this aim, for the unloading processes will be assumed the absence of dilatancy effects, i.e. $C = 0$, and an "elastic" behaviour with variable moduli Γ and μ, i.e. $B = 0$, so that the constitutive Eqs. (6.7) assume now the special form

$$\dot{p} = \Gamma(p, q, \varrho) \frac{\dot{\varrho}}{\varrho}, \qquad \mathring{T}^* = 2\mu(p, q, \varrho) D^*. \tag{6.24}$$

A leading hypothesis in the determination of the constitutive coefficients for loading and unloading processes is that every granular material has a limited range of admissible densities, with an upper bound ϱ_L that is asymptotically approached when the pressure increases, and a lower limit ϱ_M, beyond which the continuity of the body is lost. The deformability of the material, of course, decreases with the density and will be assumed that it tends to zero for $\varrho \to \varrho_L$ and to infinity as $\varrho \to \varrho_M$ (Fig. 2).

Let us suppose that, during purely hydrostatic compression of the virgin material, Γ is a function of the density only. The constitutive equations for such processes can then be written in the form

(6.25) $$\dot{p} = \frac{\Gamma}{\varrho}\dot{\varrho}, \qquad \mathring{T}^* = 0.$$

The first of (6.25) gives

(6.26) $$\frac{dp}{d\varrho} = \frac{\Gamma}{\varrho}$$

and by integration we obtain the following expression for the pressure on the virgin compression line:

(6.27) $$p(\varrho) = \int_{\varrho_M}^{\varrho} \frac{\Gamma(\eta)}{\eta}\, d\eta + p_M,$$

where $p_M = p(\varrho_M)$, while the previous considerations and the diagrams of Fig. (11) suggest the following explicit simple form

(6.28) $$p(\varrho) = \Gamma_v \frac{\varrho - \varrho_M}{\varrho_L - \varrho},$$

where Γ_v is a constant with dimension of stress.

From (6.27) and (6.28) we have

(6.29) $$\frac{\Gamma}{\varrho} = \Gamma_v \frac{\varrho_L - \varrho_M}{(\varrho_L - \varrho)^2}$$

for virgin compression processes.

Experimental pressure-density relations (Fig. 2) show that it is reasonable to assume for unloading processes a similar expression of the rate Γ/ϱ but with a greater value of the constant, so that we take

(6.30) $$\frac{\Gamma}{\varrho} = \Gamma_u \frac{\varrho_L - \varrho_M}{(\varrho_L - \varrho)^2}$$

with $\Gamma_u > \Gamma_v$.

For general loading processes, observing that the rate $dp/d\varrho = \Gamma/\varrho$ decreases with the distance from the virgin compression line tending to reach the value (6.29), we will assume

(6.31) $$\frac{\Gamma}{\varrho} = \Gamma_v \left[\left(\frac{\varrho - \varrho_M}{\varrho_L - \varrho}\right)\frac{\Gamma_v}{p}\right]^r \frac{\varrho_L - \varrho_M}{(\varrho_L - \varrho)^2},$$

where r is a dimensionless constant. For virgin processes by (6.29) it is $\frac{\varrho - \varrho_M}{\varrho_L - \varrho}\frac{\Gamma_v}{p} = 1$ and (6.31) reduces to (6.29). Thus we can summarize:

(6.32) $$\Gamma(p, \varrho) = \begin{cases} \Gamma_v \left[\left(\frac{\varrho - \varrho_M}{\varrho_L - \varrho}\right)\frac{\Gamma_v}{p}\right]^r \frac{\varrho_L - \varrho_M}{(\varrho_L - \varrho)^2}\varrho, & \text{if} \quad \operatorname{tr}(TD) \geqslant 0, \\ \Gamma_u \frac{\varrho(\varrho_L - \varrho_M)}{(\varrho_L - \varrho)^2}, & \text{if} \quad \operatorname{tr}(TD) \leqslant 0. \end{cases}$$

Let us now analyze the response to purely deviatoric loading deformation processes, characterized by the same stretch history but different values of the density.

Equation (6.19) can be written in the form

$$\dot{q} = 2\mu\left(1-\frac{q^2}{k^2}\right)\frac{\operatorname{tr}(T^*D^*)}{(\operatorname{tr}T^{*2})^{1/2}}. \tag{6.33}$$

With the initial condition $T^*(0) = 0$, we have $q(0) = 0$ and

$$\dot{q}(0) = 2\mu\delta(D^*), \tag{6.34}$$

where

$$\lim_{T^*\to 0}\frac{\operatorname{tr}(T^*D^*)}{(\operatorname{tr}T^{*2})^{1/2}} = \delta(D^*) \geqslant 0. \tag{6.35}$$

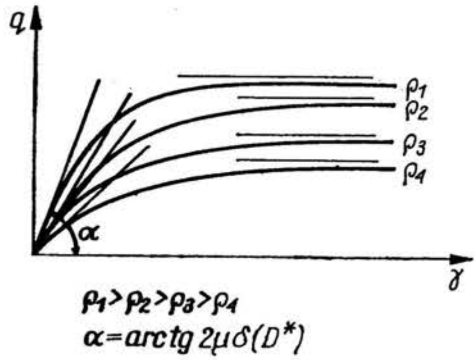

FIG. 10.

The non-negative function δ is linear and such that $D^* = 0 \Rightarrow \delta(D^*) = 0$. Typical test results are plotted in Fig. (10) and when interpreted in terms of (6.34) and (6.35), justify the assumption of the following expression for k and μ:

$$k = k_c\frac{\varrho-\varrho_M}{\varrho_L-\varrho}, \tag{6.36}$$

$$\mu = \mu_T\frac{\varrho-\varrho_M}{\varrho_L-\varrho}, \tag{6.37}$$

where k_c and μ_T are constants with dimension of stress. The constitutive parameters whose value must be determined experimentally for each material are therefore Γ_v, Γ_u, r, ϱ_M, ϱ_L, k_c, μ_T, ψ, b. The design of suitable experimental procedures for such determination will be discussed in a next paper.

Comparison with experimental data shows that as possible values for the constitutive parameters can be assumed

$$\varrho_M = 1.2,\quad \varrho_L = 2.2,\quad \mu_T = 600,\quad k_c = 90,$$

$$\Gamma_v = 400,\quad \Gamma_u = 4000,\quad r = 5,\quad b = 10000,\quad \psi = 1.$$

With such a choice, the constitutive equations can be integrated to give local stress-strain relations. The procedure is illustrated in detail in [17]. Here, only some results for spherical compression and simple contraction under constant pressure or constant density are reported. Initial states are always spherical. For spherical and constant pressure processes, also unloading-reloading paths are shown. The results are illustrated in Figs. 11–16.

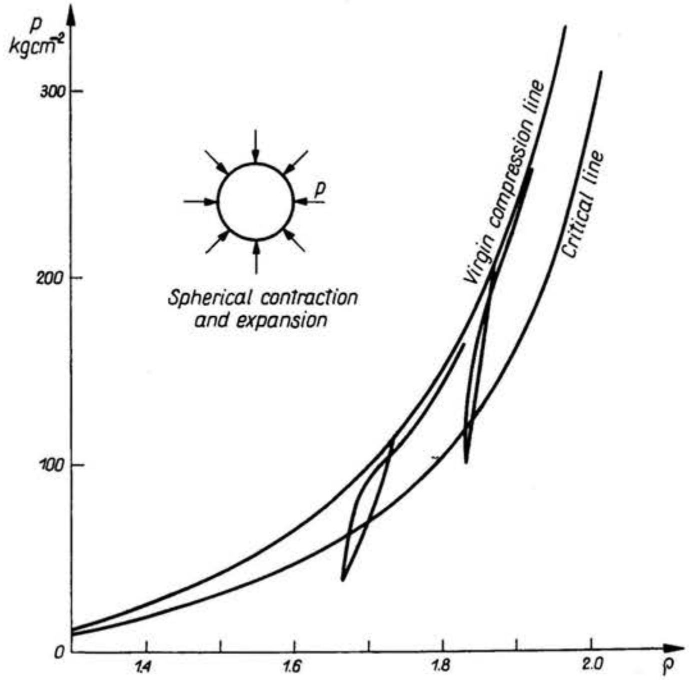

FIG. 11.

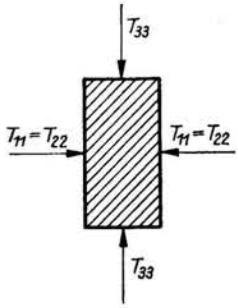

FIG. 12.

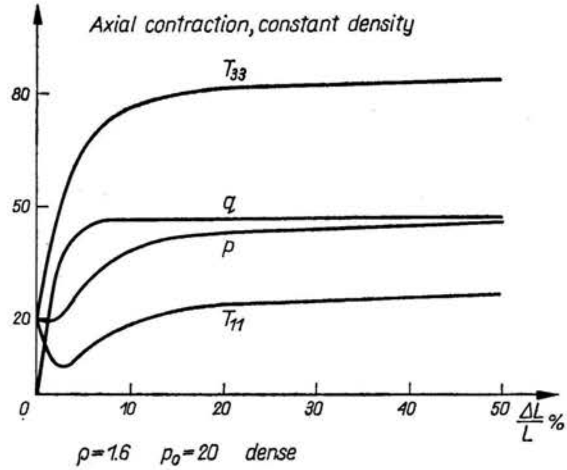

FIG. 13.

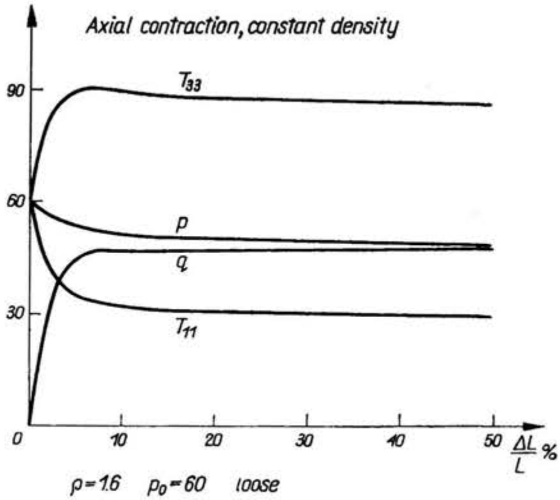

FIG. 14.

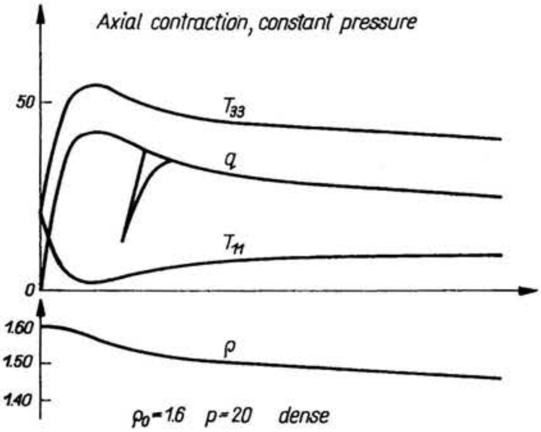

FIG. 15.

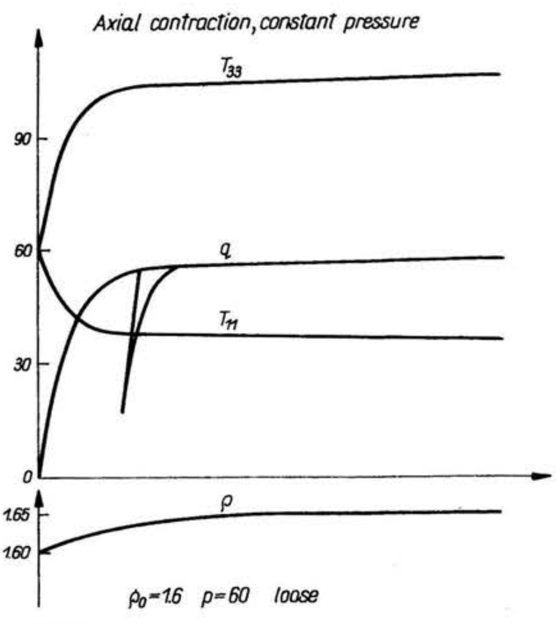

FIG. 16.

7. Conclusions

The present theory is characterized by the inclusion of the powerfully simplifying, but realistic, critical state conception in the general framework of the theory of rate type materials. It is an attempt to provide a model of the mechanical behaviour of granular media for finite deformations whose main features will be listed below:

1. No true elastic range is exibited.

2. Account is given of the dilatancy effects under constant mean pressure, through the coupling of deviatoric and spherical parts of stress response.

3. Purely spherical processes do not affect deviatoric stresses.

4. Linear elastic behaviour in the neighbourhoods of states with zero deviatoric stresses, if the density is constant.

5. Continuous transition from elastic behaviour (when $T^* \approx 0$) to perfectly plastic behaviour as deviatoric deformations increase under constant pressure or constant density.

The determination of stress-strain relations for relevant homogeneous deformation processes, to be compared with experimental tests, the discussion of the procedures as well as the solution of boundary value problems for non-uniform motions, will be the subject of next works.

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