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Hyperbolic flows in ideal plasticity

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A CERTAIN class of three-dimensional solutions is found for a system of ideal plastic flow. These solutions have a degenerate hodograph which is a minimal surface namely a catenoid. Also all simple waves (three-dimensional) for this system are considered.

Wyznaczono pewną klasę rozwiązań trójwymiarowych dla problemu idealnego przepływu plastycznego. Hodograf tych rozwiązań jest zdegenerowany i przedstawia sobą pewną powierzchnię minimalną, mianowicie katenoidę. Rozważono także wszystkie (trójwymiarowe) fale proste tego układu.

Определен некоторый класс трехмерных решений для задачи идеального пластического течения. Годограф этих решений вырожден и представляет собой некоторую минимальную поверхность, а именно катеноид. Обсуждены также все (трехмерные) простые волны этой системы.

1. Introduction

IN THIS paper we deal with the equations of an ideal rigid-plastic flow in three-dimensional space. A certain class of their solutions is distinguished by algebraical conditions which appear in the case of hyperbolic systems [7]. Therefore, we define such solutions as hyperbolic solutions. The meaning of this term is explained in the next section. All solutions obtained have a degenerate deviators S_{ij} of the stress-tensor, i.e., $\det \|S_{ij}\| = 0$ which assures the existence of the characteristic vectors. Among these solutions are all the plane flows for quasi-static case ($\varrho = 0$), the simple (not necessarily plane) waves which are expressed analytically, and a certain class of nonplanar solutions having catenoids as hodographs. The last solutions together with plane flows form a class of the so-called "double waves" obtained when the hyperbolic system of two quasi-linear equations with two dependent and two independent variables is solved. For such systems the method of characteristics or the method of Riemann invariants is well-known [5]. These systems may also be linearized by inverting the role of dependent and independent variables.

In Sec. 2 the basic method which is used for the construction of the solutions is explained. Three stages of such construction may be distinguished; the first is purely algebraic, the second derives simple waves and the last leads us to the "double waves".

The notions of exterior product and exterior derivative, which are used in Sec. 2, are explained briefly in the Appendix. More details can be found in [4].

2. The basic method

The system of quasi-linear differential equations of the first order considered here[1]

$$a^{s\nu}_j(u^1, \ldots, u^l)u^j_{,\nu}(x^1, \ldots, x^n) = 0, \qquad \begin{array}{l} \nu = 1 \ldots n, \\ j = 1 \ldots l, \\ s = 1 \ldots m, \end{array} \tag{2.1}$$

is said to be nonelliptic at the point $u = (u^1, \ldots, u_l)$ if there exists at least one pair of real vectors

$$\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n), \quad \boldsymbol{\gamma} = (\gamma^1, \ldots, \gamma^l)$$

such that

$$a^{s\nu}_j(u^1, \ldots, u^l)\gamma^j \lambda_\nu = 0. \tag{2.1a}$$

The vectors $\boldsymbol{\lambda}$, $\boldsymbol{\gamma}$ will be said to be "knotted" characteristic vectors and the matrix $L^j_\nu = \gamma^j \lambda_\nu$ will be called a "simple integral element" at the point u. We denote $\boldsymbol{\gamma}(u) \rightleftharpoons \boldsymbol{\lambda}(u)$ if, and only if, they satisfy (2.1a). The matrix L^j_ν is an integral element of (2.1) at a point u iff $a^{s\nu}_j(u) L^j_\nu = 0$.

All simple elements at any point u generate a vector space over the real numbers, which will be denoted by Q_1. Hence Q_1 contains all linear combinations of simple elements. A solution for which $u^j_{,\nu}(x) \in Q_1$, or more precisely, for which there exist at each point $x \in D$ such simple elements that

$$u^j_{,\nu}(x) = \xi^1(x)\underset{1}{\gamma^j}(u)\lambda^1_\nu(u) + \ldots + \xi^k(x)\underset{(k)}{\gamma^j}(u)\lambda^k_\nu(u),$$

will be said to be a hyperbolic solution. It may be shown [7] that solutions of hyperbolic systems have this property. Denote by E the space of $x^1, x^2, \ldots, x^4$, we have

THEOREM 1. *Each field of simple elements* (e.g. $\gamma^j(u)\lambda_\nu(u)$) *of the Eqs.* (2.1) *generates a class of solutions (the so-called "simple waves"). Indeed, let* $u^j = f^j(R)$ *be a solution of the system of ordinary differential equations*

$$\frac{d}{dR}u^j = \gamma^j(u) \tag{2.2}$$

which describes the curve tangent to $\boldsymbol{\gamma}(u)$ *in the l-dimensional space* H^l *of* $u^1 \ldots u^l$, *and let* $\varphi(\cdot)$ *be an arbitrary differentiable function of one variable, then the following formulae*

$$\begin{aligned} u^j &= (f^j R), \\ R &= \varphi(\lambda_\nu(f(R))x^\nu) \end{aligned} \tag{2.3}$$

define the solution of the Eqs. (2.1) *in the domain of* E *in which the Eqs.* (2.3) *can be solved for u.*

By differentiation we have

$$R_{,\nu} = \dot{\varphi}\frac{\lambda_\nu}{1-\dot{\varphi}\lambda_{\mu,R}x^\mu}$$

and using (2.2) we check that

[1] Throughout this paper we use the summation convention for lower and upper repeated indices except the case when one of these indices is taken in brackets e.g. $\lambda_\nu x^\nu = \lambda_1 x^1 + \ldots + \lambda_n x^{(n)}$.

$$a^{s\nu}_{j} u^j_{,\nu} \sim a^{s\nu}_{j} f^j_{,R} \lambda_\nu = 0$$

because the vector $\gamma = f_{,R}$ and $\lambda(f(R))$ are knotted. Solution given by expressions (2.3) is constant on the $(n-l)$-dimensional planes perpendicular to $\lambda(f(R))$. In fact, a displacement along the direction dx orthogonal to λ does not change the value of R and thus the values u^j and the vector $\lambda(f(R))$ remain constant.

In many cases a class of fields of simple elements corresponds to the same characteristic vector field $\gamma(u)$. For example, let $\gamma^j \underset{1}{\lambda_\nu}, \gamma^j \underset{2}{\lambda_\nu}, \ldots, \gamma^j \underset{k}{\lambda_\nu}$ be the independent fields of simple elements which have the same characteristic vector field γ and let $\varphi(\cdot, \ldots, \cdot)$ be a differentiable function of k variables; then:

THEOREM 2. *The following expressions*

$$u^j = f^j(R), \qquad R = \varphi(\underset{1}{\lambda_\nu} x^\nu, \ldots, \underset{k}{\lambda_\nu} x^\nu), \tag{2.4}$$

where $f(R)$ satisfies (2.2) *and* $\underset{i}{\lambda} = \underset{i}{\lambda}(f(R))$, *define the simple wave solution of Eqs.* (2.1) *in a certain domain of E.*

This may also be checked by differentiation.

Some generalizations of these solutions the are-so called double waves ([1, 2]). Let G: $u^j = f^j(R^1, R^2)$ be a surface on which two fields of simple elements are defined by relations:

$$\underset{1}{\gamma}^j = f^j_{,R^1}, \qquad \underset{2}{\gamma}^j = f^j_{,R^2}, \tag{2.5}$$

where $\underset{1}{\gamma}(f)$, $\underset{2}{\gamma}(f)$ are the characteristic vectors in the space H^l. Let λ^1, λ^2 be the corresponding characteristic vectors from E, i.e., $\underset{i}{\gamma} \rightleftharpoons \lambda^i$. Thus $\underset{1}{\gamma^j} \lambda^1_\nu$, $\underset{2}{\gamma^j} \lambda^2_\nu$ are the fields of simple elements defined on G. We will seek the solution in the following way: At first, the hodograph surface G and the function $f(R^1, R^2)$ satisfying the conditions (2.5) must be constructed, then two functions $R^1(x)$, $R^2(x)$ must be found the gradients of which are the characteristic vectors λ^1, λ^2 knotted with $f_{,R^1}, f_{,R^2}$ at the point $u = f(R^1(x), R^2(x))$ so they form a pair of simple elements on G. Clearly, the expression $u^j = f^j(R^1(x), R^2(x))$ is a solution of the system (2.1). We restrict our attention to the case in which for any characteristic direction $\gamma(u)$ we have only one direction $\lambda(u)$ such that $\gamma \rightleftharpoons \lambda$. If λ is a characteristic vector, then for a real number ξ the vector $\xi\lambda$ is also a characteristic vector knotted with γ. Therefore to find the functions $R^1(x)$, $R^2(x)$ we have to solve the Pfaff equations:

$$\begin{aligned} dR^1 &= \xi^1 \lambda^1_\nu \big(f(R^1, R^2)\big) dx^\nu, \\ dR^2 &= \xi^2 \lambda^2_\nu \big(f'(R^1, R^2)\big) dx^\nu, \end{aligned} \qquad dx^1 \wedge \ldots \wedge dx^n \neq 0, \tag{2.6}$$

where ξ^1 and ξ^2 are also unknown functions of $x^1, \ldots, x^n$. The Eqs. (2.6) mean that the gradients of R^1 and R^2 are the characteristic vectors. Let d denote the exterior derivative with respect to variables $x^1, \ldots, x^n$. dR^1 and dR^2 as well as dx^ν may be treated as the exterior derivatives of zero-forms: R^1, R^2 and x^ν. Exterior differentiation of (2.6) yields:

$$d\xi^1 \wedge \lambda^1 + \xi^1 d\lambda^1 = 0,$$
$$d\xi^2 \wedge \lambda^2 + \xi^2 d\lambda^2 = 0,$$

where $A \wedge B$ denotes the exterior product of two forms $A = A_\nu dx^\nu$, $B = B_\nu dx^\nu$. Here and in what follows symbol $\boldsymbol{\lambda}$ denotes the form $\lambda_\nu dx^\nu$, except the cases when we call it the vector $\boldsymbol{\lambda}$(2).

Exterior multiplication of the first equation by λ^1 and the second by λ^2 yields:

$$\xi^1 \lambda^1 \wedge d\lambda^1 = 0,$$
$$\xi^2 \lambda^2 \wedge d\lambda^2 = 0.$$

The coefficients ξ^s cannot vanish identically for nondegenerate double waves and thus the last equations imply $\lambda^s \wedge d\lambda^s = 0$. By the Eqs. (2.6) we have

$$\lambda^1 \wedge d\lambda^1 = \sum_s \xi^s \lambda^1 \wedge \lambda^s \wedge \lambda^1_{,R^s} = \xi^2 \lambda^1 \wedge \lambda^2 \wedge \lambda^1_{,R^2},$$
$$\lambda^2 \wedge d\lambda^2 = \sum_s \xi^s \xi^2 \wedge \lambda^s \wedge \lambda^2_{,R^s} = \xi^1 \lambda^2 \wedge \lambda^1 \wedge \lambda^2_{,R^1}$$

and hence, for these same arguments as above, the following conditions must be true for the double wave solution:

$$\lambda^1 \wedge \lambda^2 \wedge \lambda^1_{,R^2} \equiv 0,$$
$$\lambda^2 \wedge \lambda^1 \wedge \lambda^2_{,R^1} \equiv 0. \tag{2.7}$$

These conditions were obtained also by BURNAT [2, 3] in a different purely geometrical way. The following theorem (see also [8]) states that these conditions are also sufficient.

THEOREM 3. *If the conditions* (2.7) *are satisfied then there exists a family of double waves which depends on two function of one variable, moreover, if the conditions* (2.7) *are not satisfied the double waves with hodograph G do not exist.*

Assuming that dx in Eqs. (2.6) is orthogonal to the vectors $\boldsymbol{\lambda}^1$, $\boldsymbol{\lambda}^2$ we, can see that the following lemma is true.

LEMMA. *The solutions of the Pfaff equations* (2.6) (*if they exist*) *are constant along the directions orthogonal to the vectors* λ^1, λ^2 *and therefore the double waves have also this property.*

This fact allows us to change the Pfaff equations into a system of two hyperbolic equations. Assume for instance, that the plane x^1, x^2 contains none direction orthogonal to $\boldsymbol{\lambda}^1$ and $\boldsymbol{\lambda}^2$ in the considered domain of R^1, R^2-plane. The functions $R^1(x)$, $R^2(x)$ restricted to x^1, x^2-plane denoted by $\tilde{R}^1(x)$, $\tilde{R}^2(x)$ satisfy the following Pfaff equations by virtue of (2.6):

$$d\tilde{R}^1 = \tilde{\xi}^1(x^1, x^2)(\tilde{\lambda}^1_1 dx^1 + \tilde{\lambda}^1_2 dx^2),$$
$$d\tilde{R}^2 = \tilde{\xi}^2(x^1, x^2)(\tilde{\lambda}^2_1 dx^1 + \tilde{\lambda}^2_2 dx^2). \tag{2.8}$$

By elimination of ξ^1, ξ^2 from (2.8) we arrive at the equivalent system of two hyperbolic equations:

$$\tilde{R}^1_{,1} \tilde{\lambda}^1_2 - \tilde{R}^1_{,2} \tilde{\lambda}^1_1 = 0,$$
$$\tilde{R}^2_{,1} \tilde{\lambda}^2_2 - \tilde{R}^2_{,2} \tilde{\lambda}^2_1 = 0, \tag{2.9}$$

where the components of $\tilde{\lambda}^1_\nu$, $\tilde{\lambda}^2_\nu$ depend on $\tilde{R}^1$, $\tilde{R}^2$ in the same manner as λ^1_ν, λ^2_ν depend

(2) The form $\lambda_\nu dx^\nu$ is called the covector and the differential forms dx^ν form a basis of dual space (E^n).

on R^1, R^2 in Eqs. (2.6). The solution of the Eqs. (2.9) depends on the initial value problem which is given by two functions of one variable. By extension of $\tilde{R}^1(x^1, x^2)$ and $\tilde{R}^2(x^1, x^2)$ in such a way that they are constant on the $(n-2)$-dimensional planes orthogonal to $\lambda^1(f(\tilde{R}))$, $\lambda^2(f(\tilde{R}))$ at each point (x^1, x^2), we obtain the functions $R^1(x)$, $R^2(x)$. To prove Theorem 3, we must show that so obtained functions $R^1(x)$, $R^2(x)$ satisfy the Eqs. (2.6).

P r o o f. Assume that $\tilde{R}^1(x^1, x^2)$, $\tilde{R}^2(x^1, x^2)$ is a solution of the Eqs. (2.9). We introduce new variables z^1, z^2 instead of x^1, x^2 by the relations

$$z^a = c^a_s(R)\lambda^s_\nu(f(R))x^\nu, \qquad a, s = 1, 2, \quad \nu = 1, \ldots, n,$$

where $c^a_r\lambda^s_a = \delta^s_r$ $a, s, r = 1,2$. By definition we see that z^a is constant on the $(n-2)$-dimensional planes orthogonal to $\lambda^1(f(R))$, $\lambda^2(f(R))$. Moreover, for $x = (x^1, x^2, 0, \ldots, 0)$, we have

$$z^a = x^a, \quad a = 1,2.$$

The functions $R^1(x)$, $R^2(x)$ we define by implicite relations

$$R^s(x, \ldots, x^n) = \tilde{R}^s(z_1, z^2).$$

To check the Eqs. (2.6) we return to the conditions (2.7). The geometrical meaning of these conditions is the linear dependence among the appropriate vectors and hence they mean that there exist such coefficients $\alpha^{as}(R)$, $\beta^{as}(R)$ that

$$\lambda^s_{,R^a} = \alpha^{as}\lambda^s + \beta^{as}\lambda^a.$$

By differentiation of (2.10) we get

$$R^s_{,\nu} = \tilde{R}^s_{,z^a}z^a_{,\nu} = \xi^s\lambda^s_a(c^a_r\lambda^r_\mu x^\mu)_{,\nu} = \xi^s\lambda^s_a c^a_r\lambda^r_\nu + \xi^s\lambda^s_a(c^r_a\lambda^r_\mu)_{,R^b}R^b_{,\nu}x^\mu,$$
$$a, b, r, s = 1, 2, \quad \mu, \nu = 1, \ldots, n.$$

On the other hand we have

$$\lambda^s_a(c^a_r\lambda^r_\mu)_{,R^b} = c^a_r(-\lambda^s_{a,R^b}\lambda^r_a + \lambda^s_a\lambda^r_a\lambda^r_{\mu,R^b}) = c^a_r\lambda^s_{a,R^b}\lambda^r_\mu + \lambda^s_{\mu,R^b} = -c^a_r(\alpha^{bs}\lambda^s_a + \beta^{bs}\lambda^b_a)\lambda^r_\mu + \lambda^s_{\mu,R^b}$$
$$= \begin{cases} -(\delta^s_r\alpha^{sb} + \delta^b_r\beta^{bs})\lambda^r_\mu + \alpha^{bs}\lambda^s_\mu + \beta^{bs}\lambda^b_\mu \equiv 0, & s \neq b, \\ A^s_\mu \quad \text{if} \quad s = b, \end{cases}$$

where by A^s_μ we denote the value of $\lambda^{(s)}_a(c^a_r\lambda^r_\mu)_{,R^s}$. In consequence we attain:

$$R^s_{,\nu} = \xi^s\lambda^s_\nu + A^s_\mu x^\mu R^s_{,\nu} \qquad \text{and hence} \qquad R^s_{,\nu} = \frac{\xi^s}{1 - A^s_\mu x^\mu}\lambda^s_\nu$$

which proves that gradient of R^s is proportional to λ^s.

It is often not convenient to start from the construction of the surface G and apply (2.7) afterwards. In such a case the conditions (2.7) may be developed as follows:

$$\lambda^1 \wedge \lambda^2 \wedge \lambda^1_{,R^2} = \lambda^1 \wedge \lambda^2 \wedge \lambda^1_{,u^i}\underset{2}{\gamma^i}|_G = 0,$$
$$(2.11) \qquad \lambda^2 \wedge \lambda^1 \wedge \lambda^2_{,R^1} = \lambda^2 \wedge \lambda^1 \wedge \lambda^2_{,u^i}\underset{1}{\gamma^i}|_G = 0.$$

These conditions hold also when the magnitudes of $\underset{s}{\gamma}$ are changed and thus they depend only on the directions of $\underset{s}{\gamma}$. Suppose that the considered simple elements are defined in a neighbourhood D of the surface G then (2.11) are fulfilled when stronger conditions hold:

$$
(2.12)\qquad
\begin{aligned}
&\Delta_2^1 = \lambda^1 \wedge \lambda^2 \wedge \lambda^1_{,\underset{2}{\gamma}} = 0,\\
&\Delta_1^2 = \lambda^2 \wedge \lambda^1 \wedge \lambda^2_{,\underset{1}{\gamma}} = 0, \qquad \lambda_{,\gamma} = \lambda_{,u^i}\gamma^i
\end{aligned}
$$

in the region D. These conditions, together with Frobenius condition:

$$[\underset{1}{\gamma}, \underset{2}{\gamma}] = \underset{2}{\gamma}_{,\underset{1}{\gamma}} - \underset{1}{\gamma}_{,\underset{2}{\gamma}} \in \{\underset{1}{\gamma}, \underset{2}{\gamma}\}\,(^3)$$

which assures existence of the surfaces tangent to the vector fields $\underset{1}{\boldsymbol{\gamma}}, \underset{2}{\boldsymbol{\gamma}}$, are the equations for simple elements. If these are satisfied then there exists a family of double waves connected with each such hodograph surface G and dependent on two functions of one variable. To conclude this section the following theorem will be proved:

THEOREM 4. *Each solution $u(x)$ constructed from two simple elements, i.e., $u^s_{,\nu}(x) = \xi^1(x)\underset{1}{\gamma^j}\lambda^1_\nu + \xi^2(x)\underset{2}{\gamma^j}\lambda^2_\nu$ for which rank $\|u^j_{,\nu}\| = 2$, is locally a double wave of the form presented above.*

P r o o f. Since the rank of the integral element is 2 then the image of the transformation $u(x)$ is a two-dimensional surface G. Vectors $\underset{1}{\boldsymbol{\gamma}}, \underset{2}{\boldsymbol{\gamma}}$ are tangent to G and the curves tangent respectively to $\underset{1}{\boldsymbol{\gamma}}$ and $\underset{2}{\boldsymbol{\gamma}}$ can be taken (at least locally) as the coordinate lines on G. This permits us to introduce the functions $u^j = f^j(R^1, R^2)$ satisfying the condition (2.5). The equations $u^j(x) = f^j(R^1, R^2)$ define locally the functions $R^1(x)$, $R^2(x)$. Since the vectors $f_{,R^s}$ and $\underset{s}{\boldsymbol{\gamma}}$ have the same directions, then by the equality

$$\xi^1\underset{1}{\gamma^j}\lambda^1_\nu + \xi^2\underset{2}{\gamma^j}\lambda^2_{\nu_i} = f^j_{,R^1}R^1_{,\nu} + f^j_{,R^2}R^2_{,\nu}$$

the gradient of $R^s(x)$ must be proportional to $\boldsymbol{\lambda}^s(f(R^1\ R^2))$, which completes the proof.

3. Flow equations and their simple elements

Equations of an ideal plastic flow, which will be considered here have the form [4] (⁴),

$$
(3.1)\qquad
\begin{aligned}
&\text{(a)}\quad \sigma_{,i} + S_{ij,j} = \varrho\left(\frac{\partial v_i}{\partial t} + v_j v_{i,j}\right),\\
&\text{(b)}\quad v_{i,j} + v_{j,i} = MS_{ij},\\
&\text{(c)}\quad S_{ij}S_{ij} = 2k^2, \quad (= \text{const}),\\
&\text{(d)}\quad M(t, \overline{x}) \geqslant 0,
\end{aligned}
$$

(³) $\{\alpha, \ldots, \beta\}$ — denotes the linear space generated by $\alpha, \ldots, \beta$.

(⁴) In this section we use also the summation convention for the repeated indices in the same position e.g. $v_i v_i = v_1 v_1 + v_2 v_2 + v_3 v_3$.

where σ is the pressure; (S_{ij}) — deviator of the symmetric stress tensor: $S_{ij} = S_{ji}$, $S_{ii} = 0$, M — one of the unknown functions, and v — velocity of the flow.

In this form the equations (b) and (c) are not homogeneous. However the Eq. (b) can be written in the homogeneous form:

$$\frac{v_{i,j}+v_{j,i}}{S_{ij}} = \frac{v_{k,l}+v_{l,k}}{S_{kl}}, \quad i,j,k,l = 1,2,3.$$

For the homogeneous equations (a) and (b) the results of the previous section can be applied. Let the components of the vectors introduced in the previous section be denoted in this case by $(\gamma_i, \gamma_{ij}, \gamma_\sigma)$ which correspond to the unknown variables (v_i, S_{ij}, σ). Also let $\lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ correspond to the independent variables $(x^0, x^1\, x^2\, x^3) = (t, x, y, z)$. To determine the simple elements we must substitute $\gamma^J \lambda_\nu$ for $u^J_{,\nu}$. After this, the equation (b) gives

$$\frac{\gamma_i \lambda_j + \gamma_j \lambda_i}{S_{ij}} = \frac{\gamma_k \lambda_l + \gamma_l \lambda_k}{S_{kl}}, \quad i,j,k,l = 1,2,3$$

which is equivalent to the following relation:

$$\gamma_i \lambda_j + \gamma_j \lambda_i = m S_{ij} \tag{3.2}$$

with appropriate coefficient m.

From this follows immediately that all hyperbolic solutions have a degenerate deviator S_{ij}, i.e. rank $||S_{ij}|| = 2$. This is also true for nonelliptic solutions, i.e. solutions for which the Eqs. (2.1) possess the characteristic vectors.

If rank $||S_{ij}|| = 2$ then taking into account the symmetry properties of S we can decompose it into the sum:

$$S_{ij} = k(\alpha_i \beta_j + \beta_i \alpha_j), \tag{3.3}$$

where $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ are three-dimensional vectors for which we can put $\boldsymbol{\alpha}^2 = \boldsymbol{\beta}^2$. Further by definition of S_{ij}, we have

$$0 = S_{ii} = 2k\alpha_i \beta_i \quad \text{hence} \quad \boldsymbol{\alpha} \perp \boldsymbol{\beta};$$

also $S_{ij} S_{ij} = 2k^2 \boldsymbol{\alpha}^2 \boldsymbol{\beta}^2$ whence by the Eq. (c) we have $\boldsymbol{\alpha}^2 = \boldsymbol{\beta}^2 = 1$. After substitution of (3.3) into (a) and (b) they take the form

$$\begin{aligned} \sigma_{,i} + k[(\alpha_i \beta_j)_{,j} + (\alpha_j \beta_i)_{,j}] &= \varrho[v_{i,o} + v_j v_{i,j}], \\ v_{i,j} + v_{j,i} &= km(\alpha_i \beta_j + \alpha_j \beta_i), \end{aligned} \tag{3.4}$$

together with the additional conditions $\alpha_i \alpha_i = \beta_i \beta_i = 1$, $\alpha_i \beta_i = 0$ which lead by differentiation to:

$$\begin{aligned} \alpha_i \alpha_{i,\nu} &= 0, \\ \beta_i \beta_{i,\nu} &= 0, \\ \alpha_i \beta_{i,\nu} + \beta_i \alpha_{i,\nu} &= 0, \end{aligned} \qquad \begin{aligned} i &= 1,2,3, \\ \nu &= 0,1,2,3. \end{aligned} \tag{3.5}$$

Now, let the components of the ten-dimensional vector $\boldsymbol{\gamma}$ corresponding to the new unknown variables $(v_i, \alpha_i, \beta_i, \sigma)$ $\quad i = 1,2,3$ be denoted by $(\gamma_i, A_i, B_i, \gamma_\sigma)$.

The equations for simple elements are now

(i) $$\gamma_\sigma \lambda_i + k[(\lambda_j \beta_j) A_i + (\lambda_j \alpha_j) B_i + (B_j \lambda_j)\alpha_i + (A_j \lambda_j)\beta_i] = \varrho\gamma_i(\lambda_0 + v_j \lambda_j),$$

(ii) $$\lambda_i \gamma_j + \lambda_j \gamma_i = mk(\alpha_i \beta_j + \beta_i \alpha_j),$$

(iii) $$\lambda_\nu (A_i \alpha_i) = 0,$$

(iv) $$\lambda_\nu (B_i \beta_i) = 0,$$

(v) $$\lambda_\nu (\alpha_i B_i + \beta_i A_i) = 0.$$

From the Eq. (ii) if follows that

$$1.\ \lambda_i = \alpha_i \quad \gamma_i = \nu_1 \beta_i \qquad \text{or} \qquad 2.\ \lambda_i = \beta_i \quad \gamma_i = \nu_2 \alpha_i$$

with arbitrary ν_1, ν_2.

Applying the first possibility to the Eq. (i) and using the Eq. (iii) and the Eq. (iv) we find

(3.6) $$(\gamma_\sigma + kB_i \alpha_i)\alpha_j + kB_j = \nu_1 \varrho q_\alpha \beta_j,$$

where $q_\alpha = \lambda_0 + v_i \alpha_i$. The scalar multiplication of the Eq. (3.6) by $\boldsymbol{\beta}$ yields: $\varrho q_\alpha = 0$, hence $q_\alpha = 0$ or $\varrho = 0$. The last case $\varrho = 0$ corresponds to the quasi-static flows. From (3.6) it follows that

$$B_i = \mu_B \alpha_i, \quad \gamma_\sigma = -2\mu_B,$$

where μ_B — arbitrary coefficient.

Introducing the vector $\boldsymbol{\delta} = \boldsymbol{\alpha} \times \boldsymbol{\beta}$ one can fulfil the remaining equations (iv), (v) if, and only if: $A_i = -\mu_B \beta_i + C_1 \delta_i$ (C_1 — arbitrary).

Finally, the simple elements are determined by

1) $$\left\{\begin{array}{l} \lambda = \begin{cases} (-v_i\alpha_i, \alpha_1, \alpha_2, \alpha_3) & \text{if} \quad \varrho \neq 0, \\ (\alpha_1, \alpha_2, \alpha_3) & \text{if} \quad \varrho = 0, \end{cases} \\ \text{since time does not appear in the equations for } \varrho = 0 \text{, we have} \\ \qquad \gamma_i = \nu_1 \beta_i, \quad \gamma_\sigma = -2\mu_B k, \\ \qquad A_i = -\mu_B \beta_i + c_1 \delta_i, \quad B_i = \mu_B \alpha_i, \quad i = 1, 2, 3, \end{array}\right.$$

(3.7) or

2) $$\left\{\begin{array}{l} \lambda = \begin{cases} (-v_i\beta_i, \beta_1, \beta_2, \beta_3) & \text{if} \quad \varrho \neq 0, \\ (\beta_1, \beta_2, \beta_3) & \text{if} \quad \varrho = 0, \end{cases} \\ \qquad \gamma_i = \nu_2 \alpha_i, \quad \gamma_\sigma = -2\mu_A k, \\ \qquad A_i = \mu_A \beta_i, \quad B_i = -\mu_A \alpha_i + C_2 \delta_i. \end{array}\right.$$

To sum up we can construct the hyperbolic solutions by using two simple elements of the form 1) and 2) (3.7). Therefore the hyperbolic solution can represent the simple or double waves only.

4. Simple waves

The ordinary differential equations (2.2) which we have to integrate take the following form for the first of possible vectors (3.7):

$$\text{(4.1)}\qquad \begin{aligned} &\text{a)} && \mathbf{v}_{,R} = \nu_1 \boldsymbol{\beta}, \\ &\text{b)} && \boldsymbol{\alpha}_{,R} = -\mu_B \boldsymbol{\beta} + c_2 \boldsymbol{\delta}, \\ &\text{c)} && \boldsymbol{\beta}_{,R} = \mu_B \boldsymbol{\alpha}, \\ &\text{d)} && \sigma_{,R} = -2\mu_B, \end{aligned}$$

where ν_1, μ_B, C_2 are arbitrary functions of R. We can choose the parameter R to make $\nu_1 = 1$. This choice does not restrict the freedom of the solution as it follows from the form of simple waves (2.3) (2.4). If the vector $\boldsymbol{\beta}$ is treated as the tangent and $\boldsymbol{\alpha}$ as the normal versor then the system of equations (4.1) a)–c) represents the Frenet formulae and due to this fact an arbitrary curve $\mathbf{v} = \mathbf{v}(R)$ in three-dimensional space of (v_1, v_2, v_3) is the solution of the Eqs. (4.1). Hence, if $\nu_1 = 1$, then μ_B must be interpreted as the torsion of this curve. Thus by Theorem 1 we state:

THEOREM 5. *If $v_i = v_i(s)$, $i = 1, 2, 3$ is an arbitrary curve parametrized by its length and $\varphi(\cdot)$ is an arbitrary differentiable function, then the following formulae*:

$$\begin{aligned} v_i &= v_i(s) \\ \beta_i &= \dot{v}_i(s) \end{aligned} \qquad s = \varphi(\ddot{v}_i x^i) \text{ — for quasi-static case } \varrho = 0,$$

or

$$\mu_B \alpha_i = \ddot{v}_i \qquad s = \varphi(\ddot{v}_i(x^i - v_i t)) \text{ — for nonstatic case}$$

(dot denotes the differentiation with respect to s) define the simple wave solutions of the Eqs. (3.1) a), b), c), *in the region D, where the formulae are well defined.*

In both quasi-static and non-static cases the deviator of the stress tensor is defined by the expression:

$$S_{ij}(s) = k\varepsilon \sqrt{|\ddot{\mathbf{v}}|^{-2}}(\dot{v}_i \ddot{v}_j + \dot{v}_j \ddot{v}_i)$$

and the pressure by σ

$$\sigma(s) = [\sigma_0 - 2\varepsilon k \int |\mathbf{v}| ds,$$

where $\varepsilon = \pm 1$.

We have such two possibilities because the expression $\mu_B \boldsymbol{\alpha} = \ddot{\mathbf{v}}(s)$ defines only the product $\mu_B \boldsymbol{\alpha}$. Up to now we have not considered the condition (d) of (3.1).

The quantity M can be computed from our solutions and then we get in the case $\varrho = 0$

$$M = \frac{v_{i,j} + v_{j,i}}{S_{ij}} = \frac{\varepsilon}{k} |\ddot{\mathbf{v}}| \frac{\varphi'}{1 - \ddot{v}_i x^i \varphi'}.$$

The sign of this quantity depends on the region in the space of (x^1, x^2, x^3) and can be changed by the choice of ε. Similarly, for the non-static case, we find

$$M = \frac{\varepsilon}{k} |\ddot{\mathbf{v}}| \varphi' / [1 - (\ddot{v}_i x^i - (\ddot{v}_i v_i)_{,s} t)\varphi^1], \qquad i = 1, 2, 3.$$

The inequality $M \geqslant 0$ can also be fulfilled by appropriate choice of ε. Note that the inertial forces do not appear for the time-dependent simple waves, i.e. for $\varrho \neq 0$.

5. Double waves

Generally, the hyperbolic solutions of the Eqs. (3.1) can be constructed from two simple elements of Sec. 3. By Theorem 4 it should be possible to obtain such solutions using the method of Sec. 2.

One group of conditions (2.11) for the existence of hyperbolic double waves is

$$\Delta^r_s(u) = \lambda^r \wedge \lambda^s \wedge \lambda^r_{\underset{s}{,\gamma}} \equiv 0, \qquad \begin{matrix} r, s = 1, 2, \\ r \neq s, \end{matrix}$$

where $\boldsymbol{\gamma} = (\gamma_i, A_i, B_i, \gamma_\sigma)$ and $\mathbf{u} = (\mathbf{v}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \sigma)$ are the ten-dimensional vectors of the previous sections. We must distinguish the quasi-static case $\varrho = 0$ when the vector $\boldsymbol{\lambda}$ becomes three-dimensional and the remaining case, when $\varrho \neq 0$ and $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ is a four-dimensional vector. The conditions (2.11) are automatically satisfied for the quasi-static case ($\varrho = 0$).

Indeed we have

$$\text{(5.1)} \qquad \begin{aligned} \Delta^1_2 &= \boldsymbol{\alpha} \wedge \boldsymbol{\beta} \wedge \left(\frac{\partial \boldsymbol{\alpha}}{\partial \alpha^i} \underset{2}{A_i} \right) = \mu_A \boldsymbol{\alpha} \wedge \boldsymbol{\beta} \wedge \boldsymbol{\beta} \equiv 0, \\ \Delta^2_1 &= \boldsymbol{\beta} \wedge \boldsymbol{\alpha} \wedge \left(\frac{\partial \boldsymbol{\beta}}{\partial \beta_i} \underset{1}{B_i} \right) = \mu_B \boldsymbol{\beta} \wedge \boldsymbol{\alpha} \wedge \boldsymbol{\alpha} = 0 \end{aligned}$$

by the property of the exterior product: $\boldsymbol{\alpha} \wedge \boldsymbol{\alpha} = 0$.

For the second case $\varrho \neq 0$, as it will be shown in the Appendix, we have no possibility to satisfy (2.11) and thus in this case the double waves cannot exist. Let us return to quasi-static solutions. We have to find the function $u = f(R^1, R^2)$ which satisfies the conditions (2.5) and so by (3.7)

$$\begin{aligned} d\mathbf{v} &= \nu_1 \boldsymbol{\beta}\, dR^1 + \nu_2 \boldsymbol{\alpha}\, dR^2, \\ d\boldsymbol{\alpha} &= (-\mu_B \boldsymbol{\beta} + C_1 \boldsymbol{\delta}) dR^1 + \mu_A \boldsymbol{\beta}\, dR^1, \\ d\boldsymbol{\beta} &= \mu_B \boldsymbol{\alpha}\, dR^1 + (-\mu_A \boldsymbol{\alpha} + C_2 \boldsymbol{\delta}), \\ d\sigma &= -2k\mu_B dR^1 - 2k\mu_A dR^2. \end{aligned}$$

Since $\mu_B = -\dfrac{1}{2k}\sigma_{,R^1}$, $\mu_A = -\dfrac{1}{2k}\sigma_{,R^2}$, hence the last equations may be reduced to the following:

$$\text{(5.2)} \qquad \begin{aligned} &\text{(i)} \quad d\mathbf{v} = \nu_1 \boldsymbol{\beta} dR^1 + \nu_2 \boldsymbol{\alpha}\, dR^2, \\ &\text{(ii)} \quad d\boldsymbol{\alpha} = \left(\frac{1}{2k} \sigma_{,R^1} \boldsymbol{\beta} + C_1 \delta \right) dR^1 - \frac{1}{2k} \sigma_{,R^2} \boldsymbol{\beta}\, dR^2, \\ &\text{(iii)} \quad d\boldsymbol{\beta} = -\frac{1}{2k} \sigma_{,R^1} \boldsymbol{\alpha}\, dR^1 + \left(\frac{1}{2k} \sigma_{,R^2} \boldsymbol{\alpha} + C_2 \boldsymbol{\delta} \right) dR^2. \end{aligned}$$

The equations

$$(\nu_1 \boldsymbol{\beta})_{,R^2} = (\nu_2 \boldsymbol{\alpha})_{,R^1},$$

$$\left(\frac{1}{2k}\sigma_{,R^1}\boldsymbol{\beta}+C_1\boldsymbol{\delta}\right)_{,R^2} = \left(-\frac{1}{2k}\sigma_{,R^2}\boldsymbol{\beta}\right)_{,R^1},$$

$$\left(-\frac{1}{2k}\sigma_{,R^1}\boldsymbol{\alpha}\right)_{,R^2} = \left(\frac{1}{2k}\sigma_{,R^2}\boldsymbol{\alpha}+C_2\boldsymbol{\delta}\right)_{,R^1}$$

are the integrability conditions for the Eqs. (5.2). Since using (5.2) we have

$$\boldsymbol{\delta}_{,R^1} = (\boldsymbol{\alpha}\times\boldsymbol{\beta})_{,R^1} = \boldsymbol{\alpha}_{,R^1}\times\boldsymbol{\beta}+\boldsymbol{\alpha}\times\boldsymbol{\beta}_{,R^1} = C_1\boldsymbol{\delta}\times\boldsymbol{\beta} = -C_1\boldsymbol{\alpha},$$

$$\delta_{,R^2} = (\boldsymbol{\alpha}\times\boldsymbol{\beta})_{,R^2} = -C_2\boldsymbol{\beta},$$

then, using the Eqs. (5.2), the integrability conditions lead us to

$$\nu_{1,R^2}\boldsymbol{\beta}+\nu_1\left(\frac{\sigma_{,R^2}}{2k}\boldsymbol{\alpha}+C_2\boldsymbol{\delta}\right) = \nu_{2,R^1}\boldsymbol{\alpha}+\nu_2\left(\frac{\sigma_{,R^1}}{2k}\boldsymbol{\beta}+C_1\boldsymbol{\delta}\right),$$

(5.2a)
$$\sigma_{,R^1R^2}\boldsymbol{\beta}+\frac{1}{2k}C_2\sigma_{,R^1}\boldsymbol{\delta}+C_{1,R^2}\boldsymbol{\delta}-C_1C_2\boldsymbol{\beta} = 0,$$

$$\sigma_{,R^1R^2}\boldsymbol{\alpha}+\frac{1}{2k}C_1\sigma_{,R^2}\boldsymbol{\delta}+C_{2,R^1}\boldsymbol{\delta}-C_1C_2\boldsymbol{\alpha} = 0.$$

Comparing the coefficients with the vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\delta}$ in the Eqs. (5.2a), we get the following equations

(5.3)

a) $\nu_{1,R^2} = \dfrac{1}{2k}\nu_2\sigma_{,R^1},$ a′) $C_{1,R^2} = -\dfrac{1}{2k}C_2\sigma_{,R^1},$

b) $\nu_{2,R^1} = \dfrac{1}{2k}\nu_1\sigma_{,R^2},$ b') $C_{2,R^1} = -\dfrac{1}{2k}C_1\sigma_{,R^2},$

c) $\nu_1 C_2 = \nu_2 C_1,$ c′) $\dfrac{1}{k}\sigma_{,R^1R^2} = C_1C_2.$

To obtain certain geometrical properties of the surfaces described by the Eqs. (5.2), we shall consider them. From the first Eq. (5.2) we see that the vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ are tangent to, and $\boldsymbol{\delta} = \boldsymbol{\alpha}\times\boldsymbol{\beta}$ is perpendicular to the surface $\mathbf{v} = \mathbf{v}_i'(R^1, R^2)$. The curves tangent to $\boldsymbol{\beta}$ are given by $R^1 = \text{const}$ and the curves tangent to $\boldsymbol{\alpha}$ by $R^2 = \text{const}$. The next two equations ii) and iii) of (5.2) tell us that the curvature radii of these curves are tangent to the surface. Such curves are called the asymptotic curves [6]. Further, since $\boldsymbol{\alpha} \perp \boldsymbol{\beta}$ then the asymptotic curves form an orthogonal net on the surface. This is possible only in the case of a surface for which the average curvature of the surface at each point is zero which means that the principal curvatures of the surface have equal norms but different signes. These surfaces have an interesting property. They are the surfaces possessing the minimal surface area spanned on arbitrary closed curves. In experiment such one can be obtained when a closed contour is dipped into a soap solution. The liquid forms a film which by the surface tension assumes as a position of equilibrium the shape of which is that of minimal surface spanned on the contour. If we assume $C_1 \neq 0$, $C_2 \neq 0$, the Eqs. (5.3) can be written in logarithmic form:

$$\text{(5.4)}\quad \begin{aligned} &\text{a)}\quad (\ln\nu_1)_{,R^2} = \frac{1}{2k}\frac{\nu_2}{\nu_1}\sigma_{,R^1}, &&\text{a}')\quad -(\ln C_1)_{,R^2} = \frac{1}{2k}\frac{C_2}{C_1}\sigma_{,R^1},\\ &\text{b)}\quad (\ln\nu_2)_{,R^1} = \frac{1}{2k}\frac{\nu_1}{\nu_2}\sigma_{,R^2}, &&\text{b}')\quad -(\ln C_2)_{,R^1} = \frac{1}{2k}\frac{C_1}{C_2}\sigma_{,R^2},\\ &\text{c)}\quad \frac{\nu_1}{\nu_2} = \frac{C_1}{C_2}, &&\text{c}')\quad \frac{1}{k}\sigma_{,R^1R^2} = C_1C_2. \end{aligned}$$

Hence

$$(\ln\nu_1)_{,R^2} = -(\ln C_1)_{,R^2},$$
$$(\ln\nu_2)_{,R^1} = -(\ln C_2)_{,R^1},$$

and we get the further relations between C_1, C_2 and ν_1, ν_2

$$\nu_1 = \frac{1}{C_1}M_1(R_1{}^1), \qquad \nu_2 = \frac{1}{C_2}M_2(R^2)$$

and thus

$$C_1/C_2 = [M_2(R^2)/M_1(R^1)]^{1/2},$$

where $M_1(R^1)$, $M_2(R^2)$ are arbitrary functions.

Using these relations, the Eqs. (5.4) may be reduced to:

$$(\ln C_1)_{,R^2} + \frac{1}{2k}G\sigma_{,R^1} = 0,$$

$$[\ln(GC_1)]_{,R^1} + \frac{1}{2k}G^{-1}\sigma_{,R^2} = 0,$$

$$\frac{1}{k}\sigma_{,R^1R^2} = C_1^2G,$$

where $G = \sqrt{M_1/M_2}$.

By differentiation and by the last equation one can eliminate σ to obtain

$$\text{(5.5)}\quad \begin{aligned} &[G^{-1}(\ln C_1)_{,R^2}]_{,R^2} + \frac{1}{2}GC_1^2 = 0,\\ &[G(\ln GC_1)_{,R^1}]_{,R^1} + \frac{1}{2}GC_1^2 = 0. \end{aligned}$$

Multiplying the first equation by $G^{-1}(\ln C_1)_{,R^2}$ we can integrate it to yield:

$$\text{(5.6)}\qquad C_1 = \sqrt{2}\left[\operatorname{ch}\left\{K\left(\int G\,dR^2 - l\right)\right\}\right]^{-1},$$

where $K = K(R^1)$ $l = l(R^1)$ are arbitrary functions of R^1. We still have the freedom to make the transformation

$$R^{1\prime} = \theta^1(R^1), \qquad R^{2\prime} = \theta^2(R^2)$$

which does not change curves tangential to $\underset{1}{\gamma}$, $\underset{2}{\gamma}$ and therefore the relation (2.5) remain valid. By using this freedom we can attain

$$\text{(5.7)}\qquad G = \{M_2(R^2)/M_1(R_1^1)\}^{1/2} = (aR^2+b)(cR^1+d), \qquad a, b, c, d \text{ — constants}$$

The function C_1 from (5.6) and (5.7) can be substituted into the second Eq. (5.5) to determine $K(R^1)$ and $l(R^1)$. *It seems*, there is no other solution besides the case when $a = c = 0$. Hence we can take $G = 1$ and then the Eqs. (5.5) may be solved to yield:

$$C_1 = \sqrt{2}[\mathrm{ch}\{A(R^1+R^2)\}]^{-1}, \quad A = \mathrm{const}.$$

Deeper analysis of the Eqs. (5.2) and (5.3) shows that the surface $v^i = v^i(R^1, R^2)$ in this case is axially symmetric. On the other hand, it must be a minimal surface. Besides the family of plane surfaces there is only other family of such surfaces; these are the catenoids which, in the appropriate Cartesian coordinate system, take the following parametrical form:

$$\begin{aligned} v_1 &= \varrho\cos\varphi, \\ v_2 &= \varrho\sin\varphi, \quad \varrho = A\,\mathrm{ch}\frac{w}{A}. \\ v_3 &= w, \end{aligned} \tag{5.8}$$

The parallels $w = \mathrm{const}$ and the meridians $\varphi = \mathrm{const}$ are the curvature lines. In the case of minimal surfaces the asymptotic curves intersect the curvature lines at an angle of 45°. Hence for the asymptotic curves we have:

$$\left(\frac{\partial \mathbf{v}}{\partial\varphi}\right)^2 d\varphi^2 = \left(\frac{\partial \mathbf{v}}{\partial w}\right)^2 dw^2$$

and in the system of asymptotic curves the expressions (5.8) take the form, Fig. 1:

$$\begin{aligned} v_1 &= \varrho\cos(R^1-R^2), \\ v_2 &= \varrho\sin(R^1-R^2), \\ v_3 &= A(R^1-R^2), \quad \varrho = A\,\mathrm{ch}(R^1+R^2). \end{aligned}$$

Since $\boldsymbol{\beta} \sim v_{,R^1}$ and $\alpha \sim v_{,R^2}$. Normalizing we obtain

$$\boldsymbol{\alpha} = \frac{1}{\sqrt{2}}\left(\mathrm{th}\frac{w}{A}\cos\varphi+\sin\varphi,\ \mathrm{th}\frac{w}{A}\sin\varphi-\cos\varphi,\ \frac{1}{\mathrm{ch}\frac{w}{A}}\right),$$

$$\boldsymbol{\beta} = \frac{1}{\sqrt{2}}\left(\mathrm{th}\frac{w}{A}\cos\varphi-\sin\varphi,\ \mathrm{th}\frac{w}{A}\sin\varphi+\cos\varphi,\ \frac{1}{\mathrm{ch}\frac{w}{A}}\right),$$

where $w = A(R^1+R^2)$ $\varphi = R^1-R^2$.

The solutions are constant along the direction $\boldsymbol{\delta}$ which is defined by

$$\boldsymbol{\delta} = \boldsymbol{\alpha}\times\boldsymbol{\beta} = \left(\frac{\cos\varphi}{\mathrm{ch}\frac{w}{A}},\ \frac{\sin\varphi}{\mathrm{sh}\frac{w}{A}},\ -\mathrm{th}\frac{w}{A}\right).$$

By the conclusion from Theorem 3, the Pfaff equations

$$\begin{aligned} dR^1 &= \xi^1\alpha_i(R^1, R^2)dx^i, \\ dR^2 &= \xi^2\beta_i(R^1, R^2)dx^i, \end{aligned} \quad dx^1\wedge dx^2\wedge dx^3 \neq 0, \tag{5.9a}$$

which are to be solved, are equivalent to the system of two hyperbolic equations

$$\begin{aligned} \tilde{R}^1_{,x} &= \tilde{R}^1_{,z}[\mathrm{sh}(R^1+R^2)\sin(R^1-R^2)+\cos(R^1-R^2)\mathrm{ch}(R^1+R^2)], \\ \tilde{R}^2_{,x} &= \tilde{R}^2_{,z}[\mathrm{sh}(R^1+R^2)\sin(R^1-R^2)-\cos(R^1-R^2)\mathrm{ch}(R^1+R^2)], \end{aligned} \tag{5.9}$$

where we have used the variables $x^1 = x$, $x^3 = z$ instead of x^1, x^2 as it is in Theorem 3. These equations can be solved by the method of characteristics and then, extending the solution in such a way that it is constant along the direction $\boldsymbol{\delta}$, we obtain the functions $R^1(x)$, $R^2(x)$. The deviator of the stress tensor is given by the formula

$$S_{ij} = k(\alpha_i\beta_j+\alpha_j\beta_i),$$

and since

$$\sigma_{,R^1} = 2A\,\mathrm{th}(R^1+R^2), \qquad \sigma_{,R^2} = 2A\,\mathrm{th}(R^1+R^2),$$

then

$$\sigma = \sigma_0-2A\ln\mathrm{ch}(R^1+R^2)$$

is the expression for the pressure. Some difficulties arise in the neighbourhood of the curve $\varphi = R^1-R^2 = 0$ on the hodograph, where the vector $\boldsymbol{\delta}$ lies in x^1, x^3 plane and the solution cannot be extended. To avoid this in the neighbourhood $\varphi = 0$, we can exchange the plane x, z by another one, for instance y, z. We know the Cauchy data for a new system which has appeared so, if we know the solution on the common part of the planes.

6. Plane flows

The problem of plane flows which correspond to the case $C_1 = 0$, $C_2 = 0$ was excluded from the previous considerations in Sec. 5, where we have assumed $C_1\cdot C_2 \neq 0$. From the Eqs. (5.2), specialized for the plane flows, we get

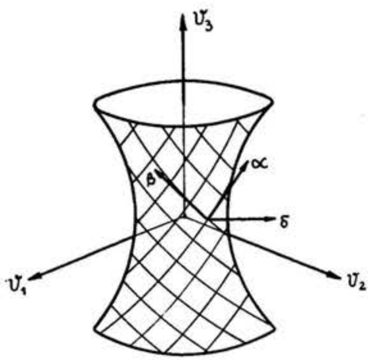

Fig. 1. Catenoid with the lines R^1 and $R^2 = \mathrm{const}$.

$$(6.1)\qquad \begin{aligned} d\mathbf{v} &= \nu_1\boldsymbol{\beta}dR^1+\nu_2\boldsymbol{\alpha}dR^2,\\ d\boldsymbol{\alpha} &= \frac{1}{2k}(\sigma_{,R^1}dR^1-\sigma_{,R^2}dR^2)\boldsymbol{\beta},\\ d\boldsymbol{\beta} &= \frac{1}{2k}(\sigma_{,R^1}dR^1-\sigma_{,R^2}dR^2)\boldsymbol{\alpha}, \end{aligned}$$

and from the integrability condition (5.3),

$$\nu_{1,R^2} = \frac{1}{2k}\nu_2\sigma_{,R^1},$$

$$\nu_{2,R^1} = \frac{1}{2k}\nu_1\sigma_{,R^2},$$

$$\sigma_{,R^1R^2} = 0.$$

The function $\sigma = \varphi_1(R^1)+\varphi_2(R^2)$, where φ_1, φ are arbitrary differentiable functions, is the general solution of the last equation. Taking advantage of this fact we can simplify the Eqs. (6.1) by introducing the new function $\psi = \frac{1}{2k}[\varphi_1(R^1)-\varphi_2(R^2)]$. Then the last two Eqs. (6.1) take the form

$$d\boldsymbol{\alpha} = \boldsymbol{\beta}d\psi,$$

$$d\boldsymbol{\beta} = \boldsymbol{\alpha}d\psi,$$

and they describe the rotation parametrized by the angle ψ of the orthogonal basis $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$.

Hence we can assume the expressions

$$\boldsymbol{\alpha} = (\cos\psi, \sin\psi), \quad \boldsymbol{\beta} = (-\sin\psi, \cos\psi)$$

as their solution. The first Eq. (6.1) is equivalent to the linear hyperbolic system:

$$\begin{aligned} \boldsymbol{\alpha} \cdot \mathbf{v}_{,R^1} &= 0, \\ \boldsymbol{\beta} \cdot \mathbf{v}_{,R^2} &= 0. \end{aligned} \tag{6.2}$$

The Pfaff Eqs. (2.6) which are now:

$$dR^1 = \xi^1 \alpha_i dx^i, \quad dx^1 \wedge dx^2 \neq 0,$$

$$dR^2 = \xi^2 \beta_i dx^i, \quad i = 1, 2,$$

we transform into the hyperbolic quasi-linear system

$$\begin{aligned} \beta_i R^1_{,i} &= 0, \\ \alpha_i R^2_{,i} &= 0, \end{aligned} \quad i = 1, 2. \tag{6.3}$$

The Eqs. (6.2), (6.3) together with the relations $\psi = \frac{1}{2k}[\varphi_1(R^1) - \varphi_2(R^2)]$ describe plane flows. In the domains where $\varphi_1(R^1)$, $\varphi_2(R^2)$ are exactly monotonic, we can put $\varphi_1(R^1) \equiv R^1$, $\varphi_2(R^2) \equiv R^2$ using appropriate transformation: $R^{1'} = \theta^1(R^1)$, $R^{2'} = \theta^2(R^2)$. Then we get the following system:

$$\text{(a)} \quad \begin{cases} v_{1,R^1} = -v_{2,R^1} \operatorname{tg}\psi, \\ v_{2,R^2} = v_{2,R^2} \operatorname{ctg}\psi, \end{cases}$$

$$\text{(b)} \quad \begin{cases} R^1_{,x} = R^1_{,y} \operatorname{ctg}\psi, \\ R^2_{,x} = -R^1_{,y} \operatorname{tg}\psi, \end{cases}$$

which is the system of equations describing the plane nondegenerate flows. The more detailed analysis of plane flow is given in [3].

Appendix

A. For arbitrary 1 — forms $\lambda^1, \ldots, \lambda^k$ in n-dimensional space we have:

1) $\lambda^1 \wedge \lambda^2 = -\lambda^2 \wedge \lambda^1$ which implies that $\lambda \wedge \lambda = 0$,

2) $\lambda^1 \wedge \ldots \wedge \lambda^k$ may be identified with antisymmetric product $\lambda^1_{[\nu_1} \lambda^2_{\nu_2} \ldots \lambda_{\nu_k]}$ where [ ] denotes antisymmetrization in the indices $\nu_1 \ldots \nu_k$ and therefore for $k = n$ we simply have:

$$\lambda^1 \wedge \ldots \wedge \lambda^n = \det|\lambda^1, \ldots, \lambda^n|$$

3) For the exterior derivative we have: a) $df = f_{,\nu} dx^\nu$ if f is a function (0 — form) and b) $d\lambda = \lambda_{\nu,\mu} dx^\mu \wedge dx^\nu = \frac{1}{2} \lambda_{[\nu,\mu]} dx^\mu \wedge dx^\nu$ if λ is a 1-form. Therefore we can identify $d\lambda$ with the antisymmetric matrix $\frac{1}{2} \lambda_{[\mu,\nu]}$.

B. As we have seen in Sec. 5, the conditions (2.7) were satisfied automatically for quasi-static case $\varrho = 0$. Unfortunately, in general case $\varrho \neq 0$, there is no possibility to fulfil these equations.

In fact, we have for example

$$\lambda^1 \wedge \lambda^2 \wedge \lambda^1_{,R^2} = \begin{bmatrix} -\mathbf{v}\cdot\boldsymbol{\alpha} \\ \boldsymbol{\alpha} \end{bmatrix} \wedge \begin{bmatrix} -\mathbf{v}\cdot\boldsymbol{\beta} \\ \boldsymbol{\beta} \end{bmatrix} \wedge \begin{bmatrix} -\nu_2\boldsymbol{\alpha}^2 - \mu_A\mathbf{v}\cdot\boldsymbol{\beta} \\ -\mu_A\boldsymbol{\beta} \end{bmatrix}.$$

Denote by e_1, e_2, e_3, e_4 — the versors of the axes (t, x^1, x^2, x^3).

By identification of the last exterior product with the following determinant (which has the same symmetry properties):

$$\det\begin{vmatrix} \mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ -\mathbf{v}\cdot\boldsymbol{\alpha} & \alpha_1 & \alpha_2 & \alpha_3 \\ -\mathbf{v}\cdot\boldsymbol{\beta} & \beta_1 & \beta_2 & \beta_3 \\ -\nu_2 & 0 & 0 & 0 \end{vmatrix} = \nu_2\boldsymbol{\alpha}\times\boldsymbol{\beta} = \nu_2\boldsymbol{\delta},$$

we can see that the only possibility $\nu_2 = 0$ leads to degeneracy of the hodograph surface.

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