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Flow over an oscillating porous plate

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IN THE PRESENT work damped oscillatory motion of a porous rigid plate in the fluid of infinite extent has been studied by using the Laplace transform technique.

W pracy rozważany jest wytłumiony oscylujący ruch porowatej sztywnej płyty w cieczy o nieskończonej rozciągłości przy użyciu techniki transformacyjnej Laplace'a.

В работе рассматривается затухающее осциллирующее движение пористой жесткой плиты в неограниченной жидкости при использовании техники преобразования Лапласа.

1. Introduction

NICOLL, *et al* [1] have shown that analysis of laminar motion of fluid near an oscillating porous infinite plane is important because the Navier-Stokes equation yields an exact solution, and the decay of the amplitude of the oscillations with distance from the surface, and the effect of mass transfer on this decay gives a quantitative basis from which the effect of mass transfer on the turbulent boundary layer can be determined. This paper also generalized van DRIEST'S [2] hypothesis that, if the effect of viscous damping on mixing-length distribution is also taken into account, then it is possible to obtain the correct form of expression for velocity profiles. This generalization is for steady state flow conditions only.

The problem of oscillatory flow of an infinite fluid near a porous plate has a practical application to Fourdrinier paper machine. DEBLER and MONTGOMERY [3] have given the analysis of the flow over an oscillating plate with suction or with an intermediate film. In this analysis, the initial condition that the plane is at rest at $t = 0$ is not satisfied. This analysis is valid for suction and moderate values of blowing.

In the present work, damped oscillatory motion of a porous rigid plane in an infinite viscous fluid, with suction or blowing, is considered. It is shown that restrictions imposed in the DEBLER and MONTGOMERY [3] work are not valid and present analysis can be applied with any suction or blowing value in order to get the same results. The particular cases of the present analysis are in agreement with WATSON [4].

2. Basic equations

Let x and y denote the space coordinates measured parallel and normal to a porous plane which is initially ($t = 0$) at rest. It is further assumed that the fluid velocity components in the direction of x and y be u and v, respectively. The fluid is assumed to be incompressible and homogeneous. The fluid above the porous plane is taken to be infinite in all directions and motion takes place in a direction parallel to x-axis. The oscillating plane is taken to be smooth and "no slip" condition exists at the surface of the plane.

Due to porosity of the plane there exists a constant velocity V_0 in a direction normal to the plane. The Navier-Stokes equation expressing the conservation of momentum in x-direction can be written as

$$\frac{\partial u}{\partial t}+V_0\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}, \tag{2.1}$$

where ν is the kinematic viscosity. The equation of continuity is satisfied identically. The plane is assumed to perform exponentially decaying oscillations. The initial and boundary conditions for the problem are

$$\begin{aligned} u(y,0^-) &= 0,\\ u(0,t) &= U_0\exp\{(-a+ib)t\}.,\\ u(\infty,t) &= 0, \end{aligned} \tag{2.2}$$

where U_0 is the velocity amplitude of the oscillating plane, a is the damping parameter and b denotes the frequency of the oscillation.

We define the following non-dimensional variables

$$\bar{u}=u/U_0,\quad \bar{y}=yU_0/\nu,\quad T=U_0^2t/\nu. \tag{2.3}$$

The Eqs (2.1) and (2.3) give

$$\frac{\partial \bar{u}}{\partial T}+\gamma\frac{\partial \bar{u}}{\partial \bar{y}}=\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}, \tag{2.4}$$

where the mass transfer parameter across the plane is

$$\gamma=V_0/U_0. \tag{2.5}$$

The corresponding initial and boundary conditions are

$$\begin{aligned} \bar{u}(\bar{y},0^-) &= 0,\\ \bar{u}(0,T) &= \exp\{(-k^2+i\omega)T\},\\ \bar{u}(\infty,T) &= 0, \end{aligned} \tag{2.6}$$

where $k^2=a\nu/U_0^2$ and $\omega=b\nu/U_0^2$.

Defining Laplace transform of $\bar{u}$ as

$$U=\int_0^\infty \bar{u}e^{-pT}dT,$$

and introducing the condition $\underset{T\to\infty}{Lt}\,\bar{u}e^{-pT}=0$, The Eq. (2.4) becomes

$$\frac{d^2U}{d\bar{y}^2}-\gamma\frac{\partial U}{\partial \bar{y}}-pU=0, \tag{2.7}$$

with the boundary conditions

$$U(\infty)=0,\quad U(0)=1/(p+k^2-i\omega). \tag{2.8}$$

The solution of the Eq. (2.7) with the boundary conditions given by Eq. (2.8) is

$$U = (1/k^2+p-i\omega)\exp\left(\frac{\gamma}{2}-\frac{1}{2}\sqrt{\gamma^2+4p}\right)\bar{y}. \tag{2.9}$$

Composite-product rule [5] is used to obtain the inverse of the Eq. (2.9) and thus velocity of plate $\bar{u}$ at time T is

$$\bar{u} = (\bar{y}/2\pi^{\frac{1}{2}})\, e^{(R\bar{y}+i\omega T-k^2T)} J(T, \alpha, \bar{y}/2), \tag{2.10}$$

where $R = \gamma/2$ and

$$J(T, \alpha, \bar{y}/2) = \frac{\pi^{\frac{1}{2}}}{\bar{y}/2}\left\{\bar{e}^{\alpha\bar{y}}\operatorname{erfc}\left(\frac{\bar{y}/2}{T^{\frac{1}{2}}}-\alpha T^{\frac{1}{2}}\right)+e^{\alpha\bar{y}}\operatorname{erfc}\left(\frac{\bar{y}/2}{T^{\frac{1}{2}}}+\alpha T^{\frac{1}{2}}\right)\right\}, \tag{2.11}$$

where α is a function of mass transfer parameter, γ defined by

$$\alpha = \pm\frac{\gamma}{2}(\beta+i\beta'), \tag{2.12}$$

where

$$\beta = (1/\sqrt{2})\left[\left\{\left(1-\frac{k^2}{R^2}\right)^2+\frac{\omega^2}{R^4}\right\}^{\frac{1}{2}}+\left(1-\frac{k^2}{R^2}\right)\right]^{\frac{1}{2}}, \tag{2.13}$$

and

$$\beta' = (1/\sqrt{2})\left[\left\{\left(1-\frac{k^2}{R^2}\right)^2+\frac{\omega^2}{R^4}\right\}^{\frac{1}{2}}-\left(1-\frac{k^2}{R^2}\right)\right]^{\frac{1}{2}}. \tag{2.14}$$

The sign to be taken in the Eq. (2.12) is that of V_0, namely positive for blowing and negative for suction.

The local velocity $\bar{u}$ given by the Eq. (2.10) is plotted at various values of ωT with R as a parameter and shown in Figs. 1, 2, 3. Values of ωT chosen for these plots are $\pi/4$, $\pi/2$

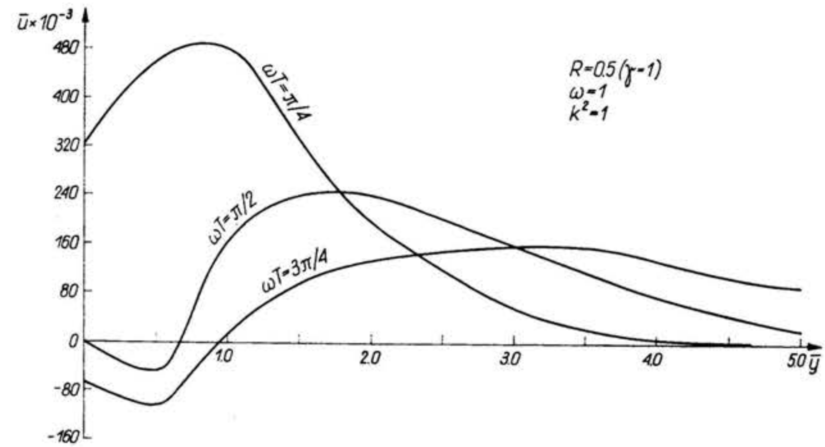

FIG. 1.

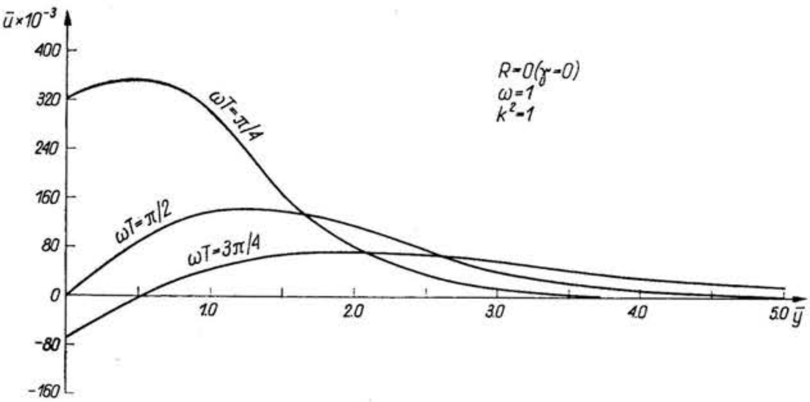

FIG. 2.

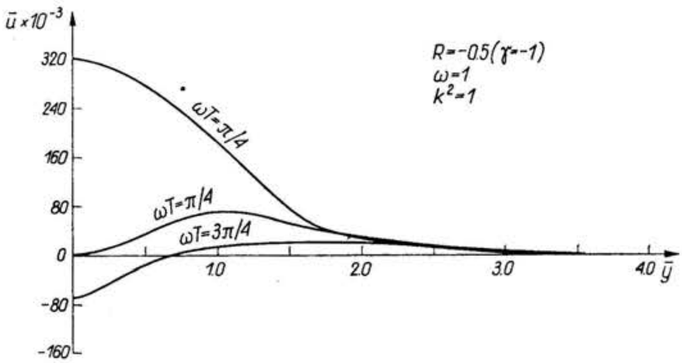

FIG. 3.

and $3\pi/4$ and $\omega = 1$. For blowing, the velocity envelope increases upto smaller values of $\bar{y}$ and then decays exponentially. For suction rapid decay starts from $\bar{y} = 0$ itself. This feature is as expected. The velocity decays more rapidly for low values of T and slowly for higher values of T. Also magnitude of $\bar{u}$ at $\bar{y} = 0$ is independent of R.

3. Node velocity

Node or phase velocity is given by

$$v^* = \frac{d}{dT}[\bar{y}(\bar{u}=0)]. \tag{3.1}$$

The Eqs. (2.10) and (3.1) give

$$v^* = (\bar{y}/2T)1/[\alpha\sqrt{\pi T}\,\mathrm{erfc}(\bar{y}/2\sqrt{T}-\alpha\sqrt{T})e^{(\bar{y}/2\sqrt{T}-\alpha\sqrt{T})^2}+1]. \tag{3.2}$$

A graph for v^* against $|R|$ is shown in Fig. 4. Node velocity gives the effects of mass transfer on the amplitude envelope. Effects of mass transfer produce some oscillations at low value of $|R|$ and then decays exponentially.

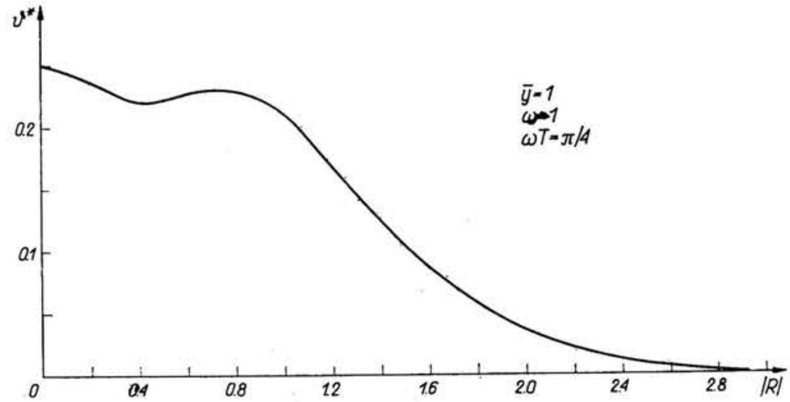

FIG. 4.

4. Drag coeffcient

The drag coefficient is defined by

$$\tau_w = \left[\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)\right]_{\bar{y}=0}. \tag{4.1}$$

The Eqs (2.10) and (4.1) yield the drag coefficient for time T as

$$\tau_w = -[(\pi T)^{-\frac{1}{2}} e^{-R^2 T} \mp Re^{-(k^2-i\omega)T} + e^{-(k^2-i\omega)T}(R^2-k^2+i\omega)^{\frac{1}{2}} \times \times \operatorname{erf}(R^2-k^2+i\omega)^{\frac{1}{2}} T^{\frac{1}{2}}]. \tag{4.2}$$

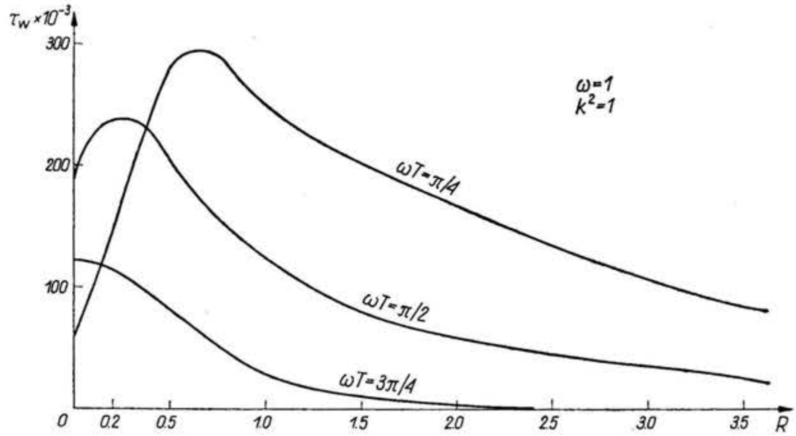

FIG. 5.

The drag coefficient obtained from Eq. (4.2) is valid for $T \neq 0$. Negative sign of R in this equation is for blowing and positive for suction. The drag coefficient, τ_w is plotted for various values of R with T as a parameter. For low values of R, τ_w, increases and then decreases exponentially (Fig. 5). The decay rate is rapid for rapid oscillations.

Figure 6 shows the variation of τ_ω for suction. It is noticed that values of τ_ω for $\omega T = \pi/2$ for suction and for blowing are same. The drag coefficient at $\omega T = \pi/4$ is negative and might cause back flow of fluid. For $\omega T = 3\pi/4$, τ_ω increases exponentially.

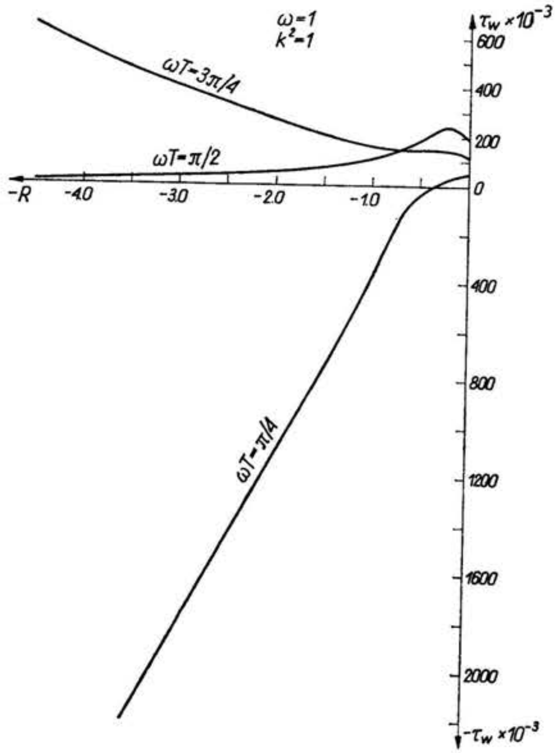

FIG. 6.

When $k^2 \neq R^2$ and $\omega \gg R^2-k^2$, real part of the Eq. (4.2) can be expressed as

$$\tau_w = -e^{-k^2T}B(\omega, \omega T, R, k, T)\sin[\omega T+\alpha(\omega, \omega T, R, k, T)], \tag{4.3}$$

where

$$B(\omega, \omega T, R, k, T) = \{[-(2\omega)^{\frac{1}{2}}(\omega T)]^2+[(2\omega)^{\frac{1}{2}}S(\omega T)\mp R+Q]^2\}^{\frac{1}{2}},$$

and

$$\alpha(\omega, \omega T, R, k, T) = \tan^{-1}\frac{(2\omega)^{\frac{1}{2}}S(\omega T)\mp R+Q}{-2(\omega)^{\frac{1}{2}}C(\omega T)}, \tag{4.4}$$

and

$$Q = (\pi T)^{-\frac{1}{2}}e^{-(R^2-k^2)T}\sec\omega T,$$

where C and S are Fresnels integrals.

For large values of T, the Eq. (4.4) simplifies to

$$B(\omega, R) = [\omega+(2\omega)^{\frac{1}{2}}(\mp R)+R^2]^{\frac{1}{2}},$$

and

$$\alpha(\omega, R) = \pi/2+\tan^{-1}\frac{(\omega/2)^{\frac{1}{2}}}{(\omega/2)^{\frac{1}{2}}\mp(R)^{\frac{1}{2}}}. \tag{4.5}$$

For $R = 1$, the Eq. (4.5) reduces to the results by WATSON [4] and also shows that τ_w leads ωT to $\pi/2$. The different values of phase angle in the present work are due to the motion of the plane in infinite stationary fluid.

For zero mass transfer or for solid plate ($R = 0$), the Eq. (4.5) gives the results obtained by STUART [6]. The phase between ω and ωT is $3\pi/4$.

For suction or blowing, the Eq. (4.3) gives τ_w as

$$\tau_w = Ae^{i(\omega T+\theta)}, \tag{4.6}$$

where

$$A = -e^{-k^2T}[\omega+2\omega(\mp R)(\omega/2)^{\frac{1}{2}}+R^2]^{\frac{1}{2}},$$

and

$$\theta = \tan^{-1}\frac{(\omega/2)^{\frac{1}{2}}}{(\omega/2)^{\frac{1}{2}}\mp R},$$

the alternate sign being opposite to that of V_0. For zero mass transfer, real part of the Eq. (4.6) is

$$\tau_w = -e^{-k^2T}\sqrt{\omega}\cos(\omega T+\pi/4).$$

Putting $\sqrt{\omega} = \sqrt{b\nu}/U_0$,

$$\tau_w = e^{-k^2T}(\sqrt{b\nu}/U_0)\sin(\omega T-\pi/4). \tag{4.7}$$

The magnitude of drag coefficient is $e^{-k^2T}\sqrt{b\omega}/U_0$ and phase difference between shear and velocity is $\pi/4$. For $k^2 = 0$, the Eq. (4.7) reduces to the result of STOKES.

5. Power

The power input to the fluid per cycle is

$$P = -\int\limits_0^{2\pi/\omega} \bar{u}(0, T)(\tau_w)_{\bar{y}=0}dT. \tag{5.1}$$

Substituting values of $\bar{u}(0, T)$ from the Eq. (2.6) and $(\tau_w)_{\bar{y}=0}$ from the Eq. (4.2), the expression for P is

$$P = -(\sqrt{(R^2+k^2-i\omega)}/\sqrt{\pi})\left(\operatorname{erf}\sqrt{\left\{\frac{2\pi}{\omega}(R^2+k^2-i\omega)\right\}}+\frac{(\mp R)}{2(k^2-i\omega)}\right.$$
$$\left.-\frac{e^{-(4\pi/\omega)(k^2-i\omega)}}{2(k^2-i\omega)}\left[\mp R+\sqrt{(R^2-k^2+i\omega)}\left(\operatorname{erf}\sqrt{\left\{\frac{2\pi}{\omega}(R^2-k^2+i\omega)\right\}}\right]\right). \tag{5.2}$$

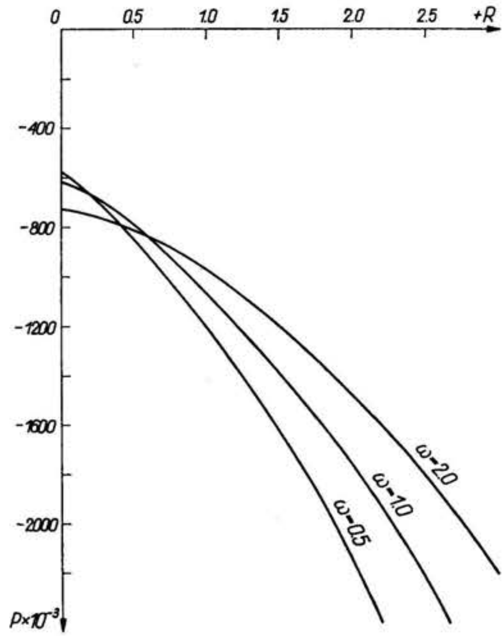

FIG. 7.

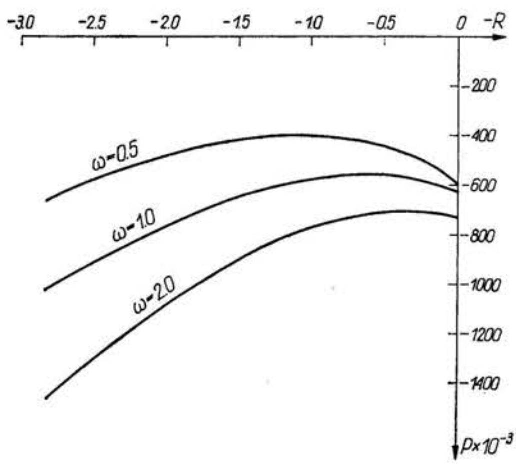

FIG. 8.

The Eq. (5.2) is plotted for various values of R taking ω as parameter. Figure 7 shows the variation of power for blowing and Fig. 8 for suction. The power input increases with ω for suction but decreases for blowing. P also increases exponentially with R but the rate of increase is slower in case of suction than for blowing.

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