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Regular simple wave interactions

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IN THE PAPER the method of simple wave interactions for non-elliptic first-order systems with arbitrary number of independent variables is described. This method is a generalization of the method of Riemann invariants applied in a theory of supersonic stationary plane potential and nonstationary one-dimensional gas flows.

W pracy omówiono metodę współdziałania fal prostych dla nieeliptycznych układów pierwszego rzędu w przypadku dowolnej ilości zmiennych niezależnych. Metoda ta jest uogólnieniem metody inwariantów Riemanna stosowanej w teorii naddźwiękowych, stacjonarnych, płaskich, potencjalnych oraz niestacjonarnych, jednowymiarowych przepływów gazu.

В работе обсужден метод взаимодействия простых волн для неэллиптических систем первого порядка в случае произвольного количества независимых переменных. Метод является обобщением метода инвариантов Римана, который применяется в теории плоских, стационарных, сверхзвуковых, потенциальных, а также одномерных, нестационарных течений газа.

Notations

R^n	n-dimensional Euclidean space of points $x = (x^1, \dots, x^n)$, independent variables of the set of Eqs. (1.1),
H^l	l-dimensional Euclidean space of points $u = (u^1, \dots, u^l)$, dependent variables of the set of Eqs. (1.1),
$H^l \otimes R^n$	the tensor product of H^l and R^n,
$R^n \ominus A$	the space of all vectors $y \in R^n$ satisfying $(y, a) = 0$ for each $a \in A$, where A a linear subspace of R^n,
$\lambda = (\lambda_1, \dots, \lambda^n)$	characteristic vectors in R^n of the set of Eqs. (1.1), see [1],
$\gamma = (\gamma^1, \dots, \gamma^l)$	characteristic vectors in H^l of the set of Eqs. (1.1), see [1],
$\lambda \rightleftharpoons \gamma$	knotted characteristic vectors λ and γ, cf. [1],
Γ	characteristic curve in H^l of the set of Eqs. (1.1), see [1],
$r\Vert a^i_j \Vert$	order of the matrix $\Vert a^i_j \Vert$,
$T_p(M)$	space, tangent to the manifold M at the point $p \in M$,
$N_p(M)$	space, normal to the manifold M at the point $p \in M$,
$[\underset{1}{a}, \dots, \underset{k}{a}]$	space, spaned by the vectors $\underset{1}{a}, \dots, \underset{k}{a}$,
$[\underset{1}{b} \vert \dots \vert \underset{k}{b}]$	space spaned by linearly independent vectors $\underset{k}{b}, \dots, \underset{1}{b}$,
$\dfrac{\partial f(x)}{\partial a} = f_a(x) = a^i f_{x_i}$	where $f(x)$ is a function and $a = (a^1, \dots, a^n)$ an arbitrary vector,
$u_a(x) = (u^1_a(x), \dots, u^l_a(x))$	where $u(x) = (u^1(x), \dots, u^l(x))$ is a maping R^n into H^l,
$(a, b) = a^1 b^1 + \dots + a^n b^u$	scalar product of the vectors $a = (a^1, \dots, a^n)$ and $b = (b^1, \dots, b^n)$.

1. Introduction

THE OBJECT of the present paper is to discuss the properties of the solutions of the set of equations

$$a_j^{sl}(u^1, \ldots, u^l)u^j_{x^i} = 0, \quad s, j = 1, \ldots, l; \quad i = 1, \ldots, n, \tag{1.1}$$

representing the interaction of simple waves and to devise methods for obtaining such solutions. This will be a generalization of the interaction method for two independent variables discussed in Ref. [1] and [13], In the case of two independent variables this method is the principal tool in the theory of plane potential and stationary flows and one-dimensional and nonstationary gas flows (see [10]).

It will also be shown how to solve the problem of ω-property of a solution of the set of Eqs. (1.1) using the method of simple wave interaction. To define the notion of ω-property, let us consider a region $G \subset R$ split up into two regions D and $\underset{\omega}{D}$ (Fig. 1) by

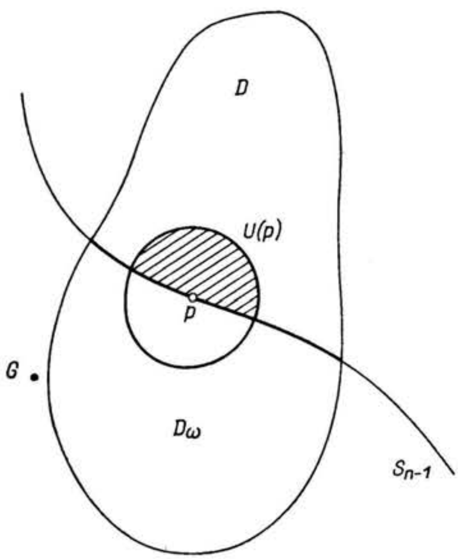

FIG. 1.

the manifold S_{n-1}. Let us consider the solution satisfying the conditions

$$u(x) \in C(G) \cap C^1(\overline{D}_\omega) \cap C^1(\overline{D}), \quad r\|u^j_{x^i}(x)\|_{x \in \overline{D}_\omega} = \omega. \tag{1.2}$$

It is said that the solution $u(x)$ satisfying the conditions (1.2) has the ω-property at the point $p \in S_{n-1}$ if there exist the manifolds $\underline{M}$, $\overline{M} \subset H^l$ such that in a certain neighbourhood $U(p) \subset R^n$ of the point p(Fig. 1) we have

$$\underline{M} \subset u(\overline{D} \cap U(p)) \subset \overline{M} \tag{1.3}$$

and

$$\omega - 1 \leqslant \dim \underline{M}, \quad \dim \overline{M} \leqslant \omega + 1. \tag{1.4}$$

A solution $u(x)$ satisfying the conditions (1.2) will be said to the have the ω-property if there exists in any neighbourhood $V \subset S_{n-1}$ a $p \in V$, in which $u(x)$ has the ω-property.

From the condition (1.3) it follows that

$$T_{u(x)}(\underline{M}) \subset [u_{x^1}(x), \ldots, u_{x^n}(x)] \subset T_{u(x)}(\overline{M}),$$

for $x \in \overline{D} \cap U(p)$, where $T_u(M)$ denotes the space tangent to the manifold M at the point $u \in M$, therefore we have the inequality

$$\omega - 1 \leqslant r\|u^j_{x^i}(x)\| \leqslant \omega + 1.$$

A solution $u = u(x)$, $x \in D$ will be said to be of order k if, for $x \in D$, $r\|u^j_{x^i}(x)\| = k$. Solutions of order k are also referred to as k-fold waves. For solutions of the first and second order the terms of simple wave (see [1]) and double wave are also used.

The problem of ω-property was discussed for the set of Eqs. (1.1) and two independent variables, $n = 2$, in Ref. [1], from the point of view of simple wave interaction. The present paper is a continuation of that paper. Without additional discussion we shall use the same notations and the information contained in the introduction and the first part of Ref. [1], concerning simple waves.

Similarly to the case of two independent variables (see [1]) we shall introduce in the general case of the set of Eqs. (1.1) three types of simple waves interaction.

A solution $u(x)$ represents a simple wave interaction if it is of the C^1 class and is constructed of integral elements having the form

$$L^j_i = \sum_{\sigma=1}^{q} \alpha_\sigma \underset{\sigma}{\gamma}^j \overset{\sigma}{\lambda}_i \tag{1.5}$$

where $\underset{\sigma}{\gamma} \rightleftharpoons \overset{\sigma}{\lambda}$, $q < +\infty$. The class of such solutions will be denoted by F.

Let $\mathfrak{G}_r$, $n \geqslant r \geqslant 1$ denote an r-dimensional manifold in H^l satisfying the condition

$$T_u(\mathfrak{G}_r) = [\underset{1}{\gamma}(u)| \ldots |\underset{r}{\gamma}(u)],$$

where $\underset{\sigma}{\gamma}$ denotes characteristic vectors in H^l. In addition it will be assumed that there exists for $u \in \mathfrak{G}_r$ a set of characteristic vectors $\overset{1}{\lambda}(u), \ldots, \overset{r}{\lambda}(u)$ such that each $k \leqslant n$ of them are linearly independent and $\underset{i}{\gamma}(u) \rightleftharpoons \overset{i}{\lambda}(u)$. In this way every manifold $\mathfrak{G}_r$ prescribes a set of integral elements in the form

$$L^i_j(u) = \sum_{\sigma=1}^{r} \alpha_\sigma \underset{\sigma}{\gamma}^j(u) \lambda^\sigma_i(u), \quad u \in \mathfrak{G}_r. \tag{1.6}$$

It may happen that $T_u(\mathfrak{G}_r) = [\underset{1}{\gamma}(u), \ldots, \underset{m}{\gamma}(u)]$, $m > r$, where the characteristic vectors $\underset{i}{\gamma}$ are linearly independent by pairs. Then, from the vectors $\underset{i}{\gamma}$, we can select, in many different ways, r linearly independent. Each such choice will correspond to a different set of integral elements (1.6).

A solution $u(x)$, $x \in D$ will be referred to as an interaction of r independent simple waves, if $u \in C^1(D)$ and if there exists a manifold $\mathfrak{G}_r$, such that $u(D) \subset \mathfrak{G}_r$ and $u(x)$ is constructed of integral elements of the form (1.6) A set of such solutions will be denoted, for a prescribed manifold, by $F(\mathfrak{G}_r)$. Solutions of the type of k-fold wave belonging to the classes F and $F(\mathfrak{G}_r)$ will be denoted by $F_k(k \leqslant n)$ and $F_k(\mathfrak{G}_r)$ $(k \leqslant r)$, respectively.

An interaction of the third type is a *regular interaction of simple waves*. By contrast with the case of $n = 2$ (see [1]) correct formulation and study of this notion requires somewhat more detailed considerations, in general. This problem will be discussed in Sec. 3.

We shall now prove teh following theorem (see [3]).

THEOREM 1. *If the set of Eqs.* (1.1) *is hyperbolic, all its solutions are of class F.*

The class F is empty for elliptic sets of equations. If the set (1.1) is non-elliptic, that is it has at least one family of characteristic vectors $\lambda(u) \rightleftharpoons \gamma(u)$ of class C^1, there exists an infinite number of solutions of the type F_1 (simple waves), therefore the class F is not empty.

By virtue of the Theorem 1 it may be said that solutions of the type F are hyperbolic solutions or hyperbolic waves of the system considered.

Proof of Theorem 1. The set of Eqs. (1.1) is $\xi = (\xi_1, \ldots, \xi_n)$ — hyperbolic (ξ is a non-characteristic vector) if the following conditions are satisfied:

1. For every $\alpha = (\alpha_1, \ldots, \alpha_n)$, $(\alpha, \xi) = 0$ the characteristic polynomial $P(\tau) = \det |A(\tau)|$, where $A_j^s(\tau) = a_j^{si}(\alpha_i + \tau\xi_i)$ has real roots only. They will be denoted by $\tau^1, \ldots, \tau^l$, $\tau^\sigma \leqslant \tau^\varrho$ for $\sigma < \varrho$.

2. Vectors $\mu = (\mu_1, \ldots, \mu_l)$ satisfying for a certain τ^σ the equation

$$\mu A(\tau^\sigma) = 0$$

determine an l-dimensional space.

The condition 2 is equivalent to the following condition. Vectors $\gamma = (\gamma^1, \ldots, \gamma^l)$ satisfying for a certain τ^σ the equation

$$A(\tau^\sigma)\gamma = 0$$

determine an l-dimensional space.

It can easily be verified that the vectors γ are characterictic vectors of the set of Eqs. (1.1) in H^l. $\lambda^\sigma = \alpha + \tau^\sigma\xi$ are chacteristic vectors in R^n, and $\lambda^\sigma \rightleftharpoons \underset{\sigma}{\gamma}$ if $A(\tau^\sigma)\underset{\sigma}{\gamma} = 0$. It is obvious that $H^l = [\underset{1}{\gamma}|\ldots|\underset{l}{\gamma}]$.

The hyperbolic system (1.1) has, therefore, the following simple integral elements (see [1]);

$$\underset{\sigma}{\gamma} \otimes \overset{\sigma}{\lambda} = \underset{\sigma}{\gamma^j}\lambda_i^\sigma = \underset{\sigma}{\gamma} \otimes \alpha + \tau^\sigma \underset{\sigma}{\gamma} \otimes \xi, \tag{1.7}$$

where $\sigma = 1, \ldots, l$, and α is any vector satisfying the condition $(\alpha, \xi) = 0$.

Let us denote by $\mathfrak{M} \subset H^l \otimes R^n$ an $l(n-1)$-dimensional space of all the integral elements of the set (1.1) If $\mathfrak{M}_1 \subset \mathfrak{M}$ denotes a space of integral elements in the form (1.5), it suffices for the Theorem 1 to be proved to show that $\mathfrak{M}_1 = \mathfrak{M}$ or, which is equivalent,

$$\dim \mathfrak{M}_1 \geqslant l(n-1). \tag{1.8}$$

However, $\underset{\sigma}{\gamma} \otimes \xi$ and $\underset{\sigma}{\gamma} \otimes \alpha$ are orthogonal in $H^l \otimes R^n$, and all the matrices $\underset{\sigma}{\gamma} \otimes \alpha$ determine a space $H^l \otimes A$, where $A = R^n \ominus [\xi]$, therefore $H^l \otimes A \subset \mathfrak{M}_1$. Hence (1.8) because $\dim H^l \otimes A = l(n-1)$. This ends the proof of Theorem 1.

Some of the results obtained in the present work were given in Refs. [2, 4] without proof. The relation between the notion of simple wave interactions and double hyperbolic waves considered by J. H. GIESE [5] was discussed in Ref. [6]. The works [7, 8] and [9] are devoted to the consideration, by E. CARTAN's method, of solutions of class F, assuming that the coefficients and solutions of the Eqs. (1.1) are analytic. In these works a number of interesting solutions of sets of equations of gas dynamics were found.

By analysing equations of gas dynamics the authors of [10] formulated a number of questions concerning solutions constant along planes of fixed dimension. An answer to these questions is contained among other things, in the present paper.

All the considerations of the present paper are of a local character. As regards the function and manifold considered it will be assumed tacitly that they are of class C^1, if there are no other remarks.

2. The R-property

In the present section we shall discuss certain properties of the solutions of the Eqs. (1.1) of the type $F(\mathfrak{G}_r)$, $r \leqslant n$. Let $\underset{1}{\gamma}(u), \ldots, \underset{r}{\gamma}(u)$; $\overset{1}{\lambda}(u), \ldots, \overset{r}{\lambda}(u)$ be vectors such that $T_u(\mathfrak{G}_r) = [\underset{1}{\gamma}(u)| \ldots |\underset{r}{\gamma}(u)]$, $\underset{i}{\gamma} \rightleftharpoons \overset{i}{\lambda}$. The symbol $\Gamma_i \subset \mathfrak{G}_r$ will denote the characteristic curves (see [1]) tangent to $\underset{i}{\gamma}(u)$. We shall introduce the following notations

$$\sum\nolimits_{n-r}(u) = [\underset{1}{\sigma}(u)|\ldots|\underset{n-r}{\sigma}(u)] = R^n \ominus [\overset{1}{\lambda}(u)| \ldots |\overset{r}{\lambda}(u)].$$

The plane in R^n, passing through the point x and tangent to $\sum_{n-r}(u)$ will be denoted by $\sum_{n-r}^x(u)$. Similarly: $L_r(u) = [\overset{1}{\lambda}(u)|\ldots|\overset{r}{\lambda}(u)]$, $L_r^x(u)$ a plane tangent to $L_r(u)$ passing through x. Let $\underset{i}{c}(u)$, $i = 1, \ldots, r$, denote a vector basis in $L_r(u)$ such that $(\underset{i}{c}(u), \overset{j}{\lambda}(u)) = \delta_i^j$. Let us set

$$P_{n-r+1}^i(u) = \left[\sum\nolimits_{n-j}(u)|\underset{i}{c}(u)\right].$$

We have the following lemma

LEMMA 1. *If* $u(x) \in F_r(\mathfrak{G}_r)$, $r \leqslant n$, *then*

$$u_a(x) = \alpha \underset{i}{\gamma}, \quad \alpha \neq 0$$

when, and only when, $a \in P_{n-r+1}^i(u(x))$, a $\notin \sum_{n-r}(u(x))$.

The proof of this lemma follows immediately from the definition of the class $F_r(\mathfrak{G}_r)$.

It will be shown that for $u \in F(\mathfrak{G}_r)$, $x \in D$, the planes $\sum_{n-r}^x(u(x))$ stratify(¹) the region D and

$$u(x) = \text{const} = u(\underset{0}{x}). \tag{2.1}$$

(¹) A family $\mathcal{R}$ of manifolds $M_k \subset \mathcal{R}$ of fixed dimensions k stratifies a region $D \subset R^n$ if: 1) through every point $p \in D$ passes exactly one manifold $M_k \in \mathcal{R}$, 2) the family $\mathcal{R}$ can be parametrized by $c^1, \ldots, c^{n-k}$ so that the manifold $M_k(c^1, \ldots, c^{n-k})$ in D can be written in the form $\Phi^\mu(x^1, \ldots, x^n) = c^\mu$, $\mu = 1, \ldots, n-k$. The notion of a family of manifolds stratifying a given manifold is defined in a similar manner.

for

$$x \in \sum\nolimits_{n-r}^{\overset{x}{0}} (u(\underset{0}{x})).$$

Indeed, for $\sigma(u) \in \sum_{n-r}(u)$ $(\sigma(u) \neq 0)$, $\sigma(u(x))$ is a vector field in D of class C^1. Let us denote by $x = x(s)$ the integral curves of that field, $dx/ds = \sigma(u(x))$. From the definition of the class $F(\mathfrak{G}_r)$ it follows that $u_\sigma = 0$, therefore $du(x(s))/ds = 0$. Thus, $u(x(s)) = \text{const}$ and the curve $x = x(s)$ is a straight line. Hence follows immediately the Eq. (2.1) We can also see that $\sum_{n-r}(u(x))$ stratify D.

Let us denote by $W(\mathfrak{G}_r)$ a class of mapings (not necessarily solutions) $v(x) \in C^1(D)$, $D \overset{v}{\to} \mathfrak{G}_r$ and $v(x)$ satisfies the condition (2.1).

If $r = 1$ and, therefore, $\mathfrak{G}_r$ is a characteristic curve $\Gamma \subset H$, the mapings $v(x) \in W(\Gamma)$ are solutions (constant or simple waves, see [1]). In general, $r > 1$, this is not the case. Our task is to give an answer to the question as to when the mapings $W(\mathfrak{G}_r)$ are solutions.

It will be said that a maping $v(x) \in W(\mathfrak{G}_r)$, $x \in D$ has the R-property in the region D if the following sets of Pfaff's forms

$$\text{(2.2j)} \qquad \overset{\mu}{\lambda}_i(v(x))dx^i = 0, \quad \mu = 1, i = 1, \ldots, j-1, j+1, \ldots, r,$$

$j = 1, \ldots, r$, are integrable in the region D.

We have the following simple lemma.

LEMMA 2. *A maping $v \in W(\mathfrak{G}_r)$, $r > 1$, has the R-property in the region D when and only when there exist r families of $n-r+1$ dimensional manifolds C^j_{n-r+1}, $j = 1, \ldots, r$, which stratify D and are such that*

$$N_x(C^j_{n-r+1}) = [\overset{1}{\lambda}(v(x))| \ldots |\overset{j-1}{\lambda}(v(x))||\overset{j+1}{\lambda}(v(x))| \ldots |\overset{r}{\lambda}(v(x))],$$

$T_x(C^j_{n-r+1}) = P^j_{n-r+1}(v(x))$. *The generators of the manifold C^j_{n-r+1} are the planes $\sum^x_{n-r}(v(p))$.*

Let us observe that in the case of $n = r = 2$ (the case considered in [1]), the planes $\sum^q_{n-r}(v(p))$ are reduced to points and each maping $v(x)$, $D \overset{v}{\to} \mathfrak{G}_2$ is of the class $W(\mathfrak{G}_2)$. Moreover, each maping has the R-property because for any j the set (2.2j) is reduced to a single form with two independent variables x^1, x^2. The manifold C^j_{n-r+1} are curves.

For $r = 1$, $\mathfrak{G}_1 = \Gamma$, the set (2.2) contains no form, it can therefore be assumed that each maping $v(x) \in W(\Gamma)$ has the R-property. Lemma 2 remains valid because $C^1_{n-1+1} = R^n$.

The result of Lemma 2 may be formulated in another manner.

Let us denote by $\mathscr{L}^p_r(v)$, for $p \in R^n$ and $v \in H^l$, any r-dimensional manifold in R^n such that $T_p(\mathscr{L}^p_r(v)) = L_r(v)$. To such manifold belongs, in particular, $L^p_r(v)$. Let $v(x) \in W(\mathfrak{G}_r)$, $x \in D$, and let $V \subset H^l$ denote the neighbourhood of the point $v(p)$, $p \in D$. If $u \in V$ and V is sufficiently small, the spaces

$$\text{(2.3)} \qquad E_i(u, p, x) = P^i_{n-r+1}(u) \cap T_x(\mathscr{L}^p_r(v(p))), \quad i = 1, \ldots, r,$$

are one-dimensional. Let $0 \neq c_i\ (u, p, x) \in E^i(u, p, x)$. Let $\underset{i}{C}(p) \subset \mathscr{L}^p_r(v(p))$ denote curves tangent to the vectors $\underset{i}{c}(v(x), p, x)$, $x \in \mathscr{L}^p_r(v(p))$.

Let us pass through every point $q \in \underset{i}{C}(p)$ a plane $\sum^q_{n-r}(v(q))$ (Fig. 2) and consider the $(n-r+1)$-dimensional manifold thus formed

$$C^i_{n-r+1}(p) = \bigcup_{q \in \underset{i}{C}(p)} \sum\nolimits^q_{n-r}(v(q)).$$

The manifolds $C^i_{n-r+1}(p)$ stratify the neighbourhood of the manifold $\mathscr{L}^p_r(v(p))$. The manifold $C^i_{n-r+1}(p)$ depend, in general, in an essential manner on p. Let us formulate now the following obvious lemma.

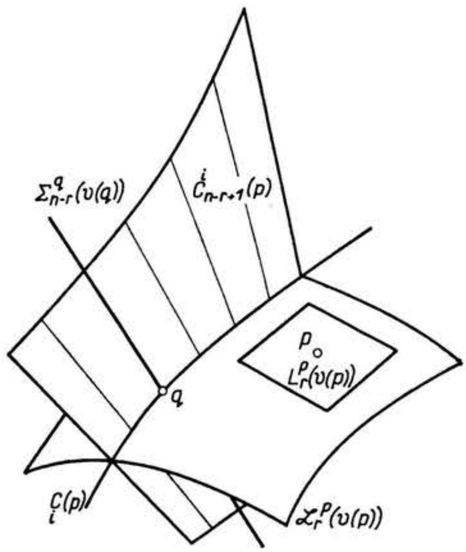

FIG. 2.

LEMMA 3. *A maping $v(x) \in W(\mathfrak{G}_r)$, $x \in D$, $r > 1$ has the R-property if, and only if, $C^i_{n-r+1}(p)$ are developable for a certain $p \in D$. Then $C^i_{n-r+1}(p)$ are independent of p.*

We have the following theorem

THEOREM 2. *If $u \in F_r(\mathfrak{G}_r)$, then $u(x)$ has the R-property, C^i_{n-r+1} are developable characteristic manifolds and*

$$u(C^i_{n-r+1}) \subset \Gamma_i \subset \mathfrak{G}_r, \quad i = 1, \dots, r. \tag{2.4}$$

Proof. For the maping $u = u(x)$ and any point $p \in D$ let us consider the manifold $\mathscr{L}^p_r(u(p))$, the curves $\underset{i}{C}(p)$ and the manifold $C^i_{n-r+1}(p)$. From the definition of the class $F_r(\mathfrak{G}_r)$ it follows that for $\underset{i}{d} = \underset{i}{c}(u(x), p, x)$ we have $u_{\underset{i}{d}} = \alpha \underset{i}{\gamma}$, $\alpha \neq 0$. Therefore $u(\underset{i}{C}(p)) \subset \Gamma_i$ and

$$u(C^i_{n-r+1}(p)) \subset \Gamma_i. \tag{2.5}$$

Since $u(x) \in W(\mathfrak{G}_r)$, then, by virtue of Lemma 3 it suffices to show that $C^i_{n-r+1}(p)$ are developable. From (2.5) and the construction of the manifold $C^i_{n-r+1}(p)$ it follows that

$$T_x(C^i_{n-r+1}) \underset{x \in \sum^q_{n-r}(u(q))}{=} \left[\sum\nolimits^q_{n-r}(u(q)) | a(x)\right],$$

where $ua(x) = \alpha \underset{i}{\gamma}$, $\alpha \neq 0$. By virtue of Lemma 1

$$a(x) \in P^i_{n-r+1}(u(x)) = \left[\sum\nolimits^q_{n-r}(u(x)) | \underset{i}{c}(u(x))\right], a(x) \notin \sum\nolimits^x_{n-r}(u(x)),$$

therefore, as a result of the fact that $u(x) = \text{const} = u(q)$ for $x \in \sum^q_{n-r}(u(q))$, we find

$$T_x(C^i_{n-r+1}) \underset{x \in \sum^q_{n-r}(u(q))}{=} \left[\sum\nolimits^q_{n-r}(u(q)) | c(\underset{i}{u}(q))\right]$$

thus ending the proof of Theorem 2.

Let us now prove the following theorem.

THEOREM 3. *$u(x) \in F(\mathfrak{G}_r)$, if $u \in C^1$ and*

A. $$u(\underset{i}{C}(p)) \subset \Gamma_i \subset \mathfrak{G}_r, \quad i = 1, \ldots, r,$$

for a certain p and all the curves $\underset{i}{C}(p) \subset \mathscr{L}^p_r(u(p))$.

B. $$u(x) = \text{const} = u(\underset{0}{x}), \quad \text{for} \quad x \in \sum\nolimits^{\overset{x}{0}}_{n-r} (u(\underset{0}{x}))$$

C. *$u(x)$ has the R-property.*

The Theorems 2 and 3 is a direct generalization of the Theorem 4 (or Theorem 3) of [1] to the case of many independent variables. Indeed, for $n = r = 1$, and the Theorems 2 and 3 can be expressed in the form of the following corollary.

COROLLARY 1. *If $n = 2$, we have $u(x^1, x^2) \in F(\mathfrak{G}_2)$ when and only when*

$$u(\underset{i}{C}) \subset \Gamma_i \subset \mathfrak{G}_2, \quad i = 1, 2.$$

for any characteristic curve $\underset{i}{C} \subset R^2$ normal to $\overset{i}{\lambda}(u(x))$, where $\overset{i}{\lambda}(u) \rightleftharpoons (\overset{i}{\gamma})(u)$, $T_u(\mathfrak{G}_2) =$ $= [\underset{1}{\gamma}(u) | \underset{2}{\gamma}(u)]$.

P r o o f of the Theorem 3. From the condition C it follows that for the manifold C^i_{n-r+1} we have

$$T_x(C^i_{n-r+1}) = [\underset{1}{\sigma}(u(x)) | \ldots | \underset{n-r}{\sigma}(u(x)) | \underset{i}{c}(u(x))]$$

and from the conditions A and B it follows that $u(C^i_{n-r+1}) \subset \Gamma_i$, $i = 1, \ldots, r$ and $u_{\underset{m}{\sigma}} = 0$, $m = 1, \ldots, n-r$, therefore $u_{\underset{i}{c}}(x) = \overset{i}{\beta}(x) \underset{i}{\gamma}(u(x))$, where $\overset{i}{\beta}(x)$ are continuous functions and

$$\underset{i}{c}(u) \in L_r(u) \ominus [\overset{1}{\lambda}(u) | \ldots | \overset{i-1}{\lambda}(u) | \overset{i+1}{\lambda}(u) | \ldots | \overset{r}{\lambda}(u)].$$

Hence, immediately,

$$u^j_{x^i}(x) = \sum_{\sigma=1}^{r} \overset{\sigma}{\beta} \underset{\sigma}{\gamma^j}(u(x) \lambda^\sigma_i(u(x))$$

which ends the proof of Theorem 3.

Theorem 3 reduces the problem of construction of solutions $u(x) \in F(\mathfrak{G}_r)$ to the construction of the maping of the manifold $\mathscr{L}_r^p$ in $\mathfrak{G}_r$, which satisfies A. It is not convenient to verify the condition C, however, therefore we introduce the following definition.

A manifold $\mathfrak{G}_r$ has the R-property if each representation $v(x) \in W(\mathfrak{G}_r)$ satisfying the conditions A and B has the R-property.

Thus for a manifold $\mathfrak{G}_r$ having the R-property, the verification of the condition C is not necessary.

It can also be easily verified that the Theorem 3 can also be given the following equivalent form:

THEOREM 3_A. *$u(x) \in F(\mathfrak{G}_r)$, if $u \in C^1$, $u(x) \in W(\mathfrak{G}_r)$, $\mathfrak{G}_r$ has the R-property and for the manifold C^i_{n-r+1} we hawe the condition*

$$u(C^i_{n-r+1}) \subset \Gamma_i \subset \mathfrak{G}_r, \quad i = 1, \ldots, r.$$

We have the following theorems:

THEOREM 4. *A manifold $\mathfrak{G}_r$, $r > 1$, has the R-property if*

$$\frac{\partial \overset{\mu}{\lambda}(u)}{\partial \underset{\nu}{\gamma}(u)} \in [\overset{1}{\lambda}(u)| \ldots |\overset{r}{\lambda}(u)],$$

$$u \in \mathfrak{G}_r, \quad \mu = 1, \ldots, r, \quad \nu = 1, \ldots, r, \quad \mu \neq \nu.$$

THEOREM 5. *A manifold $\mathfrak{G}_r$, $r > 1$ has the R-property, if the class $F_r(\mathfrak{G}_r)$ is not empty.*

Both theorems will be proved simultaneously. The proof will be done by considering the integrability of a set of forms (2.2j) for any $v(x) \subset W(\mathfrak{G}_r)$ satisfying the condition A.

For any point $\underset{0}{x} \in D$ let us introduce a new variable $t = (t^1, \ldots, t^n)$:

$$x = x(t) = \sum_{i=1}^{r} t^i \underset{i}{c}\big(v(\underset{0}{x})\big) + \sum_{j=1}^{n-r} t^{j+r} \sigma_j\big(v(\underset{0}{x})\big).$$

If by

$$\underset{i}{\overset{*}{c}}(t) = \big(\underset{i}{\overset{*1}{c}}(t), \ldots, \underset{i}{\overset{*n}{c}}(t)\big), \underset{j}{\overset{*}{\sigma}}(t) = (\underset{j}{\overset{*}{\sigma}}{}^1(t), \ldots, \underset{j}{\overset{*}{\sigma}}{}^n(t)),$$

we denote the coordinates of the vectors $\underset{i}{c}\big(v(x)\big)$ and $\underset{j}{\sigma}\big(v(x)\big)$ in the new set of coordinates it can easily be verified that the condition of integrability of the set of forms (2.2j) at the point $\underset{0}{x}$ can be expressed thus

$$\frac{\partial \underset{\omega}{\overset{*}{\sigma}}(t_0)}{\partial t^j} \in [\underset{j}{\overset{*}{c}}(t_0)|\underset{1}{\overset{*}{\sigma}}(t_0)| \ldots | \underset{n-r}{\overset{*}{\sigma}}(t_0)], \quad \omega = 1, \ldots, n-r,$$

where $x_0 = x(t_0)$. If now we return to the variables $x^1, \ldots, x^n$ and set $\underset{0}{x} = x$, then for any $v(x) \in W(\mathfrak{G}_r)$ we obtain the following necessary and sufficient condition of integrability of the set (system) of forms (2.2j)

$$(2.6) \qquad \frac{\partial \underset{\omega}{\sigma}\big(v(x)\big)}{\partial \underset{j}{c}} \in \big[\underset{j}{c}\big(v(x)|\underset{1}{\sigma}\big(v(x)\big)| \ldots | \underset{n-1}{\sigma}\big(v(x)\big)\big],$$

$$x \in D, \quad \omega = 1, \ldots, n-r.$$

If the maping $v(x)$ satisfies the condition A, then (2.6) follows from the condition

$$\frac{\partial \underset{\omega}{\sigma}(u)}{\partial \underset{j}{\gamma}(u)} \in \left[\underset{j}{c}(u)|\underset{1}{\sigma}(u)| \dots | \underset{n-1}{\partial} \sigma(u)\right], \tag{2.7}$$

$$u \in \mathfrak{G}_r, \omega = 1, \dots, n-r.$$

On differentiiating the equations $(\underset{\omega}{\sigma}(u), \lambda^\mu(u)) = 0$, $u \in \mathfrak{G}_r$, $\omega = 1, \dots, n-r$, $\mu = 1, \dots, r$ in the direction of the vector $\gamma(u)$ we infer immediately that the condition (2.7) can be expressed in the following equivalent form

$$\frac{\partial \overset{\mu}{\lambda}(u)}{\partial \underset{j}{\gamma}(u)} \in [\overset{1}{\lambda}(u)| \dots |\overset{r}{\lambda}(u)], \quad u \in \mathfrak{G}_r, \quad \mu = 1, \dots, r, \quad \mu \neq j.$$

This ends the proof of Theorem 4.

To demonstrate the Theorem 5 let us observe that if there exists at least one maping $v(x) \in F_r(\mathfrak{G}_r)$, $x \in D$, we have

$$\frac{\partial \underset{\omega}{\sigma}(v(x))}{\partial \underset{j}{c}(v(x))} = \overset{j}{\alpha}(x) \frac{\partial \underset{\omega}{\sigma}(v(x))}{\partial \underset{j}{\gamma}(v(x))}, \quad \overset{j}{\alpha} \neq 0. \tag{2.8}$$

Next, by virtue of the Theorem 2, $v(x)$ has the R-property, therefore the condition (2.6) is satisfied. If now by $\mathfrak{G}_r$ we denote a manifold $\mathfrak{G}_r = v(D)$, (2.7) follows from (2.8), which proves the Theorem 5.

3. Regular interaction of simple waves

We shall now be concerned with the notion of regular interaction of simple waves for any number of independent variables. In the case of $n = 2$ this notion was introduced in Ref. [1] by saying that a solution $u = u(x^1, x^2)$ is a regular interaction of simple waves, if $u \in F_2(\mathfrak{G}_2)$. Two-dimensional manifold $\mathfrak{G}_2$ were termed characteristic in the space H^l and denoted by the symbol $\mathfrak{H}_2$. It was also shown that the following theorem is valid.

COROLLARY 2. *A maping $u = u(x^1, x^2)$, $x \in D$ is, for the set of* Eqs. (1.1), $n = 2$, *a regular interaction of simple waves, if and only if $u(D) = \mathfrak{H}_2$ and, for any characteristic curve*

$$\Gamma \subset \mathfrak{H}_2 \subset H^l$$

in the neighbourhood of the characteristic curve

$$C = u^{-1}(\Gamma) \subset R^2$$

there exists a $v(x) \in F_1(\Gamma)$ (simple wave) such that

$$v(x) = u(x), \quad x \in C.$$

Let us introduce, for any n, the following definition of the r-dimensional characteristic manifold $\mathfrak{H}_r \subset H^l$, $r \leqslant n$.

A *manifold* $\mathfrak{G}_r$, $r \leqslant n$ *is the characteristic* $\mathfrak{H}_r$ if the following two conditions are satisfied.

1. $\mathfrak{G}_r$ is representable parametrically: $u = u(\mu^1, \ldots, \mu^r)$, so that $u_{\mu^i} = \underset{i}{\gamma}$.

$$T_{u(\mu)}(\mathfrak{G}_r) = [\underset{1}{\gamma}(u(\mu))| \ldots |\underset{r}{\gamma}(u(\mu))].$$

Thus, for every $k: 0 < k < r$, $\mathfrak{G}_r$ is stratified by $\binom{k}{n}$ different families of manifold $\overset{*}{\mathfrak{G}}_k \subset \mathfrak{G}_r$ such that $T(\overset{*}{\mathfrak{G}}_k) = [\gamma_{i_1}| \ldots |\gamma_{i_k}]$.

2. If $u(x) \in F_r(\mathfrak{G}_r)$, then, for every manifold $\overset{*}{\mathfrak{G}}_k \subset \mathfrak{G}_r$ in the neighbourhood of an $(n-r+k)$-dimensional characteristic manifold $C_{n-r+k} = u^{-1}(\overset{*}{\mathfrak{G}}_k)$, there exists a $v(x) \in \in F_k(\overset{*}{\mathfrak{G}}_k)$ such that

$$v(x) = u(x), \quad x \in C_{n-r+k}.$$

Let us observe that the condition 2 is always satisfied for $k = 1$. Indeed, if $u(x) \in F_r(\mathfrak{G}_r)$, $\overset{*}{\mathfrak{G}}_1 = \Gamma_l \subset \mathfrak{G}_r$, we have $u^{-1}(\overset{*}{\mathfrak{G}}_1) = C_{n-r+1}$. A simple wave such that $v(x) = u(x)$ for $x \in C_{n-r+1}$ can be defined on the basis of the Theorem 1 of Ref. [1], in the neighbourhood C_{n-r+1}, as follows

$$v(x) = \text{const} = u(\underset{0}{x})$$

for x belonging to a $r-1$-dimensional plane π such that $\underset{0}{x} \in \pi$ and

$$T(\pi) = [\underset{1}{c}(u(\underset{0}{x}))| \ldots |\underset{i-1}{c}(u(\underset{0}{x}))|\underset{i+1}{c}(u(\underset{0}{x}))| \ldots |\underset{r}{c}(u(\underset{0}{x}))].$$

We have the following theorem.

THEOREM 6. *A manifold* $\mathfrak{G}_r$ *is the characteristic manifold* $\mathfrak{H}_r$, *when and only when the condition 1 is satisfied and all the manifold* $\overset{*}{\mathfrak{G}}_k \subset \mathfrak{G}_r$, $1 < k < r$, *have the R-property.*

Before we proceed to prove this theorem let us discuss a number of inferences.

From the Theorems 4 and 6 it follows that $\mathfrak{G}_r$ is a characteristic manifold $\mathfrak{H}_r$ if, for $u \in \mathfrak{G}_r$,

$$\frac{\partial \overset{\mu}{\lambda}(u)}{\partial \underset{\nu}{\gamma}(u)} \in [\overset{\mu}{\lambda}(u)|\overset{\nu}{\lambda}(u)] \tag{3.1}$$

$$\mu, \nu = 1, \ldots, r; \quad \mu \neq \nu.$$

In Refs. [7, 8 and 9] it was shown that the condition (3.1) is equivalent to the involutivity of the sets of Pfaff's forms leading to the construction of solutions of the class $F(\mathfrak{G}_r)$.

The notion of regular interaction of simple waves can be introduced on the basis of the Theorem 6 as follows. *A maping* $u(x)$ *is a regular interaction of r independent simple waves if* $u(x) \in F_r(\mathfrak{H}_r)$. From (3.1) it follows that the manifolds $\mathfrak{G}_r$ *is a characteristic manifolds* $\mathfrak{H}_r$ *if and only if all the two-dimensional manifolds* $\mathfrak{G}_2 \subset \mathfrak{G}_r$ *are characteristic manifolds* $\mathfrak{H}_2$.

We have therefore the following corollary which is a generalization of the Corollary 2.

COROLLARY 3. *A maping* $u = u(x)$, $x \in D$ *is, for the Eqs.* (1.1) *a regular interaction of r independent simple waves when and only when* $u(D) = \mathfrak{H}_r$ *and, for any characteristic manifolds* $\overset{*}{\mathfrak{H}}_k \subset \mathfrak{H}_r$, $0 < k < r$ *in the neighbourhood of the manifold* $C_{n-r+k} = u^{-1}(\mathfrak{H}_k^*)$ *there exists a solution* $v(x) \in F_k(\overset{*}{\mathfrak{H}}_k)$ *such that*

$$v(x) = u(x), \quad x \in C_{n-r+k}.$$

The manifold C_{n-r+k} *are characteristic and developable.*

To prove the Theorem 6 it suffices to prove the following theorem.

THEOREM 7. *If* $u(x) \subset F_r(\mathfrak{G}_r)$, $\overset{*}{\mathfrak{G}}_k \subset \mathfrak{G}_r$, $0 < k < r$, $C_{n-r+k} = u^{-1}(\overset{*}{\mathfrak{G}}_k)$, *then there exists in the neighbourhood of* C_{n-r+k} *a solution* $v(x) \in F_k(\overset{*}{\mathfrak{G}}_k)$ *such that*

$$v(x) = u(x), \quad x \in C_{n-r+k}$$

if, and only if, $\overset{*}{\mathfrak{G}}_k$ *has the R-property.*

The necessity is a consequence of the Theorem 5.

The sufficiency will be proved for $k = r-1$. The proof of the general case is analogous.

Let us assume that $T_u(\overset{*}{\mathfrak{G}}_{r-1}) = [\underset{1}{\gamma}(u)| \dots | \underset{r-1}{\gamma}(u)]$. Then, for $u \in \overset{*}{\mathfrak{G}}_{r-1}$, we have

$$\overset{*}{\sum}_{n-(r-1)}(u) = R^n \ominus [\overset{1}{\lambda}(u)| \dots | \overset{r-1}{\lambda}(u)],$$

$$L_{r-1}^*(u) = [\lambda^1(u)| \dots | \overset{r-1}{\lambda}(u)].$$

The symbol $\overset{*}{\mathscr{L}}{}^p_{r-1}(u)$ will denote any $(r-1)$-dimensional manifold in R^n such that $p \in \overset{*}{\mathscr{L}}{}^p_{r-1}(u)$ and $T_p\ (\overset{*}{\mathscr{L}}{}^p_{r-1}) = L^*_{r-1}(u)$. From the fact that $u(x) \in F_r(\mathfrak{G}_r)$ it follows that

$$\overset{*}{\mathscr{L}}{}^{\overset{x}{0}}_{r-1}(\underset{0}{u}) = L_r^{\overset{x}{0}}(\underset{0}{u}) \cap C_{n-1},$$

where

$$\underset{0}{u} = u(\underset{0}{x}), \quad C_{n-1} = u^{-1}(\overset{*}{\mathfrak{G}}_{r-1}), \quad \underset{0}{x} \in C_{n-1}.$$

In the neighbourhood of $\overset{*}{\mathscr{L}}{}^{\overset{x}{0}}_{r-1}(\underset{0}{u})$ we shall now construct the solution $v(x) \in F_{r-1}(\overset{*}{\mathfrak{G}}_{r-1})$. First, the maping $v(x)$ will be defined for $x \in \overset{*}{\mathscr{L}}{}^0_{r-1}$ by setting $v(x) = u(x)$.

Let us denote by $\overset{*}{\underset{i}{C}}(\underset{0}{x}) \subset \overset{*}{\mathscr{L}}{}^{\overset{x}{0}}_{r-1}(\underset{0}{u})$, $i = 1, \dots, r$, the curves occurring in the Theorem 3 in the case of the manifold $\overset{*}{\mathfrak{G}}_{r-1}$ and the maping $u(x)$. These curves are tangent to the vector field $\overset{*}{\underset{i}{c}}\,(v(x), \underset{0}{x}, x)$, where

$$[\overset{*}{\underset{i}{c}}(u, \underset{0}{x}, x)] = \Big[\overset{*}{\sum}_{n-r+1}(u)|\underset{i}{c}(u)\Big] \cap T_x(\overset{*}{\mathscr{L}}{}^{\overset{x}{0}}_{r-1}(\underset{0}{u})) = \Big[\sum_{n-r}(u)|\underset{i}{c}(u)\Big] \cap T_x(L_r^{\overset{x}{0}}(\underset{0}{u})).$$

Thus the curves $\underset{i\ 0}{\overset{*}{C}}(x)$ are $\underset{i\ 0}{C}(x)$ for the manifold $\mathfrak{G}_r$ and the maping $u(x)$:

$$\underset{i\ 0}{\overset{*}{C}}(x) = \underset{i\ 0}{C}(x) \subset \underset{0}{\overset{*x}{\mathscr{L}}}{}^{0}_{r-1}(u) \subset \underset{0}{\overset{x}{L}}{}^{0}_{r}(u).$$

We have therefore

$$v(\underset{i}{\overset{*}{C}}) = u(\underset{i}{C}) \subset \Gamma_i.$$

Next, the maping $v(x)$ defined for $x \in \underset{0}{\overset{*x}{\mathscr{L}}}{}^{0}_{r-1}(u)$, $\underset{0}{\overset{x}{\mathscr{L}}}{}^{0}_{r-1}(u) \overset{v}{\leftrightarrow} \overset{*}{\mathfrak{G}}_{r-1}$ is prolonged over the neighborhood of the manifold by setting

$$v(x) = \text{const} = v(p), \quad x \in \overset{*}{\sum}{}^{p}_{n-r+1}\big(v(p)\big)$$

By virtue of the Theorem 3 and the fact that $\overset{*}{\mathfrak{G}}_{r-1}$ has the R-property, we have $v(x) \in$ $\in F_{r-1}(\overset{*}{\mathfrak{G}}_{r-1})$. To prove the theorem if suffices now to show that

$$v(x) = u(x), \quad x \in C_{n-1}.$$

This equality follows directly from the fact that C_{n-1} is stratified by $\sum^{p}_{n-r}\big(u(p)\big)$, where $p \in \underset{0}{\overset{*x}{L}}{}^{0}_{r-1}(u)$. Indeed

$$u(x) = v(x) \quad \text{for} \quad x \in \overset{*}{\sum}{}^{p}_{n-r+1}\big(u(p)\big)$$

and

$$\sum\nolimits^{p}_{n-r}\big(u(p)\big) \subset \overset{*}{\sum}{}^{p}_{n-r+1}\big(u(p)\big).$$

This ends the proof of the Theorem 7.

There is another way of explaining the geometrical sense of the notion of interaction of simple waves introduced here.

In the case of $n = 2$ and $u(x) \in F_2(\mathfrak{H}_2)$, $x \in D$ (see [1]), let us denote for $p \in D$ by $\Gamma_1(p)$, $\Gamma_2(p) \subset \mathfrak{H}_2$ the characteristing curves containing the point $u(p)$. Then the characteristic curves $C = u^{-1}\big(\Gamma_i(p)\big)$ split up the region D into four regions (Fig. 3). Let us denote one of them by $\underset{(1,2)}{G}(p)$. Region $\underset{1}{G}(p)$, $\underset{2}{G}(p)$ (Fig. 4) are adjacent to $\underset{(1,2)}{G}(p)$. In these regions $u(x)$, $x \in \underset{(1,2)}{G}(p)$ can be prolonged in a continuous manner by means of simple waves $\underset{1}{u}(x)$, $\underset{2}{u}(x)$ and $\underset{i}{u}\big(\underset{i}{G}(p)\big) \subset \Gamma_i(p)$. The solution $x \in \underset{(1,2)}{G}(p)$ is a regular interaction of simple waves $\underset{1}{u}$ and $\underset{2}{u}$.

On the basis of the Theorem 7 we can give an analogous interpretation of regular interaction of r independent waves for any n.

Let $u(x) \in F_r(\mathfrak{H}_r)$, $x \in D$. For any $p \in D$ let us denote by $\mathfrak{H}^{\mu}_{r-1}(p)$, $\mu = 1, \ldots, r$ all the $(r-1)$-dimensional characteristic surfaces such that $u(p) \in \mathfrak{H}^{\mu}_{r-1}(p) \subset \mathfrak{H}_r$. The characteristic manifolds

$$C^{\mu}_{n-1} = u^{-1}\big(\mathfrak{H}^{\mu}_{r-1}(p)\big)$$

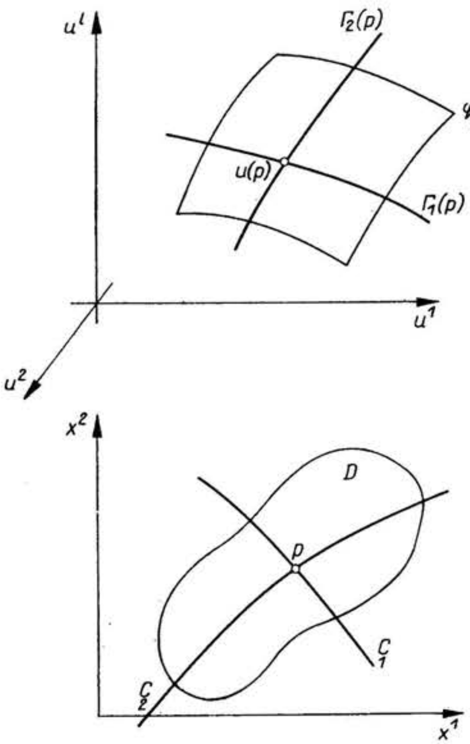

FIG. 3.

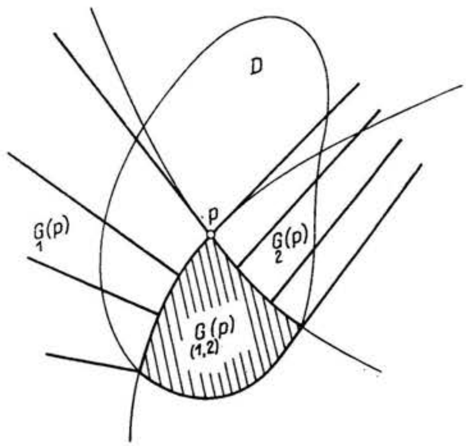

FIG. 4.

split up the region $D \subset R^n$ into 2^r regions. If any of them is denoted by $G(P)$, then, on the basis of the Theorem 7 there are regions adjacent to $G^\mu(P)$, $\mu = 1, \dots, r$, in which $u(x)$, $x \in G(p)$, can be prolonged by means of the solutions

$$u^{(\mu)}(x) \in F_{r-1}\big(\mathfrak{H}^\mu_{r-1}(p)\big), \quad x \in G^{(\mu)}(p).$$

The solution $u(x)$ is an interaction of the solutions $u^\mu(x)$, each of which is a regular interaction of $r-1$ simple waves. If now the solution $u^{(\mu)}(x)$ are treated as interactions of solutions of the type F_{r-2}, etc., we are concerned with an "interaction star" of simple waves.

As an example let us discuss an interaction star of simple waves for $r = 3$. Let $u(x) \in F_3(\mathfrak{H}_3)$, $x \in D$. And let

$$\underset{i}{\mathfrak{H}_1}(p) = \Gamma_i, \quad i = 1, 2, 3, \quad \underset{(i,j)}{\mathfrak{H}_2}(p), \quad i, j = 1, 2, 3, \quad i \neq j,$$

$p \in D$, denote one and two-dimensional characteristic manifolds such that

$$u(p) \in \underset{i}{\mathfrak{H}_1}(p) \subset \mathfrak{H}_3, \quad u(p) \in \underset{(i,j)}{\mathfrak{H}_2}(p) \subset \mathfrak{H}_3$$

(see Fig. 5). The solution $u(x)$, $x \in G(p) = \underset{(1,2,3)}{G}(p)$, can be prolonged in a continuous manner into the set

$$G = \underset{(1)}{G} \cup \underset{(2)}{G} \cup \underset{(3)}{G} \cup \underset{(31)}{G} \cup \underset{(21)}{G} \cup \underset{(32)}{G} \cup \underset{(1,23)}{G}(p)$$

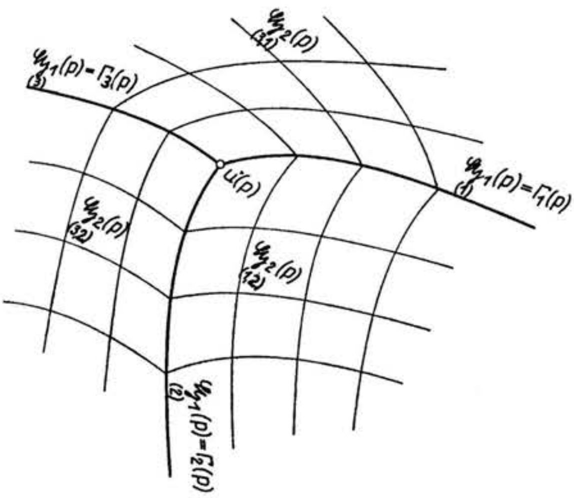

FIG. 5.

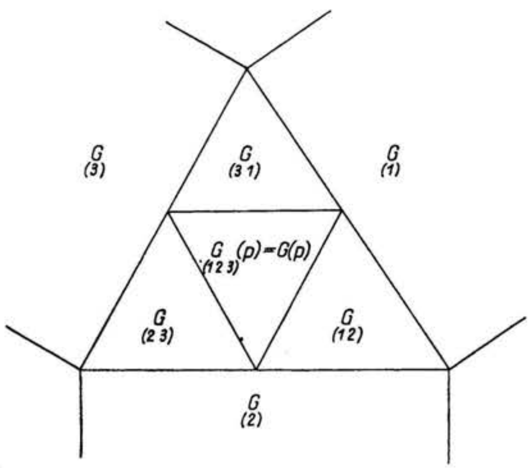

FIG. 6.

by means of the solution $v(x)$, so that

$$v(x) \in F_1\big(\Gamma_i(p)\big), \quad x \in \underset{(i)}{G}, \quad \text{and} \quad v(x) \in F_2(\mathfrak{H}_2), \quad x \in \underset{(i,j)}{G}.$$

The set G is an interaction star and is represented diagrammatically in Fig. 6. The manifolds splitting G into regions $\underset{(i)}{G}$, $\underset{(i,j)}{G}$, $\underset{(1,2,3)}{G}(p)$ are $(n-1)$-dimensional, characteristic and contain the point p.

The solution $u(x)$, $x \in \underset{(1,2,3)}{G}(p)$ is an interaction of simple waves $v(x)$, $x \in \underset{i}{G}$, $i = 1, 2, 3$. This is a direct generalization of the situation represented for $n = 2$ in Fig. 4.

In the general case $u(x) \in F_r(\mathfrak{H}_r)$, $r \leqslant n$ an interaction star can be constructed in an analogous manner.

4. The conic property

A *maping* $u = u(x)$, $D \xrightarrow{u} H^l$ *will be said to be conical* if there exists a point $y \in R^n$ such that for every straigth line l passing through the point y we have

$$u(x) = \text{const}, \quad x \in l \cap D.$$

In the present case it will be shown that the classes $F(\mathfrak{H}_r)$ include conical solutions. It will also be shown that a conical solution $u_{\text{con}}(x) \in F(\mathfrak{H}_r)$, $x \in G$, can be atteched in a simple manner to the solutions $u(x) \in F(\mathfrak{H}_r)$, $x \in D$, so that we have for a certain region $D' \subset D$, $u(D') = u_{\text{con}}(D)$.

For any manifold $\mathfrak{H}_r$ and $\sum_{n-r}(u) = [\underset{1}{\sigma}(u)| \dots | \underset{n-r}{\sigma}(u)]$, $u \in \mathfrak{H}_r$, let us denote by $l(u, \alpha, y)$, $\alpha \in R^{n-r}$, $y \in R^n$ a straight line R^n prescribed in a parametric manner thus

$$x = t \sum_{\mu=1}^{n-r} \alpha^\mu \underset{\mu}{\sigma}(u) + y.$$

A manifold $\mathfrak{H}_r$ is of cylindric type if

$$\sum\nolimits_{n-r}(u) = \text{const}, \quad u \in \mathfrak{H}_r.$$

In the opposite case, $\mathfrak{H}_r$ will be termed *non-cylindrical charakteristic manifold.*

For a non-cylindrical $\mathfrak{H}_r$ let us consider any solution $u(x) \in F(\mathfrak{H}_r)$, $x \in D$. Let $\mathscr{L}$ denote the manifold $\mathscr{L}_r^p(u(p))$. Let us select a parametric representation $\mathscr{L}$: $x = x(s^1, \dots, s^r)$, $s \in S$ so that the curves $C^i = C^i_{n-r+1} \cap \mathscr{L}$ $(i = 1, \dots, r)$ for the solution $u(x)$ are expressed thus

$$x = x^i(s) = x(\underset{0}{s}, \dots, \underset{0}{s}^{i-1}, s, \underset{0}{s}^{i+1} \dots, \underset{0}{s}). \tag{4.1}$$

The region D is stratified by the straight lines $l(u(x)\underset{0}{\alpha}, x)$ $x \in D$, where $\underset{0}{\alpha}$ is any fixed point R^{n-r}. Since the solution $u(x)$ on the straight line l is constant and all the straight lines intersect the manifold $\mathscr{L}$ at a one point, it suffices to consider the family of straight lines $l(u(x(s)), \underset{0}{\alpha}, x(s))$, $s \in S$.

For any point $y \in R^n$ and any region $D' \subset D$ let us consider the set

$$K(y, D') = \bigcup_{x \in D', \alpha \in R^{n-r}} l(u(x), \alpha, y).$$

Let the maping $M: D \xrightarrow{M} K(y, D)$ be defined in such a manner that every straight line $l(u(x(s)), \alpha, x(s))$ is mapped into a straight line $l(u(x(s)), \alpha, y)$ by a translation T satisfying the condition $Tx(s) = y$. Thus $M(\mathscr{L}) = y$ and $D' \xrightarrow{M} K(y, D')$ (see Figs. 7 and 8).

The maping M may be defined in a still different manner. Let us define:

$$x = \sum_{\omega=1}^{n-r} s^{r+\omega} \underset{\omega}{\sigma}(u(x(s^1, \dots, s^r))) + x(s^1, \dots, s^r) = A(s^1, \dots, s^n),$$

$$x = \sum_{\omega=1}^{n-r} s^{r+\omega} \underset{\omega}{\sigma}(u(x(s^1, \dots, s^r))) + y = B(s^1, \dots, s^n).$$

The maping $x = A(s)$ is, for $x \in D$, one-to-one. It can easily be verified that

$$Mx = B(A^{-1}x).$$

It will be said that a solution $u(x) \in F(\mathfrak{H}_r)$ is non-cylindric in a region D' if the maping M maps D' in a one-to-one manner into $K(y, D')$ (Fig. 7, 8).

Let us construct, for any solution $u(x) \in F(\mathfrak{H}_r)$ non-cylindric in the region D', a conical maping $u_{\text{con}}(x) \in W(\mathfrak{H}_r)$ defined for $x \in K(y, D')$ by setting

$$u_{\text{con}}(x) = u(\underset{0}{x})$$

for $x \in l\big(u(\underset{0}{x}), \alpha\, y\big) \cap K(y, D')$, where $\underset{0}{x} \in D'$ and $\alpha \in R^{n-r}$. The maping $M(x)$, $D' \xrightarrow{M} K(y, D')$ being one-to-one, the maping $u_{\text{con}}(x)$ is well defined. It is

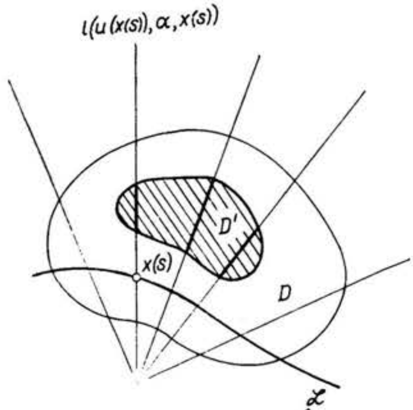

FIG. 7.

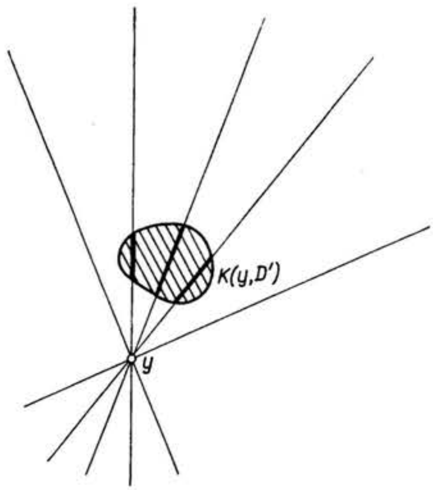

FIG. 8.

obvious that $u_{\text{con}}(x)$ is conical and $u_{\text{con}} \in W(\mathfrak{H}_r)$. The representation $u_{\text{con}}(x)$ can also be defined thus

$$u_{\text{con}}(x) = u\big(M^{-1}(x)\big), \quad x \in K(y, D').$$

We shall now prove the following theorem.

THEOREM 8. *If* $u(x) \in F(\mathfrak{H}_r)$, *we have*

$$u_{\text{con}}(x) \in F(\mathfrak{H}_r).$$

Proof. From the definition of the maping $u_{\text{con}}(x)$ it follows that, for manifolds $C^i_{n-r+1} \subset D'$ corresponding to the solution $u(x)$, the manifolds

$$E^i_{n-r+1} = M(C^i_{n-r+1}), \quad i = 1, \ldots, r,$$

are rectilinear. They stratify the region $K(y, D')$ and

$$u_{con}(E^i_{n-r+1}) \subset \Gamma_i \subset \mathfrak{H}_r, \quad i = 1, \ldots, r.$$

On the basis of Theorem 3_A it suffices to show that the manifolds E^i_{n-r+1} are identical with the manifold C^i_{n-r+1} for the maping $u_{\mathrm{con}}(x)$. In other words it suffices to show that E^i_{n-r+1} are integral manifolds of Pfaff's form

$$\lambda^\mu_\sigma\big(u_{con}(x)\big)\mathrm{d}x^\sigma = 0, \quad \mu = 1, \ldots, i-1, i+1, \ldots, r,$$

where

$$\overset{\mu}{\lambda}(u) = \big(\overset{\mu}{\lambda}_1(u), \ldots, \overset{\mu}{\lambda}_n(u)\big) \rightleftharpoons \underset{\mu}{\gamma}(u),$$

$T_u(\mathfrak{H}_r) = [\underset{1}{\gamma}(u)| \ldots |\underset{r}{\gamma}(u)]$. As the manifold C^i_{n-r+1} are the integrals of the system

$$\overset{\mu}{\lambda}\big(u(x)\big)dx^\sigma = 0, \quad \mu = 1, \ldots, i-1, i+1, \ldots, r,$$

so on the basis of the definition $u_{\mathrm{con}}(x)$ if suffices to show that

$$T_x(C^i_{n-r+1}) = T_{M(x)}(E^i_{n-r+1}), \quad \text{for } x \in D'. \tag{4.2}$$

The parameters $s^1, \ldots, s^r$ for the manifold $\mathscr{L}$ having been selected so that the curves C are expressed in the form (4.1). Hence the
manifold C^i_{n-r+1} can be expressed in the parametric form

$$x = \sum_{\omega=1}^{n-r} s^{r+\omega}\underset{\omega}{\sigma}\big(u(x(\underset{0}{s^1}, \ldots, \underset{0}{s^{i-1}}, t, \underset{0}{s^{i+1}}, \ldots, \underset{0}{s^r}))\big)+$$
$$+x(\underset{0}{s^1}, \ldots, \underset{0}{s^{i-1}}, t, \underset{0}{s^{i+1}}, \ldots, \underset{0}{s^r}) = \varphi(t, s^{r+1}, \ldots, s^n).$$

The manifold E^i_{n-r+1} is therefore written in the form

$$x = \sum_{\omega=1}^{n-r} s^{r+\omega}\underset{\omega}{\sigma}\big(u(x(\underset{0}{s^1}, \ldots, \underset{0}{s^{i-1}}, t, \underset{0}{s^{i+1}}, \ldots, \underset{0}{s^r}))\big) = \psi(t, s^{r-1}, \ldots, s^n),$$

that is

$$T_{\varphi(t,s)}(C^i_{n-r+1}) = \Big[\sum_{\omega} s^{r+\omega}\frac{\partial}{\partial t}\underset{\omega}{\sigma}+x_t|\underset{1}{\sigma}| \ldots |\underset{n-r}{\sigma}\Big].$$

From the developability of C^i_{n-r+1} we have

$$T_{\varphi(t,s)}(C^i_{n-r+1}) = T_{\varphi(t,0)}(C^i_{n-r+1}) = [x_t|\underset{1}{\sigma}| \ldots |\underset{n-r}{\sigma}].$$

Hence

$$\sum_{\omega} s^{r+\omega}\frac{\partial}{\partial t}\underset{\omega}{\sigma}\in[x_t|\underset{1}{\sigma}| \ldots |\underset{n-r}{\sigma}]. \tag{4.3}$$

Since $\psi(t, s) = M\big(\varphi(t, s)\big)$, we have

$$T_{M(\varphi(t,s))}(E^i_{n-r+1}) = \Big[\sum_{\omega} s^{r+\omega}\frac{\partial}{\partial t}\underset{\omega}{\sigma}|\underset{1}{\sigma}| \ldots |\underset{n-r}{\sigma}\Big],$$

which gives, together with (4.3), the inequality (4.2). This ends the proof of Theorem 8.

5. Double waves, free manifolds and boundary-value problems

We shall now discuss in greater detail the properties of the solutions $F(\mathfrak{H}_2)$. Such solutions are specially interesting, because it is only for them that the Theorem 2 and 3 give a simple method for constructing a solution. Let us consider a manifold

$$\mathfrak{H}_r: u = u(\mu^1, \ldots, \mu^r),\ \mu \in M \subset R^r, \quad u_{\mu i} = \underset{i}{\gamma}$$

in H^l and $\underset{0}{\mathscr{L}^p_r(u)}: x = x(s^1, \ldots, s^r)$, $s \in S \subset R^r$, in R^n. To construct a solution $u(x) \in F(\mathfrak{H}_r)$ defined in the neighbourhood of the manifold $\underset{0}{\mathscr{L}^p_r(u)}$, we shall apply the Theorem 3. The manifold $\mathfrak{H}_r$ has the R-property, therefore it suffices to construct the maping $u(x)$, $x \in \underset{0}{\mathscr{L}^p_r(u)}$, $\underset{0}{\mathscr{L}^p_r(u)} \xrightarrow{u} \mathfrak{H}_r$ so that the condition A of the Theorem 3 is satisfied. To this end let us consider the vectors

$$\underset{i}{c}(\mu, s) = \underset{i}{c}\big(u(\mu), p, x(s)\big) \subset T_{x(s)}(\underset{0}{\mathscr{L}^p_r(u)})$$

determining the spaces (2.3). The construction of the maping $\underset{0}{\mathscr{L}^p_r(u)} \xrightarrow{u} \mathfrak{H}_r$ is equivalent to that of the maping $\mu = \mu(s)$ $S \xrightarrow{\mu} M$. The characteristic curves $\Gamma_i \subset \mathfrak{H}_r$ being prescribed by the condition $\mu^j = \underset{0}{\mu^j} = \text{const}$, $j = 1, \ldots, i-1, i+1, \ldots, r$, the condition A is satisfied when and only when

$$\mu^j_{\underset{i}{\hat{c}}} = \mu^j_{s^1}\underset{i}{\hat{c}}^1(\mu, s) + \ldots + \mu^j_{s^r}\underset{i}{\hat{c}}^r(\mu, s) = 0, \tag{5.1}$$

$$i, j = 1, \ldots, r, \quad i \neq j,$$

where $\underset{i}{\hat{c}} = \big(\underset{i}{\hat{c}}^1(\mu, s), \ldots, \underset{i}{\hat{c}}^r(\mu, s)\big)$ is the vector $c(\mu, s)$ in the coordinates $s^1, \ldots, s^r$, that is $\underset{i}{c}(\mu, s) = \underset{i}{\hat{c}}^\varrho(\mu, s)x_{s^\varrho}$. Thus the construction of the solution $u \in (\mathfrak{H}_r)$ is reduced in the most general case, to the integration of a set of $r(r-1)$ quasi-linear Eq. (5.1) with r unknown functions $\mu^j(s^1, \ldots, s^r)$ $j = 1, \ldots, r$. This set of equations is always overdetermined for $r > 2$. It is only in the case of $r = 2$ that (5.1) is reduced to a set of as many equations as there are unknown functions. It is then a hyperbolic set of two equations with two dependent and two independent variables. Thus using the Theorem 3 we can construct all the solutions of the Eqs. (1.1) of the class $F(\mathfrak{H}_2)$. The functions of the set $F(\mathfrak{H}_2)$ depend on two arbitrary functions of one variable. For $r > 2$, in view of the overdetermination of the set of Eqs. (5.1), the Theorem 3 does not give a complete characteristic of the class $F(\mathfrak{H}_r)$. In Refs [7, 8, 9], using E. Cartan's methods for integrating sets of Pfaff's forms it is shown that for sets of Eqs. (1.1) with analytic coefficients and for analytic manifolds $\mathfrak{H}_r$ the functions of the set $F(\mathfrak{H}_r)$, $r \geqslant 1$ depend on r arbitrary functions of one variable.

We shall now introduce the notion of free manifolds in H^l. If we analyse the solutions $u(x)$, $x \in D$ of the equations (1.1), we can succeed in some cases in obtaining some information on the set of values before the solution is constructed. Strictly speaking we can determine a manifold $\mathfrak{F}_r$ of dimension $r < l$ such that

$$u(D) \subset \mathfrak{F}_r. \tag{5.2}$$

As an example let us consider the set of equations

$$
(5.3)\qquad
\begin{aligned}
&uc_x+vc_y+wc_z+\frac{\varkappa-1}{2}c(u_x+v_y+w_z)=0,\\
&uu_x+vu_y+wu_z+\frac{2c}{\varkappa-1}c_x=0,\\
&uv_x+vv_y+wv_z+\frac{2c}{\varkappa-1}c_y=0,\\
&uw_x+vw_y+ww_z+\frac{2c}{\varkappa-1}c_z=0,
\end{aligned}
$$

describing the isentropic flow of inviscid gas. The unknowns function c, u, v, w depend on three independent variables x, y, z, $n = 3$, $l = 4$, $\varkappa = \text{const}$.

A solution of (5.3) is a plane potential flow if $u_z = v_z = c_z = 0$, $u_y - v_x = 0$, $w = 0$. Such solutions safisfy (5.2) for the manifold $\mathfrak{F}_2(Q) \subset H^4$ prescribed by the equations

$$
(5.4)\qquad u^2+v^2+\frac{2c^2}{\varkappa-1}-Q^2=0 \quad \text{(Bernoulli's law)},\qquad w=0,
$$

where $Q > 0$ is an arbitrary constant termed maximum velocity (see [10]). For any $Q > 0$ there exists an infinite number of solutions of (5.3), the set of values of which belongs to $\mathfrak{F}_2(Q)$. The existence of the manifold $\mathfrak{F}_2(Q)$ facilitates considerably the construction of wide classes of solutions of (5.3). This was essential for the development of gas dynamics.

In this connection there arise the following interesting problems: a) What are these manifolds in H^l that may constitute sets of values of solutions of the set of Eqs. (1.1). b) What are the manifolds in H^l that contain the sets of values of an infinite family of solutions.

An r-dimensional manifold $\mathfrak{F}_r \subset H$ will be said to be a *free manifold* for the Eqs. (1.1), if there exists an infinite number of solutions $u(x)$, $x \in D$, such that the set $u(D)$ is an r-dimensional manifold ($r\|u^j_{x^i}(x)\| = r$) and $u(D) \subset \mathfrak{F}_r$.

There are the following obvious corollaries resulting from the above considerations.

COROLLARY 4. *A curve $K \subset H^l$ is a free manifold if, and only if, K is a characteristic curve Γ.*

COROLLARY 5. *The characteristic surfaces $\mathfrak{H}_2 \subset H^l$ are free manifolds.*

As an example it will be shown that part of the free manifold $\mathfrak{F}_2(Q)$, for which $c^2 > u^2+v^2$, is a characteristic manifold $\mathfrak{H}_2$. If we set $(u^1, u^2, u^3, u^4) = (c, u, v, w)$, it is easy to verify that for the set of Eqs. (5.3) the vector $\gamma = (\gamma^1, \gamma^2, \gamma^3, \gamma^4)$ is a characteristic vector, if

$$
(5.5)\qquad
\begin{aligned}
&\frac{2u^1\gamma^1}{\varkappa-1}+u^2\gamma^2+u^3\gamma^3u^4\gamma^4=0,\\
&\left(\frac{2\gamma^1}{\varkappa-1}\right)^2-(\gamma^2)^2-(\zeta\gamma^3)^2-(\gamma^4)^2=0.
\end{aligned}
$$

In addition, for the vectors (5.5) we have

$$\gamma = (\gamma^1, \gamma^2, \gamma^3, \gamma^4) \rightleftharpoons \lambda = (\gamma^2, \gamma^3, \gamma^4).$$

A space normal to $\mathfrak{F}_2(Q)$ has the form $N(\mathfrak{F}_2) = [n_1|n_2]$, where

$$n_1 = \left(\frac{2u^1}{\varkappa-1}, u^1, u^2, 0\right), \quad n_2 = (0, 0, 0, 1),$$

therefore there exist, for $U = (u^1, u^2, u^3, u^4) \in \mathfrak{F}_2(Q)$, $(u^1)^2 > (u^2)^2+(u^3)^2$, exactly two linearly independent characteristic vectors $\gamma = (\gamma^1, \gamma^2, \gamma^3, 0)$, $i = 1, 2$, satisfying (5.5) and such that

$$T_U(\mathfrak{F}_2(Q)) = [\gamma_1|\gamma_2].$$

Thus, $\mathfrak{F}_2(Q)$, $(u^1)^2 > (u^2)^2+(u^2)^3$, is a manifold $\mathfrak{G}_2$. The spaces $\sum_{n-r}(u)$ are independent of u and are one-dimensional

$$\sum\nolimits_1(u) = [(0, 0, 1)],$$

therefore the manifolds C^i_{n-r+1} are two-dimensional cylindrical surfaces, and are always developable. Thus, by virtue of Lemma 3, the manifold $\mathfrak{F}_2(Q)$, $(u^1)^2 > (u^2)^2+(u^3)^2$ has the R-property and is a cylindrical characteristic manifold $\mathfrak{H}_2(Q) \subset H^4$.

If for the manifold $\mathscr{L}^\varrho_2(u)$ we select $z = 0$, then, as can easily be verified, the hyperbolic set of Eqs. (5.1), which leads to the construction of class $F\left(\mathfrak{H}_2(Q)\right)$, has the form

$$(u^2-c^2)u_x+2uvu_y+(v^2-c^2)v_y = 0, \quad u_y-v_x = 0,$$

where $c^2 = \dfrac{\varkappa-1}{2}(Q^2-u^2-v^2)$. This is a set of equations in common use (see [10]) for the description of plane stationary and potential gas flows. For $c^2 > u^2+v^2$ it is hyperbolic.

The set of Eqs. (5.3) has many interesting manifolds $\mathfrak{H}_2$. They can be found in Refs. [7, 8, 9 and 11]. For the set of equations of magnetohydrodynamics

$$\varrho\frac{d\vec{v}}{dt}+\nabla p = -\vec{H}\times\operatorname{rot}\vec{H}, \quad \frac{\partial\varrho}{\partial t}+\operatorname{div}(\varrho\vec{v}) = 0, \quad \frac{\partial\varrho}{\partial t}-\frac{\varkappa}{\varrho}p\frac{\partial p}{\partial t} = 0,$$
$$\frac{\partial\vec{H}}{\partial t} = \operatorname{rot}(\vec{V}\times\vec{H}), \quad \operatorname{div}\vec{H} = 0$$

all the characteristic curves Γ are constructed in Ref. [12] and a number of characteristic manifolds $\mathfrak{H}_2$ are analysed.

Let us proceed now to discuss the boundary value problems for the set (1.1) which can be solved in the class $F(\mathfrak{H}_2)$. We begin with the Cauchy problem. It will be assumed, as was done above, that the surfaces $\mathscr{L} = \mathscr{L}^\varrho_2(\underset{0}{u}) \subset R^n$ and $\mathfrak{H}_2 \subset H$ are prescribed in a parametric manner $x = x(s^1, s^2)$, $s \in S$ and $u = u(\mu^1, \mu^2)$, $u_{\mu i} = \underset{i}{\gamma}$, $\mu \in M$. The solutions $u \in F(\mathfrak{H}_2)$ are constant along the planes $\sum^x_{n-2}(u(x))$. The Cauchy problems of the following type are the only problems that may be stated in the class $F(\mathfrak{H}_2)$: on the initial manifold P_{n-1}:

$$x = x(t^1, \ldots, t^{n-2}, \varrho) = \sum_{\nu=1}^{n-2} t^\nu \underset{\nu}{\sigma}\left(U(\varrho)\right)+\mathscr{X}(\varrho), \tag{5.6}$$

where $[\underset{1}{\sigma}(u)| \dots | \underset{u-2}{\sigma}(u)] = \sum_{u-2}(u)$, the solution takes the form

$$u\big((x(t, \varrho)\big) = U(\varrho). \tag{5.7}$$

This Cauchy problem is equivalent to the definition of the maping $\mathring{U}: P \xrightarrow{\mathring{U}} K$, where $P \subset R^n$ is a curve $x = \mathscr{X}(\varrho)$ and $K = \{u: u = U(\varrho)\} \subset H^t$. It is obvious that a necessary condition for our Cauchy problem to have a solution in the class $F(\mathfrak{H}_2)$ is that $K \subset \mathfrak{H}_2$.

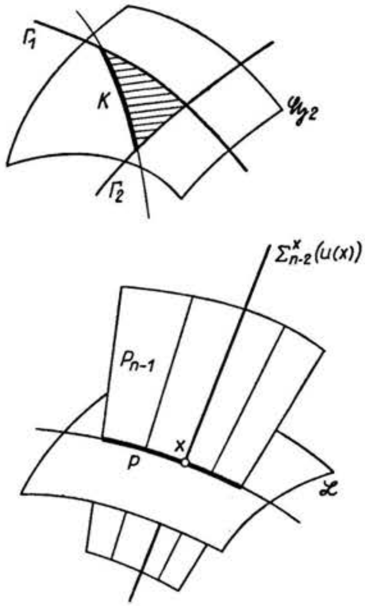

FIG. 9.

With no limitation of generality it can be assumed that $P \subset \mathscr{L}$ (Fig. 9), therefore K can be represented in the form $\mu = \mathring{\mu}(\varrho)$ and P in the form $\mathring{s} = s(\varrho)$.

Let us set $\underset{1}{\hat{c}} = (a^1(\mu, s), a^2(\mu, s))$, $\underset{2}{\hat{c}} = \big(A^1(\mu, s), A^2(\mu, s)\big)$. Then, the set of Eqs. (5.1) can be rewritten thus:

$$A^1\mu^1_{s^1} + A^2\mu^1_{s^2} = 0, \quad a^1\mu^2_{s^1} + a^2\mu^2_{s^2} = 0. \tag{5.8}$$

The construction of $u(x) \in F(\mathfrak{H}_2)$ satisfying (5.7) reduces to the obtainment of a solution $\mu = \mu(s^1, s^2)$ of the Eqs. (5.8) satisfying the initial condition

$$\mu(\mathring{s}(\varrho)) = \mathring{\mu}(\varrho). \tag{5.9}$$

The solution $u(x)$ required is obtained by setting $u(x) = u\big(\mu(s^1, s^2)\big)$ for $x = x(s^1, s^2) \in \mathscr{L}$. Next $u(x)$ is prolonged over the neighbourhood of $\mathscr{L}$ by setting $u(x) = \text{const} = u(\mathring{x})$ for $x \in \sum^{x_0}_{n-2}\big(u(x)\big)$, $\mathring{x} \in \mathscr{L}$.

Thus the Cauchy problem (5.7) for the Eqs. (1.1) in the class $F(\mathfrak{H}_2)$ reduces to the Cauchy problem (5.9) for the hyperbolic set of Eqs. (5.8). It can easily be seen that if the problem (5.7) is not characteristic for (1.1), the problem (5.9) is not characteristic for (5.8).

If the set $K \subset \mathfrak{H}_2$ is a characteristic curve Γ, the solution is a simple wave $u(x) \in F_1(\Gamma) \subset$ $\subset F(\mathfrak{H}_2)$. If K is a non-characteristic curve, $u(x)$ is a locally double wave, $u(x) \in F_2(\mathfrak{H}_2)$.

The class of solutions $F_2(\mathfrak{H}_2)$ has the property consisting in the fact that for the Cauchy problem (5.7) the set of values of the solution can easily be obtained beforehand without constructing the solution itself. Thus, for instance, if the set K is not a characteristic curve and the maping $U: P \xrightarrow{\mathring{U}} K$ is one-to-one, the set of values of the solution is the part of the manifold $\mathfrak{H}_2$ which is shaded in Fig. 9. This fact enables qualitative analysis of the properties of the class $F(\mathfrak{H}_2)$.

We shall now be concerned with mixed problems which can be solved in the class $F(\mathfrak{H}_2)$. Such problems, similarly to the Cauchy problem, are reduced to the corresponding mixed problem for the hyperbolic set of Eqs. (5.8). Therefore we shall first discuss brief funda-

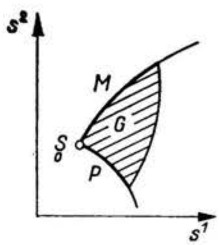

FIG. 10.

mental mixed problems for (5.8). In mathematical physics there are mixed problems of the following two fundamental types:

I. a) Consider a non-characteristic Cauchy problem for a curve P,

b) a curve M and the boundary condition

$$d_1(s)\mu^1 + d_2(s)\mu^2 = d(s) \tag{5.10}$$

to be satisfied on M by the solution.

It is required to find the solution satisfying a) and b) in the region G (Fig. 10).

II. a) For the non-characteristic Cauchy problem on a curve $\mathring{P}$ and

b) a prescribed curve M a set of values of the solution $\mathfrak{M}$ on the curve M is given.

It is required to find a solution $\mu(s)$ in the region G (Fig. 10) such that a) is satisfied and $\mu(M) = \mathfrak{M}$.

It is known that the problems I and II have a local solution, if the curve M, the condition (5.10) and the set $\mathfrak{M}$ are selected in a correct manner.

Let us proceed now to discuss these problems in the class $F(\mathfrak{H}_2)$.

If $\mathfrak{H}_2$ is cylindrical, $\sum_{n-2}(u) = \text{const}$ for $u \in \mathfrak{H}_2$ then all the solutions $u \in F(\mathfrak{H}_2)$ are constant along the same parallel $(n-2)$-dimensional planes. In the class $F(\mathfrak{H}_2)$ we can state, for a cylindrical $\mathfrak{H}_2$, mixed problems of the following types:

1. Consider a) a non-characteristic Cauchy problem (5.6) for a cylindrical manifold P_{n-1} (5.6), b) a cylindrical manifold M_{n-1}:

$$x = \sum_{\nu=1}^{n-2} t^\nu \underset{\nu}{\sigma} + M(\tau), \tag{5.11}$$

where $\underset{\nu}{\sigma} = \text{const}, \sum_{n-2} = [\underset{1}{\sigma}| \ldots |\underset{n-1}{\sigma}|] = \text{const}$ such that $M_{n-1} \cap P_{n-1} = \sum_{n-2}^{\overset{x}{\circ}}$, and the condition

(5.12) $$e_j(x)u^j = e(x),$$

where $e_j(x)$, $e(x) = \text{const}$ for $x \in \sum_{n-2}^{p}$, to be satisfied by the solution on M_{n-1}.

Our object is to find a solution in a bounded region D represented diagrammatically in Fig. 11 satisfying a) and the condition (5.12) on M_{n-1}.

2. a) The same as in the case 1. b) A cylindrical manifold M_{n-1} (5.11) is prescribed with a set of values $\mathfrak{M} \subset \mathfrak{H}_2$ to be assumed by the solution on the manifold M_{n-1}.

A solution $u(x)$ is to be found in a bounded region D (Fig. 11) such that a) is satisfied and $u(M_{n-1}) = \mathfrak{M}$.

Proceeding in the same manner as was done in the case of the Cauchy problem, it can easily be verified that mixed problems 1, 2 for cylindrical $\mathfrak{H}_2$ in the class $F(\mathfrak{H}_2)$ reduce

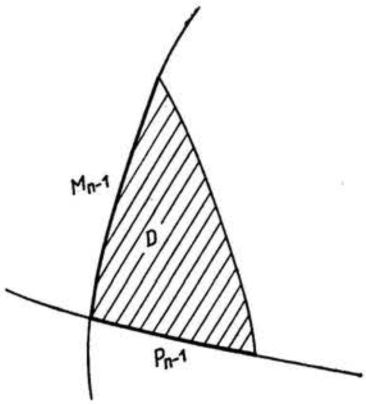

FIG. 11.

to the mixed problem I, II for the Eqs. (5.7), respectively. Thus the problems 1, 2 have a solution in $F(\mathfrak{H}_2)$ for cylindrical $\mathfrak{H}_2$.

If $\mathfrak{H}_2$ is not cylindrical, manifolds M_{n-1} in which boundary conditions may be imposed in the class $F(\mathfrak{H}_2)$ have the form

$$x = \sum_{\nu=1}^{n-2} t^{\nu} \underset{\nu}{\sigma}\big(U(\tau)\big) + M(\tau) = x(t, \tau),$$

where $U(\tau)$ is the value of the solution at a point $x = x(t, \tau)$. Thus the manifolds M_{n-1} are no more cylindrical and cannot be imposed unless the solution is known. The same applies to the condition (5.12), because the functions $e_j(x)$, $e(x)$ must be constant in the planes $\sum_{n-2}^{p}(u(p))$, which depend on the unknown solution.

Thus the mixed problems 1 and 2 cannot be stated for non-cylindrical $\mathfrak{H}_2$.

If $n = 3$ and if the Eqs. (1.1) describe a stationary flow of an inviscid material, we can solve in the class $F(\mathfrak{H}_2)$, for any manifold $\mathfrak{H}_2$, a certain mixed problem in which the kinematic condition of flow should be satisfied on the surface M_2. Let us first formulate an analogous problem for the set of Eqs. (5.8).

III. a) The same as in Case I. b) The set $\mathfrak{M}$ of values of the solution on the curve M is prescribed.

Our object is to find the curve M and a solution $\mu = \mu(s)$, $s \in G$ (Fig. 10) such that a) is satisfied, $\mu(M) = \mathfrak{M}$ and that the following condition for the normal $n(s) = (n_1(s), n_2(s))$ to M at points $s \in M$ is satisfied

$$n_1(s)h^1(s, \mu(s)) + n_2(s)h^2(s, \mu(s)) = 0, \tag{5.13}$$

where $h^1(s, \mu)$ and $h^2(s, \mu)$ given functions.

If the set $\mathfrak{M}$ is a curve the tangent to which is not parallel to the coordinate axes μ^1, μ^2 and the vector $n(\underset{0}{s})$ satisfying (5.13) for $\underset{0}{s} = M \cap P$ is not a characteristic vector of the sys-

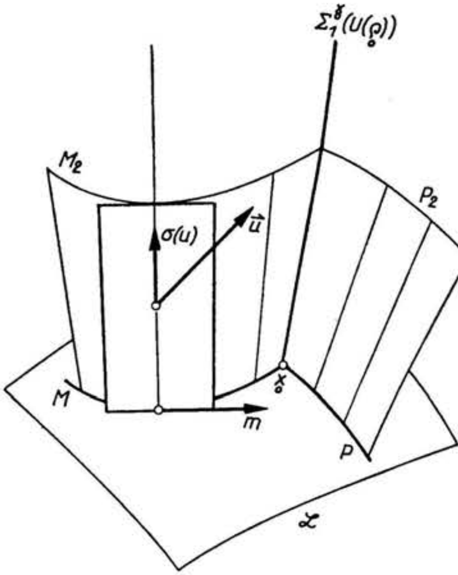

FIG. 12.

tem (5.8), we can construct locally, by the method of characteristics, a solution of the mixed problem III.

For any characteristic manifold $\mathfrak{H}_2$: $u = u(\mu)$, $n = 3$ and a set of equations (1.1) describing the motion of an inviscid material let us consider now the following mixed problem.

3. a) A Cauchy problem is stated on a rectlinear initial surface P_2

$$x = \mathscr{X}(t, \varrho) = t\sigma(U(\varrho)) + \mathscr{X}(\varrho),$$

where $[\sigma(u)] = \sum_1(u)$, $u \in \mathfrak{H}_2$:

$$u(\mathscr{X}(t, \varrho)) = U(\varrho).$$

b) There is given the set of values $\mathfrak{N} \subset \mathfrak{H}_2$ to be assumed by the solution at the rectilinear surface M_2 prescribed in the form

$$x = x(t, \tau) = \tau\sigma(W(\tau)) + K(\tau)$$

so that $M_2 \cap P_2 = \sum_1^{\overset{x}{\circ}}(U(\underset{\circ}{\varrho}))$, where $\overset{x}{\circ} = \mathscr{X}(0, \underset{\circ}{\varrho})$ (see Fig. 12).

Our object is to find a surface M_2 and a solution $u = u(x)$ defined in a bounded region D (Fig. 11) such that a) is satisfied, $u(M_2) = \mathfrak{M}$ and that the following kinematic condition is satisfied for the normal $n(x) = (n_1(x), n_2(x), n_3(x))$ to M_2 at a point $x \in M_2$:

$$n_1(x)u^1(x)+n_2(x)u^2(x)+n_3(x)u^3(x) = 0,$$

where $\vec{u} = (u^1, u^2, u^3)$ is the velocity vector of flow of the material the motion of which is described by the set of Eqs. (1.1)

Problem 3 implies problem III (Fig. 12), where

$$\mathfrak{M} = \{(\mu^1, \mu^2) : u(\mu^1, \mu^2) \in \mathfrak{N}\}, \qquad P = \{(s^1, s^2) : \mathscr{X}(s^1, s^2) \in P_2 \cap \mathscr{L}\},$$
$$M = \{(s^1, s^2) : x(s^1, s^2) \in M_2 \cap \mathscr{L}\},$$

and the vector $h(s, \mu) = (h^1(s, \mu), h^2(s, \mu))$ in the condition (5.13) is prescribed by the following equalities

$$\begin{aligned}&[\sigma(g(\mu))|\vec{u}(\mu)] \cap T_{x(s)}(\mathscr{L}) = [m(s, \mu)],\\ &m(s, \mu) = h^1(s, \mu)x_{s^1}+h^2(s, \mu)x_{s^2},\end{aligned} \tag{5.14}$$

where $\vec{u}(\mu) = (u^1(\mu), u^2(\mu), u^3(\mu))$ (Fig. 12).

However, a prolongation of the solution of the problem III into the neighbourhood of the surface $\mathscr{L}$ is not automatically a solution of the problem 3. If $s = s(\tau)$ denotes the curve M obtained by solving the problem III, the prolongation $u(x)$ into the neighbourhood of $\mathscr{L}$ is a solution of the Problem 3 when and only when the surface M_2:

$$x = t\sigma[u(x(s(\tau)))]+x(s(\tau)) \tag{5.15}$$

is developable(²). This condition is satisfied automatically in the case of cylindrical $\mathfrak{H}_2$ only.

In the case of a non-cylindrical $\mathfrak{H}_2$ a certain additional problem must be considered in order to verify the developability of the surface (5.15).

Let a function $a\ (\mu) \in R^3$ be prescribed for $\mathfrak{H}_2$: $u = g(\mu^1, \mu^2)$. Let us find curves $H: u = g((\mu(\tau))$, $H \subset \mathfrak{H}_2$ such that for each curve $K: x = x(\tau)$, $K \subset R^3$ satisfying the equation $\mathrm{d}x/\mathrm{d}\tau = a(\mu(\tau))$, the surface $M_2(K)$:

$$x = t\sigma[g(\mu(\tau))]+x(\tau) \tag{5.16}$$

is developable. A curve satisfying this condition will be referred to as *H-curve of the function $a(\mu)$ on $\mathfrak{H}_2$*.

It is obvious that for a cylindrical $\mathfrak{H}_2$ all the curves lying on $\mathfrak{H}_2$ are H-curves of any function $a(\mu)$.

For non-cylindrical $\mathfrak{H}_2$ we have the following theorem.

THEOREM 9. *If vectors $\sigma(g(\mu))$ and $a(\mu)$ are linearly independent, there is at least one H-curve of the function $a(\mu)$ passing through every point of $\mathfrak{H}_2$.*

(²) If we exclude the exceptional case of $\sigma(u) \| \vec{u}$ for $u \in \mathfrak{H}_2$.

Proof. Let a curve $K: x = x(\tau)$, $K \subset R^3$ satisfy the equation $dx/d\tau = a(\mu(\tau))$ for a curve $H \subset \mathfrak{H}_2$ prescribed in a parametric manner $\mu = \mu(\tau)$. A vector $n(t, \tau)$ normal to the surface $M_2(K)$(5.16) has the form

$$n(t, \tau) = \sigma(\mu(\tau)) \wedge \left(t\sigma_{\mu^i} \frac{d\mu^i}{d\tau} + \frac{dx}{d\tau}\right) = t\sigma(\mu(\tau)) \wedge \sigma_{\mu^i} \frac{d\mu^i}{d\tau} + \sigma(\mu(\tau)) \wedge a(\mu(\tau))),$$

where $\sigma(\mu) = \sigma[g(\mu)]$.

Thus the surface $M_2(K)$ is developable for every curve K when only when the vectors $\sigma(\mu(\tau))$, $\sigma_{\mu^i} \dfrac{d\mu^i}{d\tau}$, $a(\mu(\tau))$ are linearly dependent. In other words, H is an H-curve of the function $a(\mu)$ when and only when

$$\Delta_1(\mu) \frac{d\mu^1}{d\tau} + \Delta_2(\mu) \frac{d\mu^2}{d\tau} = 0,$$

where $\Delta_i(\mu) = \det|\sigma_{\mu^i}(\mu),\ a(\mu),\ \sigma(\mu)|$.

If $\Delta_1(\mu) = \Delta_2(\mu) = 0$, every curve $H \subset \mathfrak{H}_2$ is an H-curve of the function $a(\mu)$. In particular this is the case of cylindrical $\mathfrak{H}_2$, because then $\sigma_{\mu^i} = 0$. If $\Delta_1^2 + \Delta_2^2 \neq 0$, there is for every point $u \in \mathfrak{H}_2$ exactly one H-curve of the function $a(\mu)$ passing through that point. This ends the proof of Theorem 9.

It will now be shown that the mixed problem 3 has, for non-cylindrical $\mathfrak{H}_2$, a solution in the class $F(\mathfrak{H}_2)$, if the set $\mathfrak{N} \subset \mathfrak{H}_2$ is an H-curve. To this end let us take as a manifold $\mathscr{L}$ (Fig. 12), a plane and let us consider the relevant mixed problem III_0 implied by the problem 3. Since $\mathscr{L}$ is a plane, the vector $m(\mu)$ prescribed by the Eq. (5.14) depends on μ only. We have the following corollary.

COROLLARY 6. *If for a characteristic surface $\mathfrak{H}_2$ the set $\mathfrak{N} \subset \mathfrak{H}_2$ is an H-curve of the function $m(\mu)$, the mixed problem 3 has a solution in the class $F(\mathfrak{H}_2)$.*

Proof. It suffices to show that the surface (5.15) corresponding to the problem III_0 is developable. For the curve M: $x(\tau) = x(s(\tau))$, we have $dx/d\tau = x_{s^1}h^1 + x_{s^2}h^2 = m(\mu(\tau))$, where $\mathfrak{M} = \{\mu : \mu = \mu(\tau)\}$, $\mathfrak{N} = \{u : u = g(\mu),\ \mu \in \mathfrak{M}\}$. Next, $u(x(s(\tau)))) = g(\mu(\tau))$, therefore the surface (5.15) has the form (5.16) where $dx/d\tau = m(\mu(\tau))$. It follows that the surface (5.15) is developable, because $\mathfrak{N}$ is H-curve of the function $a(\mu)$. This ends the proof of Corollary 6.

6. The ω-property

It was shown in Ref. [1] that in the case of $n = 2$ solutions of the class $F(\mathfrak{G}_2)$, that is solutions constituting an interaction of two independent simple waves, have the ω-property for $\omega < 2$. We shall now prove a theorem which is a generalization of this fact to the case of any n.

THEOREM 10. *A solution $u(x) \in F(\mathfrak{G}_r)$, $r \leqslant n$ has the ω-property for $\omega < r$. The manifold $\overline{M}$ and $\underline{M}$ are determined in an unique manner by the set $u(S_{n-1})$.*

Proof. The following simple lemma will be made use of.

LEMMA 4. *If the maping $u(x)$, $x \in G$, where G is a region or a closed region, satisfies the condition*

$$r\|u^j_{x^l}\|_{x\in G} \equiv \omega,$$

then there exists, for any manifold $S_{n-1} \subset G$, in every neighbourhood $U \subset S_{n-1}$, a neighbourhood $V \subset U$ such that the set $u(V)$ is a manifold having a fixed dimension and

$$\dim u(V) = \omega \quad or \quad \omega - 1.$$

Let S_{n-1} be the common part of the boundaries of the regions D and D_ω involved in the condition (1.2). For vectors $\underset{i}{c}(u)$ $i = 1, \ldots, r$, such that $P^i_{n-r+1}(u) = [\sum_{n-r}(u) | \underset{i}{c}(u)]$ and for any neighbourhood $V \subset S_{n-1}$, the following two cases are the only that may occur:

$$\alpha)\ \underset{i}{c}\big(u(x)\big) \subset T_x(S_{n-1}), \quad x \in V, \quad i = 1, \ldots, r.$$

β) There exist an index i_0 and a point $\underset{\circ}{x} \in V$ such that

$$c_{i_0}\big(u(\underset{\circ}{x})\big) \notin T_{\underset{\circ}{x}}(S_{n-1}).$$

From the assumption $u(x) \in F(\mathfrak{G}_r)$, $r \leqslant n$ follows that $u_{\underset{i}{\sigma}}(x) = 0$ and $u_{\underset{i}{c}}(x) = \alpha(x)\underset{i}{\gamma}$, therefore the manifold S_{n-1} is stratified in the case α) by r families of curves $\underset{i}{C}$ such that $u(\underset{i}{C}) \subset \Gamma_i \subset \mathfrak{H}_r$. There exists a vector $\sigma(u) \in \sum_{n-r}(u)$ such that $\sigma\big(u(x)\big) \notin T_x(S_{n-1})$ and the solution $u(x)$, $x \in D$ must be constant along the straight line $x = t\sigma\big(u(p)\big) + p$, $p \in V$. Thus, by virtue of the Lemma 4, the solution has the ω-property and

$$\underline{M} = \overline{M} = u(V).$$

In the case β) we are concerned with the following possible cases. Either all the characteristic curves Γ_{i_0} having a common point with the manifold $u(V)$ belong to $u(V)$ and $u(x)$ has the ω-property, and $\underline{M}$ and $\overline{M}$ as in the case α), or there exists a neighbourhood $V' \subset V \subset S_{n-1}$ such that the curves Γ_{i_0} have at most one point common with the manifold $u(V')$. Then, passing a curve Γ_{i_0} through every point of the manifold $u(V')$, we obtain a manifold N, $\dim N = 1 + \dim u(V')$. Thus from Lemma 4 it follows that $u(x)$ has the ω-property and

$$\underline{M} = u(V'), \quad \overline{M} = N.$$

This ends the proof of Theorem 10.

The aim of Ref. [1] was to define, in term of interaction of simple waves the classes of solutions having the ω-property in the case of $n = 2$. We have inferred that solutions constituting interactions of independent waves have that property. A jump in the order of the solution by more than one (no ω-property) may occur when the solution is not an interaction of independent simple waves. We have also found that for hyperbolic sets of Eqs. (1.1) $n = 2$ all the solutions are interactions of independent waves.

In the case of any n we are concerned with a similar situation. The only difference is that solutions of hyperbolic systems of equations are not all interactions of independent waves for $n > 2$. There exist wide classes of solutions which are not interactions of independent simple waves. Such solutions may have a jump of the order more than one.

An example will now be used to show that the above phenomenon may occur for $n > 2$ even for strong hyperbolic sets of equations (1.1) with constant coefficients. To this end we will give example of a strong hyperbolic set of equations of the form (1.1), $n = 3$, for which there exist solutions $u(x) \in F$ satisfying the conditions (1.2) for $\omega = 0$, but $r\|u^j_{xi}\| = 3$, $x \in D$. A solution $u(x)$ for $x \in \overline{D}$ will not be an interaction of independent simple waves.

As an example let us consider a linearized set of equations describing a non-stationary plane isentropic gas flow.

$$\begin{aligned} &c_t + Ac_x + Bc_y + kD(u_x + v_y) = 0, \\ (6.1)\qquad &u_t + Au_x + Bu_y + \frac{D}{k} c_x = 0, \\ &v_t + Av_x + Bv_y + \frac{D}{k} c_x = 0, \end{aligned}$$

where A, B, D, k are constant numbers and c, u, v are the sought for functions of the variables (t, x, y). The set of Eqs. (6.1) is strong hyperbolic. If we use the notations $\mathscr{X} = (x^1, x^2, x^3) = (t, x, y)$, $U = (u^1, u^2, u^3) = (c, u, v)$, the vector $\gamma = (\gamma^1, \gamma^2, \gamma^3)$ is a characteristic vector of the Eqs. (6.1) in H^3, if

$$(\gamma^1)^2[(\gamma^1)^2 - k^2[(\gamma^2)^2 + (\gamma^3)^2]] = 0.$$

For $\gamma^1 \neq 0$ we have

$$\gamma \rightleftharpoons \lambda(\gamma) = \left(\frac{D}{k}\gamma^1 + A\gamma^2 + B\gamma^3, -\gamma^2, -\gamma^3\right).$$

Through every point $\underset{\circ}{U} \in H^3$ we can pass three different characteristic curves $\Gamma_1, \Gamma_2, \Gamma_3$ which have a common tangent vector at the point $\underset{\circ}{U}$. For any points $U_1 \in \Gamma_1$, $\underset{2}{U} \in \Gamma_2$, $\underset{3}{U} \in \Gamma_3$ different from $\underset{\circ}{U}$, the vectors $\underset{i}{\gamma}(\underset{i}{U})$ $(i = 1, 2, 3)$ tangent to Γ_i at the point $\underset{i}{U}$ are linearly independent. This follows from the fact that the characteristic curves Γ of the system (6.1) have the form

$$c(s) = k\int_{\underset{\circ}{s}}^{s} \sqrt{\left(\frac{d\psi}{ds}\right)^2 + \left(\frac{d\varphi}{ds}\right)^2}\, ds + \underset{\circ}{c}, \qquad u = \psi(s), \qquad v = \varphi(s),$$

where ψ and φ are arbitrary functions.

By applying the Theorem 1 of Ref. [1] we can easily construct solutions of simple wave type $u_i(x) \in F_1(\Gamma_i)$, $i = 1, 2, 3$ such that

$$u_1(x) = u_2(x) = u_3(x) = \text{const} = \underset{\circ}{U}$$

in the plane $(\lambda(\underset{\circ}{\gamma}), x - \underset{\circ}{x}) = 0$. It is not difficult to verify that the solution

$$u(x) = u_1(x) + u_2(x) + u_3(x)$$

has the properties required.

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