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A PARTIAL DIFFERENTIAL EQUATION CONNECTED WITH THE SIMPLEST CASE OF ABEL'S THEOREM.

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CONSIDER a given cubic curve cut by a line in the points (x_1, y_1), (x_2, y_2), (x_3, y_3); taking the first and second points at pleasure, these determine uniquely the third point. Analytically, the equation of the curve determines y_1 as a function of x_1, and y_2 as a function of x_2: writing in the equation

$$x_3 = \lambda x_1 + (1-\lambda)\, x_2, \quad y_3 = \lambda y_1 + (1-\lambda)\, y_2,$$

we have λ by a simple equation, and thence x_3; viz. x_3 is found as a function of x_1, x_2, and of the nine constants of the equation. Hence forming the derived equations (in regard to x_1, x_2) of the first, second, and third orders, we have $(1+2+3+4=)\, 10$ equations from which to eliminate the 9 constants; x_3, considered as a function of x_1 and x_2, thus satisfies a partial differential equation of the third order, independent of the particular cubic curve.

To obtain this equation it is only necessary to observe that we have, by Abel's theorem,

$$\frac{dx_1}{X_1} + \frac{dx_2}{X_2} + \frac{dx_3}{X_3} = 0,$$

where X_1 is a given function of x_1 and y_1, that is, of x_1; X_2 and X_3 are the like functions of x_2 and x_3 respectively. Hence, considering x_3 as a function of x_1 and x_2, we have

$$\frac{dx_3}{dx_1} = -\frac{X_3}{X_1}, \quad \frac{dx_3}{dx_2} = -\frac{X_3}{X_2},$$

and consequently

$$\frac{dx_3}{dx_1} \div \frac{dx_3}{dx_2} = \frac{X_2}{X_1};$$

where X_2, X_1 are functions of x_2, x_1 respectively: hence taking the logarithm and differentiating successively with regard to x_1 and x_2, we have

$$\frac{d}{dx_1} \frac{d}{dx_2} \log \left(\frac{dx_3}{dx_1} \div \frac{dx_3}{dx_2}\right) = 0,$$

which is the required partial differential equation of the third order.

This differential equation has a simple geometrical signification. Consider three consecutive positions of the line meeting the cubic curve in the points 1, 2, 3; 1′, 2′, 3′; 1″, 2″, 3″ respectively: *quà* equation of the third order, the equation should in effect determine 3″ by means of the other points. And, in fact, the three positions of the line constitute a cubic curve; the nine points are thus the intersections of two cubic curves, or, say, they are an "ennead" of points; any eight of the points thus determine uniquely the ninth point.