942.

ON SEMINVARIANTS.

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I WISH to prove the following negative: a given sharp seminvariant is not in every case obtainable by mere derivation from a form of the same extent and of the next inferior degree. The meaning of the statement will be explained.

According to the general theory developed in Clebsch's *Theorie der binären algebraischen Formen*, Leipzig, 1872, the covariants of a given binary quantic f are all of them obtainable, the covariants of a given degree from those of the next inferior degree, by derivation (Ueberschiebung) of these with f; viz. if the covariants of the next inferior degree are P, Q, &c., then the covariants of the degree in question are all of them included among the forms

$$\begin{array}{lll} (f, P)^0 (=fP), & (f, P)^1, & (f, P)^2, \ldots, \\ (f, Q)^0 \qquad\quad , & (f, Q)^1, & (f, Q)^2, \ldots, \\ \&c., & & \end{array}$$

the index of derivation for (f, P) being at most equal to the degree of f or to that of P, whichever of these is the smaller, and so for Q, &c. The forms thus obtained are far too numerous; but rejecting repetitions, we have a complete system of the covariants of the given degree, viz. every covariant whatever of that degree is a linear function (with numerical multipliers) of the several distinct forms thus obtained by derivation.

We can therefore, by linear combination as above, obtain all the sharp covariants of the given degree, but we may very well have a sharp covariant not included among the several distinct forms thus obtained by derivation, but only expressible as a linear combination of two or more such forms; or say we may very well have a sharp covariant not obtainable by *mere* derivation from the forms of the next inferior degree; and it is important to verify that there are sharp covariants not thus obtainable by mere derivation. I remark that the notion of a sharp covariant does not present itself in Clebsch, and that it will be presently explained.

Passing from a covariant to its leading coefficient which is a seminvariant, the statement may be applied to seminvariants; viz. it is to be verified that there are sharp seminvariants not obtainable by mere derivation from the seminvariants of the next inferior degree. But we have to introduce the notion of "extent" so as to connect the seminvariant with a quantic of some particular order, thus, if the highest letter of the seminvariant is f, we say that the extent is $=5$, and thus connect it with the quintic

$$(1,\ b,\ c,\ d,\ e,\ f \between x,\ y)^5.$$

The notion "sharp" applies to the seminvariants of a given weight. Suppose, for instance, the weight is $=8$; we have a series of initial or non-unitary terms i, cg, df, e^2, &c., and a series of final or power-ending terms e^2, cd^2, b^2d^2, c^4, &c., and we denote by $cg - c^4$ (where observe that here and in all similar cases the $-$ is not a minus sign, but is simply a stroke), the whole series of terms (including cg and c^4) which are in counter-order (CO) subsequent to cg, and in alphabetical order (AO) precedent to c^4; and so in other cases. This being so, arranging the seminvariants with their final terms in AO, we have the seminvariants

$$\begin{array}{l} i\ \ - e^2, \\ cg - cd^2, \\ df - b^2d^2, \\ e^2\ - c^4, \\ \text{\&c.,} \end{array}$$

viz. we have a seminvariant $i - e^2$ containing all or any (in fact, all) of the terms of this set as above defined; a seminvariant $cg - cd^2$ containing all or any of the terms of this set, a seminvariant $df - b^2d^2$ containing all or any of the terms of this set; and so on. These are sharp forms; a seminvariant ending in e^2, must of necessity have the leading term i, and thus belong at least to the octic

$$(1,\ b,\ c,\ d,\ e,\ f,\ g,\ h,\ i \between x,\ y)^8,$$

a seminvariant ending in cd^2 must of necessity have a leading term as high as cg, and thus belong at least to the sextic

$$(1,\ b,\ c,\ d,\ e,\ f,\ g \between x,\ y)^6.$$

Any linear combination of these would be a seminvariant $i - cd^2$, belonging to the octic, but it is not a sharp form; the final term cd^2 does not of necessity imply an initial so high as i (in fact, as we have seen, it only implies the lower initial cg): and so in other cases.

For the quintic $(1,\ b,\ c,\ d,\ e,\ f \between x,\ y)^5$, we have (for the weight 8 and degree 4) the seminvariants $df - b^2d^2$, and $e^2 - c^4$, this last belongs, of course, also to the quartic $(1,\ b,\ c,\ d,\ e \between x,\ y)^4$, it is, in fact, the squared quadrinvariant $(e - 4bd + 3c^2)^2$. I wish to show that $df - b^2d^2$ is not obtainable by mere derivation from the covariants of degree 3 of the quartic.

The quintic and its covariants up to the degree 3 are

$$A = (\begin{array}{|c|c|c|c|c|c|} 1 & 5b & 10c & 10d & 5e & f \end{array} \between x, y)^5,$$

say

$$(1, b, c, d, e, f \between x, y)^5;$$

$$B = (\begin{array}{|c|c|c|} e + 1 & f + 1 & bf + 1 \\ bd - 4 & be - 3 & ce - 4 \\ c^2 + 3 & cd + 2 & d^2 + 3 \end{array} \between x, y)^2,$$

say

$$(e - c^2,\ f - cd,\ bf - d^2 \between x, y)^2;$$

$$C = (c - b^2,\ d - bc,\ e - c^2,\ f - cd,\ bf - d^2,\ cf - de,\ df - e^2 \between x, y)^6,$$

$$D = (ce - c^3,\ cf - c^2d,\ df - cd^2,\ bdf - d^3 \between x, y)^3,$$

$$E = (f - bc^2,\ bf - c^3,\ cf - c^2d,\ df - cd^2,\ ef - d^3,\ f^2 - d^2e \between x, y)^5,$$

$$F = (d - b^3,\ e - b^2c,\ f - bc^2,\ bf - c^3,\ cf - c^2d,\ df - cd^2,$$
$$ef - d^3,\ f^2 - d^2e,\ bf^2 - de^2,\ cf^2 - e^3 \between x, y)^9.$$

Hence all the covariants of the degree 3 are A^3, AB, AC, D, E, F, where

$$AB = (e - c^2,\ f - bc^2,\ bf - c^3,\ cf - c^2d,\ bcf - cd^2,\ bdf - d^3,\ bef - d^2e,\ bf^2 - d^2f \between x, y)^7$$

$$AC = (c - b^2,\ d - bc, \ldots \between x, y)^{11};$$

and the derivatives are

$(A, A^3)^0$, (A, A^3), &c.	weights of leading coefficients are	0, 1, 2, 3, 4, 5
$(A, AB)^0$, (A, AB), &c.	....	4, 5, 6, 7, $\underline{8}$, 9
$(A, AC)^0$, (A, AC), &c.	....	2, 3, 4, 5, 6, 7
$(A, D)^0$, (A, D), &c.	....	6, 7, $\underline{8}$, 9
$(A, E)^0$, (A, E), &c.	....	5, 6, 7, $\underline{8}$, 9, 10
$(A, F)^0$, (A, F), &c.	....	3, 4, 5, 6, 7, $\underline{8}$.

The terms giving rise to a seminvariant of weight 8 are

$$(A, AB)^4 = \left\{\begin{matrix} e - c^2, & f - bc^2, & bf - c^3, & cf - c^2d, & bcf - cd^2 \\ e, & d, & c, & b, & 1 \end{matrix}\right\} = df - c^4,$$

$$(A, D)^2 = \left\{\begin{matrix} ce - c^3, & cf - c^2d, & df - cd^2 \\ c, & b, & 1 \end{matrix}\right\} = df - c^4,$$

$$(A, E)^3 = \left\{\begin{matrix} f - bc^2, & bf - c^3, & cf - c^2d, & df - cd^2 \\ d, & c, & b, & 1 \end{matrix}\right\} = df - c^4,$$

$$(A, F)^5 = \left\{\begin{matrix} d - b^3, & e - b^2c, & f - bc^2, & bf - c^3, & cf - c^2d, & df - cd^2 \\ f, & e, & d, & c, & b, & 1 \end{matrix}\right\} = df - c^4,$$

where to explain the algorithm, I remark, that if

$$A = (A_0,\ A_1,\ A_2, \ldots \between x,\ y)^5 \text{ and } D = (D_0,\ D_1,\ D_2, \ldots \between x,\ y)^3,$$

then

$$(A,\ D)^2 = D_0A_2 - 2D_1A_1 + D_2A_0,$$

represented as above by

$$(A,\ D)^2 = \begin{Bmatrix} D_0, & D_1, & D_2 \\ A_2, & A_0, & A_1 \end{Bmatrix} = \begin{Bmatrix} ce - c^3, & cf - c^2d, & df - cd^2 \\ c & b & 1 \end{Bmatrix}.$$

The result is in every case given as $df - c^4$; in each case there is only a single term $c \cdot c^3$, $= c^4$, and the term in c^4 certainly presents itself. In $(A,\ AB)^4$ there is a single term $d \cdot f$, $= df$, and in $(A,\ D)^2$ a single term df, and thus the term df certainly presents itself: in $(A,\ E)^3$ there are two terms $d \cdot f$, $= df$ and df, and it is conceivable that, inserting the proper numerical coefficients, these might destroy each other: if this were so, the form instead of being $df - c^4$ would be $e^2 - c^4$; and similarly in $(A,\ F)^5$, there are two terms df, $= df$ and df, which it is conceivable might destroy each other, and the form would then be $e^2 - c^4$. But in every case we have the term c^4, and it thus appears that the form $df - b^2d^2$ is not obtainable by mere derivation.

The form in question is in fact obtained by a linear combination of $df - c^4$ and $e^2 - c^4$, viz. writing down the leading coefficients of the covariants B^2 and H, we have

	$3H$	$-2B^2$	$=$
df	3		3
e^2	0	-2	-2
bcf	-9		-9
bde	-15	$+16$	$+1$
c^2e	$+30$	-12	-18
cd^2	-12		-12
b^3f	$+6$		$+6$
b^2ce	-15		-15
b^2d^2	$+42$	-32	-10
bc^2d	-48	$+48$	0
c^4	$+18$	-18	0

viz. the form in question $df - b^2d^2$ is $= 3df - 2e^2 - \ldots - 10b^2d^2$.