941.

NOTE ON THE PARTIAL DIFFERENTIAL EQUATION $Rr + Ss + Tt + U(s^2 - rt) - V = 0$.

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It is well known that this equation, R, S, T, U, V being any functions whatever of (x, y, z, p, q), in the case where u admits of an integral of the form $u = f(v)$ (u, v functions of x, y, z, p, q, and f an arbitrary functional symbol) can be integrated as follows; viz. taking m_1, m_2 as the roots of the quadratic equation

$$m^2 - Sm + RT - UV = 0,$$

(that is, writing $m_1 + m_2 = S$ and $m_1 m_2 = RT - UV$), then, m_1 denoting either root at pleasure, and m_2 the other root of the quadratic equation, if the system of ordinary differential equations

$$\begin{aligned} m_1 dx - R\,dy + U\,dq &= 0, \\ -T\,dx + m_2 dy + U\,dp &= 0, \\ -V\,dx + m_2 dq + R\,dp &= 0, \\ -V\,dy + T\,dq + m_1 dp &= 0, \\ -p\,dx - q\,dy + dz &= 0, \end{aligned}$$

(equivalent to three independent equations) admits of two integrals $u = \text{const.}$ and $v = \text{const.}$, the solution of the given partial differential equation is $u = f(v)$.

In fact, to prove this, we have

$$\begin{aligned} du = {} & \lambda\,(\ \ m_1 dx - R\,dy + U\,dq) \\ & + \mu\,(-T\,dx + m_2 dy + U\,dp) \\ & + \nu\,(-V\,dx + m_2 dq + R\,dp) \\ & + \rho\,(-V\,dy + T\,dq + m_1 dp) \\ & + \sigma\,(-p\,dx - q\,dy + dz), \end{aligned}$$

that is,

$$\frac{du}{dx} = \lambda m_1 - \mu T - \nu V \qquad - \sigma p,$$

$$\frac{du}{dy} = -\lambda R + \mu m_2 \qquad - \rho V - \sigma q,$$

$$\frac{du}{dz} = \qquad \sigma,$$

$$\frac{du}{dp} = \qquad \mu U + \nu R + \rho m_1 \qquad ,$$

$$\frac{du}{dq} = \lambda U \qquad + \nu m_2 + \rho T \qquad ,$$

and thence

$$\frac{du}{dx} + \frac{du}{dz} p + \frac{du}{dp} r + \frac{du}{dq} s = \lambda (m_1 + Us) + \mu (-T + Ur) + \nu (-V + Rr + m_2 s) + \rho (m_1 r + Ts),$$

$$\frac{du}{dy} + \frac{du}{dz} q + \frac{du}{dp} s + \frac{du}{dq} t = \lambda (-R + Ut) + \mu (m_2 + Us) + \nu (Rs + m_2 t) + \rho (-V + m_1 s + Tt),$$

which equations may be represented by

$$\frac{d(u)}{dx} = A\lambda + B\mu + C\nu + D\rho,$$

$$\frac{d(u)}{dy} = A'\lambda + B'\mu + C'\nu + D'\rho;$$

and for λ, μ, ν, ρ writing λ', μ', ν', ρ', we have similarly

$$\frac{d(v)}{dx} = A\lambda' + B\mu' + C\nu' + D\rho',$$

$$\frac{d(v)}{dy} = A'\lambda' + B'\mu' + C'\nu' + D'\rho';$$

whence

$$\frac{d(u)}{dx}\frac{d(v)}{dy} - \frac{d(u)}{dy}\frac{d(v)}{dx} = \begin{vmatrix} A\lambda + B\mu + C\nu + D\rho, & A\lambda' + B\mu' + C\nu' + D\rho' \\ A'\lambda + B'\mu + C'\nu + D'\rho, & A'\lambda' + B'\mu' + C'\nu' + D'\rho' \end{vmatrix}.$$

The determinant is

$$\begin{aligned} = \ & (AD' - A'D)(\lambda\rho' - \lambda'\rho) + (BD' - B'D)(\mu\rho' - \mu'\rho) \\ & + (CD' - C'D)(\nu\rho' - \nu'\rho) + (BC' - B'C)(\mu\nu' - \mu'\nu) \\ & + (CA' - C'A)(\nu\lambda' - \nu'\lambda) + (AB' - A'B)(\lambda\mu' - \lambda'\mu). \end{aligned}$$

The determinants $AD' - A'D$, &c., each of them contain the factor

$$\Theta, = Rr + Ss + Tt + U(s^2 - rt) - V;$$

viz. we have

$$AD' - A'D = m_1\Theta, \quad BC' - B'C = -m_2\Theta,$$
$$BD' - B'D = -T\Theta, \quad CA' - C'A = -R\Theta,$$
$$CD' - C'D = -V\Theta, \quad AB' - A'B = U\Theta,$$

values which give

$$(AD' - A'D)(BC' - B'C) + (BD' - B'D)(CA' - C'A) + (CD' - C'D)(AB' - A'B) = \Theta^2(-m_1m_2 + TR - VU) = 0,$$

as it should be.

Hence, when the partial differential equation $\Theta = 0$ is satisfied, we have

$$\frac{d(u)}{dx}\frac{d(v)}{dy} - \frac{d(u)}{dy}\frac{d(v)}{dx} = 0;$$

and we thence have $u = f(v)$ as the integral of the partial differential equation.

It should be possible to express analytically the conditions in order that the systems of differential equations may have one or each of them two integrals.

It is interesting to remark that, if each of the two systems of ordinary differential equations has only a single integral, these two integrals do *not* lead to the solution of the partial differential equation. Consider, for instance, the case

$$R = 0, \quad S = x + y, \quad T = 0, \quad U = 0, \quad V = p + q;$$

the partial differential equation is here

$$(x + y)s - (p + q) = 0,$$

which has an integral

$$z = (x + y)\{\phi'(x) + \psi'(y)\} - 2\{\phi(x) + \psi(y)\},$$

where ϕ, ψ are arbitrary functions: the equation in m is $m^2 - m(x + y) = 0$, the roots of which are $m = 0$, and $m = x + y$.

For $m_1 = 0$, $m_2 = x + y$, the system of differential equations becomes

$$\begin{aligned} dy &= 0, \\ -(p + q)\,dx + (x + y)\,dq &= 0, \\ -p\,dx + dz &= 0, \end{aligned}$$

which has only the integral $y = \text{const.}$; and similarly for $m_1 = x + y$, $m_2 = 0$, the system becomes

$$\begin{aligned} dx &= 0, \\ -(p + q)\,dy + (x + y)\,dp &= 0, \\ -q\,dy + dz &= 0, \end{aligned}$$

which has only the integral $x = \text{const.}$ And these two integrals $y = \text{const.}$ and $x = \text{const.}$ do not in anywise lead to the integral of the partial differential equation.

I take the opportunity of remarking that the complete system of conditions in order that the differential

$$Adx + Bdy + Cdz + Ddw$$

may be $= MdU$ is as follows: viz. writing

$$A,\ B,\ C,\ D = 1,\ 2,\ 3,\ 4;$$

$$\frac{dB}{dz} - \frac{dC}{dy},\ \frac{dC}{dx} - \frac{dA}{dz},\ \frac{dA}{dy} - \frac{dB}{dx},\ \frac{dA}{dw} - \frac{dD}{dx},\ \frac{dB}{dw} - \frac{dD}{dy},\ \frac{dC}{dw} - \frac{dD}{dz} = 23,\ 31,\ 12,\ 14,\ 24,\ 34,$$

where of course $12 = -21$, &c., and

$$\overline{123} = 1\,.\,23 + 2\,.\,31 + 3\,.\,12,\ \text{\&c.};\ \overline{1234} = 1\,.\,234 - 2\,.\,341 + 3\,.\,412 - 4\,.\,123,$$

is $= 0$ identically;

$$1234 = 12\,.\,34 + 13\,.\,42 + 14\,.\,23,$$

then the conditions equivalent to three independent conditions are

$$\overline{234} = 0,\quad \overline{341} = 0,\quad \overline{412} = 0,\quad \overline{123} = 0,\quad 1234 = 0.$$

In fact, the first four equations are

$$\begin{array}{l} .\quad 2\,.\,34 - 3\,.\,24 + 4\,.\,23 = 0, \\ -1\,.\,34\quad .\quad + 3\,.\,14 + 4\,.\,31 = 0, \\ 1\,.\,24 - 2\,.\,14\quad .\quad + 4\,.\,12 = 0, \\ -1\,.\,23 - 2\,.\,31 - 3\,.\,12\quad .\quad = 0; \end{array}$$

hence, multiplying by 1, 2, 3, 4 respectively and adding, we have the identity $\overline{1234} = 0$, so that these four are equivalent to three independent equations: and multiplying by

$$\begin{array}{l} .\quad 12\,.\,\mu - 31\,.\,\nu + 14\,.\,\rho, \\ -12\,.\,\lambda\quad .\quad + 23\,.\,\nu + 24\,.\,\rho, \\ 31\,.\,\lambda - 23\,.\,\mu\quad .\quad + 34\,.\,\rho, \\ -14\,.\,\lambda - 24\,.\,\mu - 34\,.\,\nu\quad . \end{array}$$

respectively, (where λ, μ, ν, ρ are arbitrary), we have

$$(1\,.\,\lambda + 2\,.\,\mu + 3\,.\,\nu + 4\,.\,\rho)(23\,.\,14 + 31\,.\,24 + 12\,.\,34) = 0,$$

that is,

$$23\,.\,14 + 31\,.\,24 + 12\,.\,34 = 0,\quad \text{or}\quad 1234 = 0,$$

the fifth condition.