922.

NOTE ON THE LUNAR THEORY.

In the Lunar Theory, in whatever way worked out, the values ultimately obtained for the coordinates r, v, y should of course satisfy identically the equations of motion; and that they do so, is the ultimate verification of the correctness of the results obtained. It can hardly be hoped for that such a verification will ever be made for Delaunay's results; and yet it would seem generally that the labour of such a verification of the results to any extent, while exceeding (and possibly greatly exceeding) that of obtaining these results by any method employed for that purpose, ought still to be, so to speak, a labour of the same order. And one can, moreover, imagine the process of verification so arranged as to be a process of mere routine which could be carried out by ordinary computers. But, however this may be, I think it is not without interest to exhibit the verification to a very small extent, viz. to e, m^4.

I think there is an advantage in using capital letters for the arguments, and I accordingly write G (instead of Delaunay's g), to denote the mean anomaly.

The equations of motion are :

$$\frac{d^2 r}{dt^2} - r \left\{ \cos^2 y \left(\frac{dv}{dt} \right)^2 \right\} + \frac{n^2 a^3}{r^2} = \frac{d\Omega}{dr},$$

$$\frac{d}{dt} \left\{ r^2 \cos^2 y \left(\frac{dv}{dt} \right) \right\} = \frac{d\Omega}{dv},$$

$$\frac{d}{dt} \left(r^2 \frac{dy}{dt} \right) + r^2 \sin y \cos y \left(\frac{dv}{dt} \right)^2 = \frac{d\Omega}{dy},$$

$$\Omega = \frac{m' r^2}{r'^3} \left(\tfrac{3}{2} \cos^2 H - \tfrac{1}{2} \right), \text{ where } \cos H = \cos y \cos (v - v'),$$

or say

$$\Omega = \frac{m^2 n^2 a'^3}{r'^3} r^2 \left\{ \tfrac{3}{2} \cos^2 y \cos^2 (v - v') - \tfrac{1}{2} \right\};$$

and thus the equations become

$$\frac{d^2r}{dt^2} - r\left\{\cos^2 y\left(\frac{dv}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\} + \frac{n^2a^3}{r^2} = \frac{m^2n^2a'^3}{r'^3}\, r\,\{3\cos\{\tfrac{1}{2} + \tfrac{1}{2}\cos(2v-2v')\} - \tfrac{1}{2}\},$$

$$\frac{d}{dt}\, r^2\cos^2 y\left(\frac{dv}{dt}\right)^2 = \frac{m^2n^2a'^3}{r'^3}\, r^2\,\{-\tfrac{3}{2}\cos^2 y\,\sin(2v-2v')\},$$

$$\frac{d}{dt}\left(r^2\frac{dy}{dt}\right) + r^2\sin y\cos y\left(\frac{dv}{dt}\right)^2 = \frac{m^2n^2a'^3}{r'^3}\, r^2\,\{-3\sin y\cos y\,\tfrac{1}{2} + \tfrac{1}{2}\cos(2v-2v')\}.$$

To simplify as much as possible, take the Sun's orbit to be circular, i.e. $r' = a'$, $v' = mnt$; also neglect y^2: the first and second equations are

$$\frac{d^2r}{dt^2} - r\left(\frac{dv}{dt}\right)^2 + \frac{n^2a^3}{r^2} = m^2n^2r\,\{\tfrac{1}{2} + \tfrac{3}{2}\cos(2v-2v')\},$$

$$\frac{d}{dt}\,r^2\cdot\left(\frac{dv}{dt}\right) = m^2n^2r^2\,\{-\tfrac{3}{2}\sin(2v-2v')\};$$

and if for convenience of working we write $a = 1$, $n = 1$, then the first equation may be written

$$\frac{1}{r}\frac{d^2r}{dt^2} - \left(\frac{dv}{dt}\right)^2 + \frac{1}{r^3} = m^2\,\{\tfrac{1}{2} + \tfrac{3}{2}\cos(2v-2v')\},$$

and similarly the second equation is

$$\frac{1}{r^2}\frac{d}{dt}\left(r^2\frac{dv}{dt}\right) = -\tfrac{3}{2}m^2\sin(2v-2v');$$

and the third equation may be disregarded.

The two equations should be satisfied by Delaunay's values, putting therein $e' = 0$, $y = 0$; say by the values

$$\frac{1}{r} = 1 + \tfrac{1}{6}m^2 - \tfrac{179}{288}m^4$$
$$+ (m^2 + \tfrac{19}{6}m^3 + \tfrac{131}{18}m^4)\cos 2D$$
$$+ \qquad\qquad (\tfrac{7}{8}m^4)\cos 4D$$
$$+ e(1 - \tfrac{7}{12}m^2)\qquad\qquad \cos G$$
$$+ e(\tfrac{33}{16}m^2)\qquad\qquad \cos(2D+G)$$
$$+ e(\tfrac{15}{8}m + \tfrac{187}{32}m^2)\qquad \cos(2D-G);$$

$$v = t$$
$$+ (\tfrac{11}{8}m^2 + \tfrac{59}{12}m^3 + \tfrac{893}{72}m^4)\sin 2D$$
$$+ \qquad\qquad (\tfrac{201}{256}m^4)\sin 4D$$
$$+ 2e\qquad\qquad \sin G$$
$$+ e(\tfrac{17}{8}m^2)\qquad\qquad \sin(2D+G)$$
$$+ e(\tfrac{15}{4}m + \tfrac{263}{16}m^2)\qquad \sin(2D-G);$$

where

$$D = (1-m)\,t, \qquad G = (1 - \tfrac{3}{4}m^2)\,t, \qquad (v' = mt).$$

The verification for the first equation is

	$\dfrac{1}{r}\dfrac{d^2r}{dt^2} =$	$-\left(\dfrac{dv}{dt}\right)^2 =$	$\dfrac{1}{r^3} =$	$m^2\{-\tfrac{1}{2} - \tfrac{3}{2}(\cos 2v - 2v')\} =$	
Const.		-1	$+1$		$= 0$
		$+\tfrac{1}{2}m^2$	$-\tfrac{1}{2}m^2$		$= 0$
	$+2m^4$	$-\tfrac{121}{32}m^4$	$-\tfrac{9}{32}m^4$	$-\tfrac{33}{16}m^4$	$= 0$
Cos $2D$	$+4m^2$	$-\tfrac{11}{2}m^2$	$+3m^2$	$-\tfrac{3}{2}m^2$	$= 0$
	$+\tfrac{14}{3}m^3$	$-\tfrac{85}{6}m^3$	$+\tfrac{19}{2}m^3$		$= 0$
	$+\tfrac{64}{9}m^4$	$-\tfrac{539}{18}m^4$	$+\tfrac{137}{6}m^4$		$= 0$
Cos $4D$	$+8m^4$	$-\tfrac{161}{16}m^4$	$+\tfrac{33}{8}m^4$	$-\tfrac{33}{16}m^4$	$= 0$
Cos G	e	$-4e$	$3e$		$= 0$
	$-\tfrac{9}{4}em^2$	$+3em^2$	$-\tfrac{3}{4}em^2$		$= 0$
Cos $(2D+G)$	$e\,\tfrac{193}{16}m^2$	$e\left(-\tfrac{73}{4}m^2\right)$	$e\left(\tfrac{147}{16}m^2\right)$	$e\left(-3m^2\right)$	$= 0$
Cos $(2D-G)$	$em\left(\tfrac{15}{8}\right)$	$em\left(-\tfrac{15}{2}\right)$	$em\left(\tfrac{45}{8}\right)$		$= 0$
	$em^2\left(-\tfrac{5}{32}\right)$	$em^2\left(-\tfrac{187}{8}\right)$	$em^2\left(\tfrac{657}{32}\right)$	$em^2\,(3)$	$= 0.$

The verification for the second equation is

	$\dfrac{2}{r}\dfrac{dr}{dt}\cdot\dfrac{dv}{dt} =$	$+\dfrac{d^2v}{dt^2} =$	$+\tfrac{3}{2}m^2\sin(2v - 2v') =$	
Sin $2D$	$+4m^2$	$-\tfrac{11}{2}m^2$	$+\tfrac{3}{2}m^2$	$= 0$
	$+\tfrac{26}{3}m^3$	$-\tfrac{26}{3}m^3$		$= 0$
	$+\tfrac{142}{9}m^4$	$-\tfrac{142}{9}m^4$		$= 0$
Sin $4D$	$+\tfrac{21}{2}m^4$	$-\tfrac{201}{16}m^4$	$+\tfrac{33}{16}m^4$	$= 0$
Sin G	$e\,.\,2$	$+e\,.-2$		$= 0$
	$e\,.\,3m^2$	$+e\,.\,3m^2$		$= 0$
Sin $(2D+G)$	$e\,.\,\tfrac{129}{8}m^2$	$e\,.-\tfrac{153}{8}m^2$	$e\,.\,3m^2$	$= 0$
Sin $(2D-G)$	$e\,.\,\tfrac{15}{4}m$	$e\,.-\tfrac{15}{4}m$		$= 0$
	$e\,.\,\tfrac{71}{16}m^2$	$e\,.-\tfrac{23}{16}m^2$	$e\,.-3m^2$	$= 0\,;$

and the verification is thus completed.

The following intermediate results may be recorded; $\frac{1}{r} = 1 + \chi$,

	$\chi^2 =$	$-\log r =$	$\frac{dv}{dt} =$
	$\frac{19}{36}m^4$	$\frac{1}{6}m^2 - \frac{255}{288}m^4$	$\cdot 1$
Cos $2D$	$+ \frac{1}{3}m^4$	$m^2 + \frac{19}{6}m^3 + \frac{64}{9}m^4$	$\frac{11}{4}m^2 + \frac{85}{12}m^3 + \frac{539}{36}m^4$
Cos $4D$	$+ \frac{1}{2}m^4$	$\frac{5}{8}m^4$	$\frac{201}{64}m^4$
Cos G	$+ \frac{1}{3}em^2$	$e(1 - \frac{3}{4}m^2)$	$e(2 - \frac{3}{2}m^2)$
Cos $2D + G$	$+ em^2$	$e(\frac{25}{16}m^2)$	$e(\frac{51}{8}m^2)$
Cos $2D - G$	$+ em^2,$	$e(\frac{15}{8}m + \frac{171}{32}m^2),$	$e(\frac{15}{4}m + \frac{143}{16}m^2),$

$$\frac{1}{r}\frac{dr}{dt} =$$

Sin $2D$	$2m^3 + \frac{13}{3}m^3 + \frac{71}{9}m^4$
Sin $4D$	$\frac{5}{2}m^4$
Sin G	$e(1 - \frac{3}{2}m^2)$
Sin $(2D + G)$	$e(\frac{75}{16}m^2)$
Sin $(2D - G)$	$e(\frac{15}{8}m + \frac{51}{32}m^2).$

These were, in fact, made use of for finding the foregoing values of r, $\frac{d^2r}{dt^2}$, &c.