918.

ON THE SUBSTITUTION GROUPS FOR TWO, THREE, FOUR, FIVE, SIX, SEVEN, AND EIGHT LETTERS.

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THE substitution groups for two, three, four, and five letters were obtained by Serret: those for six, seven and eight letters have recently been obtained by Mr Askwith. I wish to reproduce these results in a condensed form.

The following table shows for the several cases respectively, the orders of the several groups, and for any order the number of distinct groups. As regards the case of eight letters, the numbers mentioned do not exactly agree with Mr Askwith: he gives a few non-existent groups, and omits some which I have supplied (see *post*, the list of the groups for eight letters); and it is possible that there are other omissions: the several numbers in the column and the sum total of 155 are given subject to correction.

2		3		4		5		6		7		8	
1	2	1	3	1	2	1	5	1	2	2	6	1	2
1		1	6	3	4	2	6	1	3	1	7	8	4
		2		1	8	1	10	4	4	2	10	4	6
				1	12	1	12	3	6	9	12	25	8
				1	24	1	20	6	8	1	14	6	12
				7		1	60	1	9	2	20	1	15
						1	120	1	12	1	21	22	16
						8		1	16	7	24	3	18
								3	18	1	36	4	24
								4	24	1	40	3	30
								2	36	1	42	12	32
								2	48	1	48	5	36
								1	60	3	72	14	48
								1	72	2	120	3	60
								1	120	1	144	3	64
								1	360	1	240	3	72
								1	720	1	2520	10	96
								34		1	5040	3	120
										38		2	144
												1	168
												1	180
												3	192
												1	240
												3	288
												1	336
												3	360
												1	384
												2	576
												3	720
												1	1152
												1	1440
												1	20160
												1	40320
												155	

Here the top line shows the numbers of letters; each second column shows the order of the groups, and each first column the number of groups of the several orders: the sums at the foot of the first columns show therefore the whole number of groups, viz.

$$\frac{\text{No. of letters} = 2,\ 3,\ 4,\ 5,\ \ 6,\ \ 7,\ \ \ 8}{\text{No. of groups} = 1,\ 2,\ 7,\ 8,\ 34,\ 38,\ 155}.$$

In the enumeration of the groups, I use some notations which must be explained. For greater simplicity I omit parentheses, and write *ab, abc, ab.cd, abc.def*, &c., to denote substitutions, viz. *ab* is the interchange of *a* and *b*; *abc* the cyclical change *a* into *b*, *b* into *c*, *c* into *a*; *ab.cd* the combined interchange of *a* and *b* and of *c* and *d*; *abc.def* the combined cyclical changes *a* into *b*, *b* into *c*, *c* into *a*, and *d* into *e*, *e* into *f*, *f* into *d*; and so in other cases.

Again, (*abc*) all, means the complete group of all the substitutions

(1, *abc*, *acb*, *bc*, *ca*, *ab*)

upon the three letters; and so in other cases. In the case, however, of two letters, I write simply (ab) to denote the complete group $(1, ab)$ of substitutions, and so also for any substitution such as $ab\,.\,cd$, where the complete group is $(1, ab\,.\,cd)$, I denote the group by $(ab\,.\,cd)$. Moreover, (abc) cyc., denotes the group of cyclical substitutions $(1, abc, acb)$ upon the three letters; and so in other cases.

A group will in general contain positive and negative substitutions, and, when this is so, the positive substitutions will form a group which is denoted by the symbol, pos.; the negative substitutions (which of course do not form a group) are denoted in like manner by the symbol, neg. Thus, (abc) all, pos., or for shortness, (abc) pos., will denote the group formed by the positive substitutions of (abc) all; (abc) pos. is thus the same thing as (abc) cyc., but obviously $(abcd)$ pos. and $(abcd)$ cyc. have quite different meanings. It is to be noticed that, for any odd number of letters, the substitutions of a group (abc) cyc. are all positive, and thus (abc) cyc. pos. is the original group: for any even number of letters, the group $(abcd)$ cyc. or $(abcdef)$ cyc. contains positive and negative substitutions, but the positive substitutions thereof form a cyclical group, thus

$$(abcd) \text{ cyc. pos.} = (ab\,.\,cd) \text{ cyc.,} \quad (abcdef) \text{ cyc. pos.} = (ace\,.\,bdf) \text{ cyc.,}$$

and the notation, ( ) cyc. pos., is thus unnecessary, and it will not be used.

Substitutions or groups which have no letter in common are said to be independent. The product of two independent groups is of course a group, and the components may be called independent factors of the resultant group: we use for such a product the notation $A\,.\,B$, and call the group a composite group. If each of the groups A and B contain positive and negative substitutions, we thence derive a new group, $(A\,.\,B)$ pos., viz. the substitutions hereof are the products of a positive substitution of A and a positive substitution of B, and the products of a negative substitution of A and a negative substitution of B, say

$$(A\,.\,B) \text{ pos.} = (A \text{ pos. } B \text{ pos.}) + (A \text{ neg.})\,(B \text{ neg.}):$$

obviously the number of substitutions or order of the group $(A\,.\,B)$ pos. is one half of that of the group $(A\,.\,B)$.

A more general, but not perfectly definite, notation is that of $(A\,.\,B)$ dim., the dimidiate of the group $A\,.\,B$. Suppose, for instance, that A is a group of substitutions of the letters (a, b, c, d); and that B is the group (ef). Here if A is composed of two equal sets $1, P, Q, \ldots; R, S, T, \ldots$, where the first set $1, P, Q, \ldots$, is a group, then we have a group $1, P, Q, \ldots, ef\,.\,R, ef\,.\,S, ef\,.\,T, \ldots$, which is a group of the form in question, $(A\,.\,B)$ dim. But in some cases the group A can be in more than one way divided into two equal sets the first of which forms a group, and a further explanation of the notation is required. I do not give at present a more complete explanation, nor explain the analogous notions trisection (tris.), &c.

Substitutions or groups having a letter or letters in common are non-independent. If A, B are such groups, then the substitutions of A are not commutable with those of B, and we have not in general such a group as $B.A$ or $A.B$. It may happen that, although the individual substitutions of A are not commutable with those of B, yet that the groups are commutable, say we have $A.B = B.A$, viz. here the substitutions of $A.B$ are in a different order identical with those of $B.A$. We have in this case a group $A.B$; this is not a composite group; and the notation will never be employed without explanation.

I consider, in particular, the two groups $(abc.def)$ cyc. and (abc) cyc. (def) cyc.; in each of them, the six letters (a, b, c, d, e, f) are divided into two sets (a, b, c), (d, e, f), and the substitutions are the product of an abc-substitution into a def-substitution: viz. in the first group the substitutions are

$$1,\ abc.def,\ acb.dfe,$$

and in the second group they are

$$\begin{array}{ll} 1, & abc, \quad abc.def, \\ & acb, \quad acb.def, \\ & def, \quad abc.dfe, \\ & dfe, \quad acb.dfe. \end{array}$$

We can from each of the groups, introducing substitutions which interchange (a, b, c) with (d, e, f), or say by combination with the group $(ad.be.cf)$, derive a group of double the order; and it is worth while to consider the two cases in detail. First, for the group

$$(ad.be.cf)(abc.def) \text{ cyc.}$$

We have

Arrangements.	Substitutions.
$abcdef$,	1,
$bcaefd$,	$abc.def$,
$cabfde$,	$acb.dfe$,
$defabc$,	$ad.be.cf$,
$efdbca$,	$aecdbf$,
$fdecab$,	$afbdce$,

say

$$(ad.be.cf)(abc.def) \text{ cyc.} = (abcdef)_6.$$

This group of the order 6 is, as it happens, the group $(aecdbf)$ cyc.

It may be remarked that if, instead of $(abc.def)$ cyc., we consider the precisely similar group $(abc.dfe)$ cyc., then operating upon it with the same group $(ad.be.cf)$, we obtain quite a different form of group $(abcdef)_6$. In fact, for $(ad.be.cf)(abc.dfe)$ cyc., we have

Arrangements.	Substitutions.
abcdef,	1,
bcafde,	*abc . dfe*,
cabefd,	*acb . def*,
defabc,	*ad . be . cf*,
efdcab,	*ae . bf . cd*,
fdebca;	*af . bd . ce*;

which is not the cyclical group. See *post*, six letters, ord. $6._3$.

Next, for the group $(ad\,.\,be\,.\,cf)\,(abc)$ cyc. (def) cyc.; we have here

Arrangements.	Substitutions.
abcdef,	1,
bcadef,	*abc*,
cabdef,	*acb*,
abcefd,	*def*,
bcaefd,	*abc . def*,
cabefd,	*acb . def*,
abcfde,	*dfe*,
bcafde,	*abc . dfe*,
cabfde,	*acb . dfe*,
defabc,	*ad . be . cf*,
efdabc,	*aebfcd*,
fdeabc,	*afcebd*,
defbca,	*adbecf*,
efdbca,	*aecdbf*,
fdebca,	*af . bd . ce*,
defcab,	*adcfbe*,
efdcab,	*ae . bf . cd*,
fdecab;	*afbdce*;

say

$$(ad\,.\,be\,.\,cf)\,\{(abc)\text{ cyc. }(def)\text{ cyc.}\} = (abcdef)_{18}.$$

See *post*, six letters, ord. 18.

The groups thus obtained, with substitutions which interchange the two sets of letters, are said to be "woven" groups.

By means of the foregoing notations a large number, but by no means all, of the groups belonging to the several cases of 2, 3, 4, 5, 6, 7, and 8 letters can be represented in a very compendious form. It is, even in the case of 4 letters, proper to introduce special notations: thus for the order 4, we have $(ab)(cd)$, (or as I generally write it $(ac)(bd)$), and $(abcd)$ cyc. (which it is convenient thus to represent), and another group $(1,\ ab.cd,\ ac.bd,\ ad.bc)$: this is a woven group $(ac.bd)(ab)(cd)$, but in thus representing it, we fail to exhibit the symmetrical character of the group, and I prefer to represent it as $(abcd)_4$. Again for the order 8, there is a single group, which I write $(abcd)_8$: this can be, in regard to dimidiation, divided in three different ways into two sets of four, and to distinguish them I write

$$(abcd)_8 \text{ com.} = (1,\ ac,\ bd,\ ac.bd;\ abcd,\ adbc,\ ab.cd,\ ad.bc),$$

$$(abcd)_8 \text{ cyc.} = (1,\ abcd,\ ac.bd,\ adbc;\ ac,\ bd,\ ab.cd,\ ad.bc),$$

$$(abcd)_8 \text{ pos.} = (1,\ ab.cd,\ ac.bd,\ ad.bc;\ ac,\ bd,\ abcd,\ adcb);$$

viz. com. denotes that the first set of four is the composite group $(ac)(bd)$; cyc. that it is the cyclical group $(abcd)$ cyc.; and pos. that it is the positive group $(abcd)_4$. We have thus, in the case of 6 letters, the three dimidiation forms,

$$\{(abcd)_8 \text{ com. } (ef)\} \text{ dim.} = (ac)(bd) + ef(abcd,\ adbc,\ ab.cd,\ ad.bc),$$

$$\{(abcd)_8 \text{ cyc. } (ef)\} \text{ dim.} = (abcd) \text{ cyc.} + ef(ac,\ bd,\ ab.cd;\ ad.bc),$$

$$\{(abcd)_8 \text{ pos. } (ef)\} \text{ dim.} = (abcd)_4 + ef(ac,\ bd,\ abcd,\ adcb),$$

the last of which may be more simply written as $\{(abcd)_8(ef)\}$ pos.

In the cases of 2, 3, 4, and 5 letters, the groups are as follows:

2 letters ord.		3 letters ord.		4 letters ord.		5 letters ord	
2	(ab)	3	(abc) cyc.	2	$(ab.cd)$	5	$(abcde)$ cyc.
		6	(abc) all	4	$(ab)(cd)$	6	(abc) cyc. (de)
				,,	$(abcd)$ cyc.	,,	$\{(abc)$ all $(de)\}$ pos.
				,,	$(abcd)_4$	10	$(abcde)_{10}$
				8	$(abcd)_8$	12	(abc) all (de)
				12	$(abcd)$ pos.	20	$(abcde)_{20}$
				24	$(abcd)$ all	60	$(abcde)$ pos.
						120	$(abcde)$ all

The nature of the group is in most cases at once intelligible from the foregoing explanations: thus (ab) denotes the group $(1,\ ab)$; (abc) cyc., the cyclical group $(1,\ abc,\ acb)$: (abc) all, the group of the six substitutions $(1,\ abc,\ acb,\ ab,\ ac,\ bc)$, and so in other cases: but, when a further explanation is required, the nature of the group is merely indicated by writing down the order as a suffix: thus $(abcd)_4$ means a group of four substitutions, $(abcd)_8$ a group of 8 substitutions, and so in other cases. In the case of four letters, the necessary explanations have already been given: it may be remarked that the group $(abcd)_8$ is that of the substitutions which leave unaltered the three-valued function $ac + bd$.

Five letters. Explanations. Five letters.

ord. 10. $(abcde)_{10}$. This may be written $\{(abcde)_{20}\}$ pos., viz. it consists of the 10 positive substitutions out of the next-mentioned group of 20 substitutions. Referring to that group, and writing $\sigma = S^2 = ac\,.\,bd$, and $T = aecdb$, the present group is $(1,\ \sigma)(1,\ T,\ T^2,\ T^3,\ T^4)$, where $\sigma^2 = 1$, $T^5 = 1$, $\sigma T = T^4\sigma$, $\sigma T^4 = T\sigma$, $\sigma T^3 = T^2\sigma$, $\sigma T^2 = T^3\sigma$: or, retaining S^2 instead of σ, say the group is $(1,\ S^2)(1,\ T,\ T^2,\ T^3,\ T^4)$.

ord. 20. $(abcde)_{20}$. The substitutions, distinguishing the positive and the negative ones, are

+	+	+	−
1,	$ab\,.\,de$,	$abdce$,	$abcd$,
	$ac\,.\,bd$,	$acbed$,	$abec$,
	$ad\,.\,ce$,	$adebc$,	$acde$,
	$ae\,.\,bc$,	$aecdb$,	$aceb$,
	$be\,.\,cd$,		$adbe$,
			$adcb$,
			$aebd$,
			$aedc$,
			$bced$,
			$bdec$;

and here, if $S = abcd$, $T = aecdb$, then the group is $(1,\ S,\ S^2,\ S^3)(1,\ T,\ T^2,\ T^3,\ T^4)$, where $S^4 = 1$, $T^5 = 1$, and generally $S^\alpha T^\beta = T^{\beta\,.\,2^\alpha} S^\alpha$; that is,

	1	T	T^2	T^3	T^4
1	1	T	T^2	T^3	T^4
S	S	T^2S	T^4S	TS	T^3S
S^2	S^2	T^4S^2	T^3S^2	T^2S^2	TS^2
S^3	S^3	T^3S^3	TS^3	T^4S^3	T^2S^3

read

$$ST = T^2S,$$
$$ST^2 = T^4S, \text{ \&c.}$$

In the cases of 6, 7, and 8 letters, where the number of groups is larger, I introduce current numbers for the groups of the same order: thus, *infra*, 6 letters order 4, we have the groups 4.1, 4.2, 4.3, 4.4; and so in other cases, where there is more than one group for the same order.

Six letters.

Six letters: the groups are

ord.

2. $(ab\,.\,cd\,.\,ef)$,

3. $(abc\,.\,def)$ cyc.

4. 1 $(ab\,.\,cd)\,(ef)$,

2 $\{(ab)\,(cd)\,(ef)\}$ pos.

3 $\{(abcd)$ cyc. $(ef)\}$ pos.

4 $\{(abcd)_4\,(ef)\}$ dim.

6. 1 $(abcdef)$ cyc.

2 $(abc\,.\,def)$ all,

3 $(ad\,.\,bf\,.\,ce)\,(abc\,.\,def)$ cyc.

8. 1 $(ab)\,(cd)\,(ef)$,

2 $(abcd)$ cyc. (ef),

3 $(abcd)_4\,(ef)$,

4 $\{(abcd)_8$ com. $(ef)\}$ dim.

5 $\{(abcd)_8$ cyc. $(ef)\}$ dim.

6 $\{(abcd)_8$ pos. $(ef)\}$ dim. $= \{(abcd)_8\,(ef)\}$ pos.

9. (abc) cyc. (def) cyc.

12. $(abcdef)_{12}$,

16. $(abcd)_8\,(ef)$,

18. 1 (abc) all (def) cyc.,

2 $\{(abc)$ all (def) all$\}$ pos.,

3 $(ad\,.\,be\,.\,cf)\,\{(abc)$ cyc. (def) cyc.$\}$,

24. 1 $(abcd)$ pos. (ef),

2 $\{(abcd)$ all $(ef)\}$ pos.,

4 $(+\,abcdef)_{24}$,

3 $(\pm\,abcdef)_{24}$,

The $\pm$ and $+$ are used for distinction, the $\pm$ showing that the group is one with positive and negative substitutions, the $+$ that it is a group with positive substitutions only.

36. 1 (abc) all (def) all,

2 $(abcdef)_{36}$,

48. 1 $(abcd)$ all (ef),

2 $(abcdef)_{48}$,

60. $(abcdef)_{60}$,

72. $(abcdef)_{72}$,

120. $(abcdef)_{120}$,

360. $(abcdef)$ pos.,

720. $(abcdef)$ all.

Six letters. Explanations. Six letters.

ord. 6. 2. $(abc\,.\,def)$ all; the substitutions are those of (abc) all each, compounded with the corresponding substitution of (def) all; viz. they are

$$\begin{array}{ll} 1, & ab\,.\,de, \quad abc\,.\,def, \\ & ac\,.\,df, \quad acb\,.\,dfe, \\ & bc\,.\,ef. \end{array}$$

ord. 6. 3. $(ad\,.\,bf\,.\,ce)\,(abc\,.\,def)$ cyc. This is, in fact, the group $(ad\,.\,be\,.\,cf)\,(abc\,.\,dfe)$ cyc. explained in the introductory paragraphs; only for convenience the letters e, f have been interchanged. Making this interchange, the substitutions of the group $(ad\,.\,bf\,.\,ce)\,(abc\,.\,def)$ cyc. are

$$\begin{array}{lll} 1, & abc\,.\,def, & ad\,.\,bf\,.\,ce, \\ & acb\,.\,dfe; & ae\,.\,bd\,.\,cf, \\ & & af\,.\,be\,.\,cd. \end{array}$$

It may be added that, writing $\beta = ad\,.\,bf\,.\,ce$ and $\theta = abc\,.\,def$, the group is $(1, \beta)(1, \theta, \theta^2)$, where $\beta^2 = 1$, $\theta^3 = 1$, and $\beta\theta = \theta^2\beta$, $\beta\theta^2 = \theta\beta$. If $U = adbecf$, then $U^2 = abc\,.\,def = \theta$, and the form is $(1, \beta)(1, U^2, U^4)$: see $(abcdef)_{12}$, *infra*.

ord. 8. 4, 5, 6; the nature of these groups is explained in the introductory paragraphs.

ord. 12. $(abcdef)_{12}$. The substitutions are

$$\begin{array}{lllll} 1, & ab\,.\,ef, & ad\,.\,bf\,.\,ce, & abc\,.\,def, & adbecf, \\ & ac\,.\,de, & af\,.\,be\,.\,cd, & acb\,.\,dfe, & afcebd; \\ & bc\,.\,df, & ae\,.\,bd\,.\,cf, & & \\ & & ae\,.\,bf\,.\,cd, & & \end{array}$$

writing $U = adbecf$ and $\beta = ad\,.\,bf\,.\,ce$, this is

$$(1, \beta)(1, U, U^2, U^3, U^4, U^5),$$

where

$$U^6 = 1, \quad \beta^2 = 1, \quad \beta U = U^5\beta,$$

and thence

$$\beta(1, U, U^2, U^3, U^4, U^5) = (1, U^5, U^4, U^3, U^2, U)\,\beta.$$

ord. 18. $(ad\,.\,be\,.\,cf)\,\{(abc)$ cyc. (def) cyc.$\}$; this has been explained in the introductory paragraphs. Arranging them in a more convenient order, the substitutions are

$$\begin{array}{lllll} 1, & abc, & abc\,.\,def, & ad\,.\,be\,.\,cf, & adcfbe, \\ & acb, & abc\,.\,dfe, & ae\,.\,bf\,.\,cd, & adbecf, \\ & def, & acb\,.\,def, & af\,.\,bd\,.\,ce; & aebfcd, \\ & dfe; & acb\,.\,dfe; & & aecdbf, \\ & & & & afbdce, \\ & & & & afcebd. \end{array}$$

Six letters. ord. 24. $_3$. $(\pm\, abcdef)_{24}$: the substitutions are

+	+	+	−	−
1,	$ac\,.\,bd$,	$abe\,.\,cdf$,	$abcd$,	$ab\,.\,cd\,.\,ef$,
	$ac\,.\,ef$,	$abf\,.\,cde$,	$adcb$,	$ac\,.\,be\,.\,df$,
	$bd\,.\,ef$,	$ade\,.\,bfc$,	$aecf$,	$ac\,.\,bf\,.\,de$,
		$adf\,.\,bce$,	$afce$,	$ad\,.\,bc\,.\,ef$,
		$aeb\,.\,cfd$,	$bedf$,	$ae\,.\,bd\,.\,fc$,
		$aed\,.\,bcf$,	$bfde$,	$af\,.\,bd\,.\,ce$,
		$afb\,.\,ced$,		
		$afd\,.\,bce$.		

This includes, as part of itself, the group

$$\{(abcd)_8 \text{ cyc. } (ef)\} \text{ dim.};$$

and it may be written

$$(ade\,.\,bcf) \text{ cyc. } \{(abcd)_8 \text{ cyc. } (ef)\} \text{ dim.}$$

24. $_4$. $(+\, abcdef)_{24}$; the substitutions are

1,	$ab\,.\,cd$,	$abe\,.\,cdf$,	$abcd\,.\,ef$,
	$ac\,.\,bd$,	$abf\,.\,cde$,	$adcb\,.\,ef$,
	$ac\,.\,ef$,	$ade\,.\,bfc$,	$aecf\,.\,bd$,
	$ad\,.\,bc$,	$adf\,.\,bec$,	$afce\,.\,bd$,
	$ae\,.\,cf$,	$aeb\,.\,cfd$,	$bedf\,.\,ac$,
	$af\,.\,ce$,	$aed\,.\,bcf$,	$bfde\,.\,ac$,
	$bd\,.\,ef$,	$afb\,.\,ced$,	
	$be\,.\,df$,	$afd\,.\,bce$,	
	$bf\,.\,de$.		

This includes, as part of itself, the group $\{(abcd)_8\,(ef)\}$ pos.; and it may be written $(ade\,.\,bfc)$ cyc. $\{(abcd)_8\,(ef)\}$ pos. It consists of the positive terms of $(abcdef)_{48}$, and might thus be written $(abcdef)_{48}$ pos.

ord. 36. $(abcdef)_{36}$; this consists of the positive terms of $(abcdef)_{72}$. The group consists of the substitutions which leave unaltered $b-c\,.\,c-a\,.\,a-b\,.\,e-f\,.\,f-d\,.\,d-e$.

Six letters.

ord. 48. $(abcdef)_{48}$. The substitutions are

+	+	+	+	−	−	−	−
1,	ab . cd,	abe . cdf,	abcd . ef,	ac,	abcd,	ab . cd . ef,	abecfd,
	ac . bd,	abf . cde,	adcb . ef,	bd,	adcb,	ac . bd . ef,	abecdf,
	ac . ef,	ade . cbf,	aecf . bd,	ef;	aecf,	ac . be . df,	abfcde,
	ad . bc,	adf . cbe,	afce . bd,		afce,	ac . bf . de,	adecbf,
	ae . cf,	aeb . cfd,	bedf . ac,		bedf,	ad . bc . ef,	adfcbe,
	af . ce,	aed . cfb,	bfde . ac;		bfde;	ae . cf . bd,	aedcfb,
	bd . ef,	afb . ced,				af . ce . bd;	afbced,
	be . df,	afd . ceb;					afdceb;
	bf . de.						

ord. 60. $(abcdef)_{60}$; this consists of the positive terms of $(abcdef)_{120}$.

ord. 72. $(abcdef)_{72}$. The substitutions are

+	+	+	+	+	−	−	−	−
1,	abc,	abc . def,	ab . de,	ad . becf,	ab,	ab . def,	ad . be . cf,	adbecf,
	acb,	abc . dfe,	ab . df,	ad . bfce,	ac,	ab . dfe,	ad . bf . ce,	adbfce,
	def,	acb . def,	ab . ef,	ae . bdcf,	bc,	ac . def,	ae . bd . cf,	adcebf,
	dfe;	acb . dfe;	ac . de,	ae . bfcd,	de,	ac . dfe,	ae . bf . cd,	adcfbe,
			ac . df,	af . bdce,	df,	bc . def,	af . bd . ce,	aebdcf,
			ae . cf,	af . becd,	ef;	bc . dfe,	af . be . cd;	aebfcd,
			bc . de,	bd . aecf,		de . abc,		aecdbf,
			bc . df,	bd . afce,		de . acb,		aecfbd,
			bc . ef;	be . adcf,		df . abc,		afbdce,
				be . afcd,		df . acb,		afbecd,
				bf . adce,		ef . abc,		afcdbe,
				bf . aecd,		ef . acb;		afcebd;
				cd . aebf,				
				cd . afbe,				
				ce . adbf,				
				ce . afbd,				
				cf . adbe,				
				cf . aebd.				

This contains, as part of itself, the group (abc) all (def) all; and it may be written $(ad \,.\, be \,.\, cf)\,\{(abc)$ all (def) all$\}$. It leaves unaltered the function $abc + def$.

Six letters.

ord. 120. $(abcdef)_{120}$. The substitutions are

	+	+	+	+	−	−	−
	1,	ab . cf,	abcef,	abc . dfe,	abcd,	abcfde,	ab . cd . ef,
		ab . de,	abdce,	abd . cfe,	abdf,	abdefc,	ab . ce . df,
		ac . bd,	abefd,	abe . cdf,	abec,	abedcf,	ac . be . df,
		ac . ef,	abfdc,	abf . cde,	abfe,	abfced,	ac . bf . de,
		ad . bf,	acbed,	acb . def,	acbf,	acbdfe,	ad . bc . ef,
		ad . ce,	acdfb,	acd . bfe,	acde,	acdbef,	ad . be . cf,
		ae . bc,	acedf,	ace . bfd,	aceb,	acefbd,	ae . bd . cf,
		ae . df,	acfbe,	acf . bde,	acfd,	acfedb,	ae . bf . cd,
		af . be,	adbcf,	adb . cef,	adbe,	adbfec,	af . bc . de,
		af . cd,	adbec,	adc . bef,	adcb,	adcebf,	af . bd . ce;
		bc . df,	adcfe,	ade . bfc,	adef,	adecfb,	
		bd . ef,	adfeb,	adf . bec,	adfc,	adfbce,	
		be . cd,	aebdf,	aeb . cfd,	aebd,	aebcdf,	
		bf . ce,	aebfc,	aec . bdf,	aecd,	aecbfd,	
		cf . de;	aefcd,	aed . bcf,	aecf,	aedfcb,	
			aecdb,	aef . bdc,	aefb,	aefdbc,	
			afbde,	afb . ced,	afbc,	afbecd,	
			afcdb,	afc . bed,	afce,	afcdeb,	
			afdec,	afd . bce,	afdb,	afdcbe,	
			afecb,	afe . bcd;	afed,	afebdc;	
			bcdef,		bced,		
			bdfce,		bcfe,		
			becfd,		bdcf,		
			bfedc;		bdec,		
					bedf,		
					befc,		
					bfcd,		
					bfde,		
					cdfe,		
					cefd.		

This is the remarkable group giving rise to a six-valued function of six letters, not symmetrical in regard to five of the letters. Such a function is Six letters.

$$\begin{gathered} ab\,.\,cf\,.\,de, \\ ac\,.\,db\,.\,ef, \\ ad\,.\,ec\,.\,fb, \\ ae\,.\,fd\,.\,bc, \\ af\,.\,be\,.\,cd; \end{gathered}$$

viz. this denotes any symmetrical function of the five functions $ab\,.\,cf\,.\,de$, &c., where $ab\,.\,cf\,.\,de$ is a symmetrical function of ab, cf, de, these denoting the same symmetrical functions of a and b, of c and f, and of d and e, respectively.

Writing $S = abcd$, $T = aecdb$, $U = abfced$, the group $(1, S, S^2, S^3)(1, T, T^2, T^3, T^4)$ is convertible with the group $(1, U, U^2, U^3, U^4, U^5)$, as may be verified by means of the diagram:

	1	U	U^2	U^3	U^4	U^5
1	1	$U\,.\,1$	$U^2\,.\,1$	$U^3\,.\,1$	$U^4\,.\,1$	U^5
T	T	$U^3\,.\,S^2$	$U\,.\,S^3T^2$	$U^5\,.\,S^2T^2$	$U^2\,.\,T^4$	$U^4\,.\,ST^4$
T^2	T^2	$U^5\,.\,T^3$	$U^3\,.\,ST^2$	$U^4\,.\,S^3T^3$	$U\,.\,S^3T$	$U^2\,.\,ST$
T^3	T^3	$U^4\,.\,ST^2$	$U^5\,.\,S^3T^3$	$U^2\,.\,S^3T$	$U^3\,.\,ST$	$U\,.\,T^2$
T^4	T^4	$U^2\,.\,ST^4$	$U^4\,.\,T$	$U\,.\,S^2$	$U^5\,.\,S^3T^2$	$U^3S^2\,.\,T^2$
S	S	$U^3\,.\,ST^3$	$U^2\,.\,S^3T^4$	$U^4\,.\,S^2T^3$	$U^5\,.\,S^2T^4$	$U\,.\,S^3$
ST	ST	$U^4\,.\,T^2$	$U^3\,.\,T^3$	$U\,.\,ST^2$	$U^2\,.\,S^3T^3$	$U^5\,.\,S^3T$
ST^2	ST^2	$U\,.\,S^3T^3$	$U^4\,.\,S^3T$	$U^5\,.\,ST$	$U^3\,.\,T^2$	$U^2\,.\,T^3$
ST^3	ST^3	$U^5\,.\,S^3T^4$	$U\,.\,S^2T^3$	$U^2\,.\,S^2T^4$	$U^4\,.\,S^3$	$U^3\,.\,S$
ST^4	ST^4	$U^2\,.\,T$	$U^5\,.\,S^2$	$U^3\,.\,S^3T^2$	$U\,.\,S^2T^2$	$U^4\,.\,T^4$
S^2	S^2	$U^4\,.\,S^3T^2$	$U^2\,.\,S^2T^2$	$U^5\,.\,T^4$	$U\,.\,ST^4$	$U^3\,.\,T$
S^2T	S^2T	$U^5\,.\,S^3T$	$U^4\,.\,S^2T$	$U^3\,.\,S^2T$	$U^2\,.\,S^2T$	$U\,.\,S^2T$
S^2T^2	S^2T^2	$U^3\,.\,T^4$	$U^5\,.\,ST^4$	$U\,.\,T$	$U^4\,.\,S^2$	$U^2\,.\,S^3T^2$
S^2T^3	S^2T^3	$U\,.\,S^2T^4$	$U^3\,.\,S^3$	$U^2\,.\,S$	$U^5\,.\,ST^3$	$U^4\,.\,S^3T^4$
S^2T^4	S^2T^4	$U^2\,.\,S^3$	$U\,.\,S$	$U^4\,.\,ST^3$	$U^3\,.\,S^3T^4$	$U^5\,.\,S^2T^3$
S^3	S^3	$U^5\,.\,S$	$U^2\,.\,ST^3$	$U\,.\,S^3T^4$	$U^3\,.\,S^2T^3$	$U^4\,.\,S^2T^4$
S^3T	S^3T	$U\,.\,ST$	$U^5\,.\,T^2$	$U^4\,.\,T^3$	$U^2\,.\,ST^2$	$U^3\,.\,S^3T^3$
S^3T^2	S^3T^2	$U^4\,.\,S^2T^2$	$U\,.\,T^4$	$U^3\,.\,ST^4$	$U^5\,.\,T$	$U^2\,.\,S^2$
S^3T^3	S^3T^3	$U^3\,.\,S^3T$	$U^4\,.\,ST$	$U^2\,.\,T^2$	$U\,.\,T^3$	$U^5\,.\,ST^2$
S^3T^4	S^3T^4	$U^2\,.\,S^2T^3$	$U^3\,.\,S^2T^4$	$U^5\,.\,S^3$	$U^4\,.\,S$	$U\,.\,ST^3$

Six letters. and the group is thus

$$(1, \ldots, S^3)(1, \ldots, T^4)(1, \ldots, U^5) \text{ or } (1, \ldots, U^5)(1, \ldots, S^3)(1, \ldots, T^4),$$

or, since $(1, \ldots, S^3)(1, \ldots, T^4)$ is a group, we may invert the order of the factors $(1, \ldots, S^3)$ and $(1, \ldots, T^4)$.

But it is noticeable that we cannot write the group as

$$(1, \ldots, S^3)(1, \ldots, U^5)(1, \ldots, T^4) \text{ or } (1, \ldots, T^4)(1, \ldots, U^5)(1, \ldots, S^3).$$

The 120 substitutions of either of these products are not the 120 substitutions of the group in question; but some of these are missing altogether, and others of them occur twice repeated. And it is to be remarked also that

$$(1, \ldots, S^3)(1, \ldots, U^5) \text{ and } (1, \ldots, T^4)(1, \ldots, U^5)$$

are neither of them a group. To illustrate this, I give the following fragment of a table

	STU	*SUT*	do.	*TUS*	do.
abcdef	000	000		000	
abdfce	033	122		330,	412
abecfd	124	032,	251	343	
abfedc	105	150,	331	431,	013
acbfed	323	111		453	
acdefb	044	—		440	
adefbc	113	243		—	

viz. here the second line denotes that the substitution *abdfce*, which is $S^0T^3U^3$, is in one way SUT, viz. it is $S^1U^2T^2$: and in two ways TUS, viz. it is $T^3U^3S^0$ and also $T^4U^1S^2$. But *acdefb*, which is $S^0T^4U^4$, is not in any way SUT; and so *adefbc*, which is $S^1T^1U^3$, is not in any way TUS.

Seven letters. Seven letters.

ord.

6. 1 $(ac \,.\, bd)(efg)$ cyc.
 2 $\{(ac \,.\, bd)(efg)$ all$\}$ dim.

7. $(abcdefg)$ cyc.

10. 1 $(abcde)$ cyc. (fg),
 2 $\{(abcde)_{10}(fg)\}$ dim.

12. 1 $(ac \,.\, bd)(efg)$ all,
 2 $(ab)(cd)(efg)$ cyc.
 3 $(abcd)$ cyc. (efg) cyc.
 4 $(abcd)_4(efg)$ cyc.
 5 $\{(ab)(cd)(efg)$ all$\}$ pos.
 6 $\{(abcd)$ cyc. (efg) all$\}$ pos.
 7 $\{(abcd)_4(efg)$ all$\}$ dim.
 8 $\{(abc)$ all $(de)\}$ pos. (fg),
 9 $\{(abcd)$ pos. (efg) cyc.$\}$ tris.

Seven letters.

14. $(abcdefg)_{14}$,

20. 1 $(abcde)_{10}\,(fg)$,

2 $\{(abcde)_{20}\,(fg)\}$ pos.

21. $(abcdefg)_{21}$,

24. 1 $(ab)\,(cd)\,(efg)$ all,

2 $(abcd)$ cyc. (efg) all,

3 $(abcd)_4\,(efg)$ all,

4 $(abcd)_8\,(efg)$ cyc.

5 $\{(abcd)_8$ com. (efg) all$\}$ dim.

6 $\{(abcd)_8$ cyc. (efg) all$\}$ dim.

7 $\{(abcd)_8$ pos. (efg) all$\}$ dim.

36. $(abcd)$ pos. (efg) cyc.

40. $(abcde)_{20}\,(fg)$,

42. $(abcdefg)_{42}$,

48. $(abcd)_8\,(efg)$ all,

72. 1 $(abcd)$ all (efg) cyc.

2 $(abcd)$ pos. (efg) all,

3 $\{(abcd)$ all (efg) all$\}$ pos.

120. 1 $(abcde)$ pos. (fg),

2 $\{(abcde)$ all $(fg)\}$ pos.

144. $(abcd)$ all (efg) all,

240. $(abcde)$ all (fg),

2520. $(abcdefg)$ pos.

5040. $(abcdefg)$ all.

Seven letters. Explanations.

ord. 12. 9. $\{(abcd)$ pos. (efg) cyc.$\}$ tris. The substitutions are

$$\begin{array}{lll} 1, & ab\,.\,cd, & abc\,.\,efg, \\ & ac\,.\,bd, & acd\,.\,efg, \\ & ad\,.\,bc, & adb\,.\,efg, \\ & & bdc\,.\,efg, \\ & & acb\,.\,egf, \\ & & adc\,.\,egf, \\ & & abd\,.\,egf, \\ & & bcd\,.\,egf; \end{array}$$

Seven letters. viz. the 12 substitutions of $(abcd)$ pos. are divided into fours, which are connected with the substitutions 1, efg, egf respectively.

ord. 14. $(abcdefg)_{14}$: this is $(1, P^3)(1, Q, Q^2, Q^3, Q^4, Q^5, Q^6)$, where $P^3 = ad.be.cf$, $Q = agdecfb$, $(P^3)^2 = 1$, $Q^7 = 1$, $P^3Q = Q^5P^3$, see *infra*, ord. 42. The substitutions are

1,	$abfcedg$,	$ab.cd.fg$,
	$acgfdbe$,	$ac.bf.eg$,
	$adcbgef$,	$ad.be.cf$,
	$aebdfgc$,	$ae.bc.dg$,
	$afegbcd$,	$af.cg.de$,
	$agdecfb$,	$ag.bd.ef$,
		$bg.ce.df$.

ord. 21. $(abcdefg)_{21}$; this is $(1, P^2, P^4)(1, Q, Q^2, Q^3, Q^4, Q^5, Q^6)$, where

$$P^3 = acb.dfe,\ Q = agdecfb,\ (P^2)^3 = 1,\ Q^7 = 1,\ P^3Q = Q^5P^2;$$

see *infra* order 42. The substitutions are the positive substitutions of $(abcdefg)_{42}$.

ord. 24. The last three groups correspond to the before-mentioned three modes of division of $(abcd)_8$ into two sets of four, viz. we have,

24. 5; $\{(abcd)_8$ com. (efg) all$\}$ dim. $= \{(ac)(bd)\}(efg)$ pos.
$+ (abcd,\ adbc,\ ab.cd,\ ad.bc)(efg)$ neg.

24. 6; $\{(abcd)_8$ cyc. (efg) all$\}$ dim. $= (abcd)$ cyc. (efg) pos.
$+ (ac,\ bd,\ ab.cd,\ ad.bc)(efg)$ neg.

24. 7; $\{(abcd)_8$ pos. (efg) all$\}$ dim. $= (abcd)_4\ (efg)$ pos.
$+ (ac,\ bd,\ abcd,\ adbc)(efg)$ neg.

ord. 42. $(abcdefg)_{42}$: this is

$$(1, P, P^2, P^3, P^4, P^5)(1, Q, Q^2, Q^3, Q^4, Q^5, Q^6),$$

where $P = aecdbf$, $Q = agdecfb$, $P^6 = 1$, $Q^7 = 1$, and $PQ = Q^5P$. The substitutions are

+	+	+	−	−
1,	$abc.def$,	$abfcedg$,	$ab.cd.fg$,	$abegdf$,
	$abd.cge$,	$acgfdbe$,	$ac.bf.eg$,	$abgcfe$,
	$acb.dfe$,	$adcbgef$,	$ad.be.cf$,	$acdebg$,
	$acf.bdg$,	$aebdfgc$,	$ae.bc.dg$,	$acefgd$,
	$adb.ceg$,	$afegbcd$,	$af.cg.de$,	$adfbcg$,
	$ade.bfg$,	$agdecfb$,	$ag.bd.ef$,	$adgfec$,
	$aed.bgf$,		$bg.ce.df$,	$aecdbf$,
	$aeg.cfd$,			$aefcgb$,

Seven letters.

afc . bgd,	*afbdce*,
afg . bec,	*afdgeb*,
age . cdf,	*agbedc*,
agf . bce,	*agcbfd*,
bef . cdg,	*bcfged*,
bfe . cgd,	*bdegfc*.

The table for the combination of the powers of P and Q is

$$\begin{array}{c|ccccccc}
 & 1 & Q & Q^2 & Q^3 & Q^4 & Q^5 & Q^6 \\
\hline
1 & (1, & Q, & Q^2, & Q^3, & Q^4, & Q^5, & Q^6)\, P, \\
P & (1, & Q^5, & Q^3, & Q, & Q^6, & Q^4, & Q^2)\, P, \\
P^2 & (1, & Q^4, & Q, & Q^5, & Q^2, & Q^6, & Q^3)\, P^2, \\
P^3 & (1, & Q^6, & Q^5, & Q^4, & Q^3, & Q^2, & Q\,)\, P^3, \\
P^4 & (1, & Q^2, & Q^4, & Q^6, & Q, & Q^3, & Q^5)\, P^4, \\
P^5 & (1, & Q^3, & Q^6, & Q^2, & Q^5, & Q, & Q^4)\, P^5;
\end{array}$$

read

$$PQ = Q^5P, \quad PQ^2 = Q^3P, \text{ \&c.}$$

This completes the explanations as to the groups of seven letters.

I proceed to consider the substitution groups of eight letters. The abbreviation A. refers to Mr Askwith's paper (*Quart. Journal of Math.*, vol. XXIV. (1890), pp. 263--331). The list is

Eight letters.

ord.

2. $_1$ $(ab\,.\,cd\,.\,ef\,.\,gh)$.

4. $_1$ $(ab\,.\,cd\,.\,ef)\,(gh)$,
   $_2$ $(ab\,.\,cd)\,(ef\,.\,gh)$,
   $_3$ $\{(ab)\,(cd)\,(ef\,.\,gh)\}$ dim.,
   $_4$ $\{(abcd)$ cyc. $(ef\,.\,gh)\}$ dim.,
   $_5$ $\{(abcd)_4\,(ef\,.\,gh)\}$ dim.,
   $_6$ $(abcd\,.\,efgh)$ cyc.,
   $_7$ $\{(abcd)_4\,(ef\,.\,gh)\}$ dim.,
   $_8$ $(abcdefgh)_4$. Substitutions are

$$\begin{aligned} 1,\ & ae\,.\,bf\,.\,cg\,.\,dh, \\ & ac\,.\,eg\,.\,bd\,.\,fh, \\ & ag\,.\,ce\,.\,bh\,.\,df. \end{aligned}$$

6. $_1$ $(abc\,.\,def)$ cyc. (gh),
   $_2$ $\{(abcdef)$ cyc. $(gh)\}$ pos.
   $_3$ $\{(abc\,.\,def)$ all $(gh)\}$ dim. [not in A.],
   $_4$ $\{(abcdef)_6\,(gh)\}$ dim.

Eight letters.

8\. 1 $(ab\,.\,cd)\,(ef)\,(gh)$,

2 $\{(ab)\,(cd)\,(ef)\}$ pos. (gh),

3 $\{(abcd)$ cyc. $(ef)\}$ pos. (gh),

4 $\{(abcd)_4\,(ef)\}$ dim. (gh),

5 $(abcd)$ cyc. $(ef\,.\,gh)$,

6 $(abcd)_4\,(ef\,.\,gh)$,

7 $(1,\ abcd,\ ac\,.\,bd,\ adcb) + ef\,.\,gh\,(ac,\ bd,\ ab\,.\,cd,\ ad\,.\,bc)$,

8 $(1,\ ab\,.\,cd,\ ac\,.\,bd,\ ad\,.\,bc) + ef\,.\,gh\,(ac,\ bd,\ abcd,\ adcb)$,

9 $(1,\ ac,\ bd,\ ac\,.\,bd) + ef\,.\,gh\,(abcd,\ adbc,\ ab\,.\,cd,\ ad\,.\,bc)$ [Not in A.],

10 $(1,\ ac\,.\,bd)\,(1,\ eg\,.\,fh) + (ac,\ bd)\,(eg,\ fh) = A'\,(abcd\,.\,efgh)_8$,

11 $(1,\ ac\,.\,bd)\,(1,\ eg\,.\,fh) + (abcd,\ adcb)\,(efgh,\ ehfg) = B'\,(abcd\,.\,efgh)_8$,

12 $(1,\ ac\,.\,bd)\,(1,\ eg\,.\,fh) + (ab\,.\,cd,\ ad\,.\,bc)\,(ef\,.\,gh,\ eh\,.\,fg) = C'\,(abcd\,.\,efgh)_8$,

13 $(1,\ ac\,.\,bd)\,(1,\ eg\,.\,fh) + (ac,\ bd)\,(efgh,\ ehgf) = D'\,(abcd\,.\,efgh)_8$,

14 $(1,\ ac\,.\,bd)\,(1,\ eg\,.\,fh) + (ac,\ bd)\,(ef\,.\,gh,\ eh\,.\,fg) = E'\,(abcd\,.\,efgh)_8$,

15 $(1,\ ac\,.\,bd)\,(1,\ eg\,.\,fh) + (ab\,.\,cd,\ ad\,.\,bc)\,(efgh,\ ehgf) = F'\,(abcd\,.\,efgh)_8$,

16 $(ae\,.\,bf\,.\,cg\,.\,dh)\,(ac\,.\,bd)\,(eg\,.\,fh)$. Substitutions are

$$\begin{array}{llll} 1, & ac\,.\,bd, & ac\,.\,bd\,.\,eg\,.\,fh, & agce\,.\,bhdf, \\ & eg\,.\,fh, & ae\,.\,bf\,.\,cg\,.\,dh, & aecg\,.\,bfdh. \\ & & ag\,.\,bh\,.\,ce\,.\,df, & \end{array}$$

17 $(1,\ ac\,.\,bd,\ ac\,.\,ef,\ bd\,.\,ef) + gh\,(abcd,\ adcb,\ ab\,.\,cd\,.\,ef,\ ad\,.\,bc\,.\,ef)$,

18 $(1,\ ac\,.\,bd,\ ac\,.\,ef,\ bd\,.\,ef) + gh\,(ab\,.\,cd,\ ad\,.\,bc,\ abcd\,.\,ef,\ adcb\,.\,ef)$,

19 $(1,\ ac\,.\,bd,\ ab\,.\,cd\,.\,ef,\ ad\,.\,bc\,.\,ef) + gh\,(abcd,\ adcb,\ ac\,.\,ef,\ bd\,.\,ef)$,

20 $(abcdefgh)_8$. Substitutions are

$$\begin{array}{llll} 1, & ac\,.\,be\,.\,dg, & ac\,.\,bd\,.\,eg\,.\,fh, & afch\,.\,bgde, \\ & bd\,.\,af\,.\,ch, & & ahcf\,.\,bedg. \\ & eg\,.\,ah\,.\,cf, & & \\ & fh\,.\,bg\,.\,de, & & \end{array}$$

21 $(abcdefgh)$ cyc.,

22 $A\,(abcdefgh)_8$,

23 $B\,(abcdefgh)_8$,

24 $C\,(abcdefgh)_8$,

25 $D\,(abcdefgh)_8$.

Eight letters.

12. 1 $(abcdef)$ cyc. (gh),

2 $(abc\,.\,def)$ all (gh),

3 $(abcdef)_6\,(gh)$,

4 $(1,\ abcdef,\ ace\,.\,bdf,\ ad\,.\,be\,.\,cf,\ aec\,.\,bfd,\ afedcb)$
$\quad + gh\,(af\,.\,be\,.\,cd,\ ab\,.\,cf\,.\,de,\ ad\,.\,bc\,.\,ef,\ bf\,.\,ce,\ ac\,.\,df,\ ae\,.\,bd)$,

5 $(1,\ bf\,.\,ce,\ ac\,.\,df,\ ae\,.\,bd,\ ace\,.\,bdf,\ aec\,.\,bfd)$
$\quad + gh\,(ab\,.\,cf\,.\,de,\ ad\,.\,bc\,.\,ef,\ ad\,.\,bf\,.\,ce,\ af\,.\,be\,.\,cd,\ abcdef,\ afedcb)$,

6 $(1,\ ab\,.\,cf\,.\,de,\ ad\,.\,bc\,.\,ef,\ af\,.\,be\,.\,cd,\ ace\,.\,bdf,\ aec\,.\,bfd)$
$\quad + gh\,(bf\,.\,ce,\ ac\,.\,df,\ ae\,.\,bd,\ ad\,.\,be\,.\,cf,\ abcdef,\ afedcb)$.

15. 1 $(abcde)$ cyc. (fgh) cyc.

16. 1 $(ab)\,(cd)\,(ef)\,(gh)$,

2 $(abcd)$ cyc. $(ef)\,(gh)$,

3 $(abcd)_4\,(ef)\,(gh)$,

4 $\{(abcd)_8$ com. $(ef)\}$ dim. (gh),

5 $\{(abcd)_8$ cyc. $(ef)\}$ dim. (gh),

6 $\{(abcd)_8$ pos. $(ef)\}$ dim. (gh),

7 $(abcd)$ cyc. $(efgh)$ cyc.,

8 $(abcd)_4\,(efgh)$ cyc.,

9 $(abcd)_4\,(efgh)_4$,

10 $(acbd)_8\,(ef\,.\,gh)$,

11 $(1,\ ac\,.\,bd)\,(1,\ eg,\ fh,\ eg\,.\,fh) + (ac,\ bd)\,(ef\,.\,gh,\ eh\,.\,fg,\ efgh,\ ehgf)$,

12 $(1,\ ac\,.\,bd)\,(1,\ eg,\ fh,\ eg\,.\,fh) + (ab\,.\,cd,\ ad\,.\,bc)\,(ef\,.\,gh,\ eh\,.\,fg,\ efgh,\ ehgf)$,

13 $(1,\ ac\,.\,bd)\,(1,\ eg,\ fh,\ eg\,.\,fh) + (abcd,\ adcb)\,(ef\,.\,gh,\ eh\,.\,fg,\ efgh,\ ehfg)$,

14 $(1,\ ac\,.\,bd)\,(1,\ eg\,.\,fh,\ efgh,\ ehgf) + (abcd,\ adcb)\,(eg,\ fh,\ ef\,.\,gh,\ eh\,.\,fg)$,

15 $(1,\ ac\,.\,bd)\,(1,\ eg\,.\,fh,\ efgh,\ ehgf) + (ab\,.\,cd,\ ad\,.\,bc)\,(eg\,.\,fh,\ ef\,.\,gh,\ eh\,.\,fg)$,

16 $(1,\ ac\,.\,bd)\,(1,\ ef\,.\,gh,\ eg\,.\,fh,\ eh\,.\,fg) + (abcd,\ adcb)\,(eg,\ fh,\ efgh,\ ehfg)$,

17 $(1,\ ac\,.\,bd)\,(1,\ ef\,.\,gh,\ eg\,.\,fh,\ eh\,.\,fg) + (ab\,.\,cd,\ ad\,.\,bc)\,(eg,\ fh,\ efgh,\ ehgf)$,

18 $(1,\ ac\,.\,bd)\,(1,\ efgh,\ eg\,.\,fh,\ ehgf) + (ac,\ bd)\,(eg,\ fh,\ ef\,.\,gh,\ eh\,.\,fg)$,

19 $(1,\ ac\,.\,bd)\,(1,\ ef\,.\,gh,\ eg\,.\,fh,\ eh\,.\,fg) + (ac,\ bd)\,(eg,\ fh,\ efgh,\ ehgf)$,

20 $(ae\,.\,bf\,.\,cg\,.\,dh)\,B'\,(abcd\,.\,efgh)_8$,

21 $(ae\,.\,bf\,.\,cg\,.\,dh)\,C'\,(abcd\,.\,efgh)_8$,

22 $(aebf\,.\,cgdh)$ cyc. $(ac\,.\,bd)\,(eg\,.\,fh)$,

23 $(aebfcgdh)$ cyc. $(1,\ ac\,.\,bd)$.

Eight letters.

18. 1 (abc) cyc. (def) cyc. (gh),

2 $\{(abc)$ all $(de)\}$ pos. (fgh),

3 $\{(abc)$ pos. (def) pos.$\} + gh\,\{(abc)$ neg. (def) neg.$\} = A\,(abcdefgh)_{18}$,

4 $B\,(abcdefgh)_{18}$.

[A. gives, *loc. cit.*, p. 310, a form which is not a group.]

24. 1 $(abcdef)_{12}\,(gh)$,

2 $(abcd)$ pos. $(ef\,.\,gh)$,

3 $\{(abcd)$ all $(ef\,.\,gh)\}$ dim.

4 $(abcdefgh)_{24}$.

30. 1 $(abcde)$ cyc. (fgh) all,

2 $(abcde)_{10}\,(fgh)$ cyc.

3 $(abcdefgh)_{30}$.

32. 1 $(abcd)_8\,(ef)\,(gh)$,

2 $(abcd)_8\,(efgh)$ cyc.

3 $(abcd)_8\,(efgh)_4$,

4 $L\,\{(abcd)_8\,(efgh)_8\}$ dim.

5 $M\,\{(abcd)_8\,(efgh)_8\}$ dim.

6 $N\,\{(abcd)_8\,(efgh)_8\}$ dim.

7 $P\,\{(abcd)_8\,(efgh)_8\}$ dim.

8 $Q\,\{(abcd)_8\,(efgh)_8\}$ dim.

9 $R\,\{(abcd)_8\,(efgh)_8\}$ dim.

10 $(ae\,.\,bf\,.\,cg\,.\,dh)\,(abcd)$ cyc. $(efgh)$ cyc.

11 $(ae\,.\,bf\,.\,cg\,.\,dh)\,(abcd)_4\,(efgh)_4$.

[A. has, *loc. cit.*, p. 291 and p. 295, two forms which are identical, and, p. 275, there is a single form which is, by mistake, counted as two.]

36. 1 $\{(abc)$ all $(de)\}$ pos. (fgh),

2 $\{(abc)$ all (def) all$\}$ pos. (gh),

3 $\{(abc)$ all $(def)\,(gh)\}$ pos.

4 $(abcdef)_{18}\,(gh)$,

5 $(ad\,.\,be\,.\,cf)\,A\,(abcdefgh)_{18}$ is $= \{(abcdef)_{36}\,(gh)\}$ dim.

[The group, A., *loc. cit.*, p. 312, is derived from the non-existent group of 18, p. 310, and is thus non-existent.]

Eight letters.

48. 1 $(abcd)$ all $(ef\,.\,gh)$,

2 $(abcd)$ pos. $(ef)(gh)$,

3 $(abcd)$ pos. $(efgh)$ cyc.

4 $(abcd)$ pos. $(efgh)_4$,

5 $\{(abcd)$ all $(ef)\}$ pos. (gh),

6 $(\pm\, abcdef)_{24}(gh)$,

7 $(+\, abcdef)_{24}(gh)$,

8 $\{(abcd)$ all $(efgh)$ cyc.$\}$ dim.

9 $\{(abcd)$ all $(ef)(gh)\}$ dim.

10 $\{(abcd)$ all $(efgh)_4\}$ dim.

11 $\{(abcdef)_{48}(gh)\}$ dim.

12 $\{(abcdef)_{48}(gh)\}$ pos.

[A. *loc. cit.*, p. 272 is not a group.]

13 $(abcdefgh)_{48}$,

14 $(abcd)_{12}(efgh)_{12}$ tris.

60. 1 $(abcde)_{10}(fgh)$ all,

2 $(abcde)_{20}(fgh)$ cyc.

3 $\{(abcde)_{20}(fgh)$ all$\}$ dim.

64. 1 $(abcd)_8(efgh)_8$,

2 $(ae\,.\,bf\,.\,cg\,.\,dh)\,M\,\{(abcd)_8(efgh)_8\}$ dim.

3 $(ae\,.\,bf\,.\,cg\,.\,dh)\,N\,\{(abcd)_8(efgh)_8\}$ dim.

72. 1 (abc) all (def) all (gh),

2 $(abcdef)_{36}(gh)$,

3 $\{(abcdef)_{72}(gh)\}$ dim.

4 $(abcdefgh)_{72}$.

96. 1 $(abcd)$ all $(ef)(gh)$,

2 $(abcd)$ all $(efgh)$ cyc.

3 $(abcd)$ all $(efgh)_4$,

4 $(abcdef)_{48}(gh)$,

5 $(abcd)$ pos. $(efgh)_8$,

6 $\{(abcd)$ all $(efgh)_8$ com.$\}$ dim.

7 $\{(abcd)$ all $(efgh)_8$ cyc.$\}$ dim.

Eight letters.

96. 8 $\{(abcd)$ all $(efgh)_8$ pos.$\}$ dim.

9 $(ae \, . \, bf \, . \, cg \, . \, dh) \{(abcd)_{12} (efgh)_{12}\}$ tris.

10 $(ae \, . \, bg \, . \, cf \, . \, dh) \{(abcd)_{12} (efgh)_{12}\}$ tris.

120. 1 $(abcde)_{20} (fgh)$ all,

2 $(abcdef)_{60} (gh)$,

3 $\{(abcdef)_{120} (gh)\}$ pos.

144. 1 $(abcd)$ pos. $(efgh)$ pos.

2 $(ad \, . \, be \, . \, cf) \{(abc)$ all (def) all $(gh)\}$.

168. $(abcdefgh)_{168}$.

180. $(abcde)$ pos. (fgh) cyc.

192. 1 $(abcd)$ all $(efgh)_8$,

2 $(1, agch, bgdh, egfh) \{(abcdef)_{48} (gh)\}$ dim.

3 $(1, bd \, . \, agch, ef \, bgdh, ac \, . \, egfh) \{(abcdef)_{48} (gh)\}$ pos.

240. $(abcdef)_{120} (gh)$.

288. 1 $(abcd)$ all $(efgh)$ pos.

2 $\{(abcd)$ all $(efgh)$ all$\}$ pos.

3 $(ae \, . \, bf \, . \, cg \, . \, dh) (abcd)$ pos. $(efgh)$ pos.

336. $(abcdefgh)_{336}$.

360. 1 $(abcde)$ all (fgh) cyc.

2 $(abcde)$ pos. (fgh) all,

3 $\{(abcde)$ all (fgh) all$\}$ pos.

384. $(1, ag \, . \, bh, cf \, . \, dh, eg \, . \, fh) \{(abcdef)_{48} (gh)\}$.

576. 1 $(abcd)$ all $(efgh)$ all,

2 $(ae \, . \, bf \, . \, cg \, . \, dh) \{(abcd)$ all $(efgh)$ all$\}$ pos.

720. 1 $(abcde)$ all (fgh) all,

2 $(abcdef)$ pos. (gh),

3 $\{(abcdef)$ all $(gh)\}$ pos.

1152. $(ae \, . \, bf \, . \, cg \, . \, dh) \{(abcd)$ all $(efgh)$ all$\}$.

1440. $(abcdef)$ all (gh).

20160. $(abcdefgh)$ pos.

40320. $(abcdefgh)$ all.

Eight letters, explanations. Eight letters. Explanations. Ord. 8.

ord. 8.

$A\,(abcdefgh)_8$. The substitutions are

$$\begin{array}{l} 1,\ ab\,.\,cd\,.\,ef\,.\,gh, \\ ac\,.\,bd\,.\,eg\,.\,fh, \\ ad\,.\,bc\,.\,eh\,.\,fg, \\ ae\,.\,bf\,.\,cg\,.\,dh, \\ af\,.\,be\,.\,ch\,.\,dg, \\ ag\,.\,bh\,.\,ce\,.\,df, \\ ah\,.\,bg\,.\,cf\,.\,de. \end{array}$$

$B\,(abcdefgh)_8$. The substitutions are

$$\begin{array}{ll} 1,\ ab\,.\,ch\,.\,dg\,.\,ef, & aceg\,.\,bdfh, \\ ad\,.\,bc\,.\,eh\,.\,fg, & agec\,.\,bhdf. \\ ae\,.\,bf\,.\,cg\,.\,dh, & \\ af\,.\,be\,.\,cd\,.\,gh, & \\ ah\,.\,bg\,.\,cf\,.\,de, & \end{array}$$

$C\,(abcdefgh)_8$. The substitutions are

$$\begin{array}{ll} 1,\ ad\,.\,bg\,.\,cf\,.\,eh, & aceg\,.\,bdfh, \\ ae\,.\,bf\,.\,cg\,.\,dh, & agec\,.\,bhfd, \\ ah\,.\,bc\,.\,de\,.\,fg, & abef\,.\,cdgh, \\ & afeb\,.\,chgd. \end{array}$$

$D\,(abcdefgh)_8$. The substitutions are

$$\begin{array}{ll} 1,\ ae\,.\,bf\,.\,cg\,.\,dh, & aceg\,.\,bdfh, \\ & agec\,.\,bhfd, \\ & abef\,.\,chgd, \\ & afeb\,.\,cdgh, \\ & adeh\,.\,bgfc\,, \\ & ahed\,.\,bcfg\,. \end{array}$$

ord. 12. The last three groups, the substitutions of which are given in full, are each of them of the form $\{(abcdef)_{12}\,(gh)\}$ dim. They may be written: Ord. 12.

$\{(abcdef)_{12}$ cyc. $(gh)\}$ dim.; viz. here the six substitutions of $(abcdef)_{12}$ combined with 1 are those of $(abcdef)$ cyc.

$\{(abcdef)_{12}$ pos. $(gh)\}$ dim.; viz. here the six substitutions of $(abcdef)_{12}$ combined with 1 are the positive substitutions of $(abcdef)_{12}$.

$\{(abcdef)_{12} \pm (gh)\}$ dim.; viz. here the six substitutions of $(abcdef)_{12}$ combined with 1 are three positive and three negative substitutions.

Eight letters. Ord. 16.

ord. 16. I have not thought it necessary to devise any notation for the set of groups 11 to 19, the substitutions of which are given in full.

For the remaining groups 20 to 23:

$(ae \,.\, bf \,.\, cg \,.\, dh)\ B'\ (abcd \,.\, efgh)_8$. The substitutions are

1,	$ac \,.\, bd$,	$ac \,.\, bd \,.\, ef \,.\, gh$,	$abcd \,.\, efgh$,
	$eg \,.\, fh$,	$ae \,.\, bf \,.\, cg \,.\, dh$,	$abcd \,.\, ehgf$,
		$af \,.\, bg \,.\, ch \,.\, de$,	$adcb \,.\, efgh$,
		$ag \,.\, bh \,.\, ce \,.\, df$,	$adcb \,.\, ehgf$,
		$ah \,.\, be \,.\, cf \,.\, dg$,	$agce \,.\, bhdf$,
			$aecg \,.\, bfdh$,
			$afch \,.\, bgde$,
			$ahcf \,.\, bedg$.

$(ae \,.\, bf \,.\, cg \,.\, dh)\ C'\ (abcd \,.\, efgh)_8$. The substitutions are

1,	$ac \,.\, bd$,	$ac \,.\, bd \,.\, eg \,.\, fh$,	$aceg \,.\, bfdh$,
	$eg \,.\, fh$,	$ab \,.\, cd \,.\, ef \,.\, gh$,	$agce \,.\, bhdf$,
		$ab \,.\, cd \,.\, eh \,.\, fg$,	$afch \,.\, bedg$,
		$ad \,.\, bc \,.\, ef \,.\, gh$,	$ahcf \,.\, bgde$.
		$ad \,.\, bc \,.\, eh \,.\, fg$,	
		$ae \,.\, bf \,.\, cg \,.\, dh$,	
		$af \,.\, bg \,.\, ch \,.\, de$,	
		$ag \,.\, bh \,.\, ce \,.\, df$,	
		$ah \,.\, bg \,.\, cf \,.\, de$,	

$(aebf \,.\, cdgh)$ cyc. $(ac \,.\, bd)(ef \,.\, gh)$. The substitutions are

1,	$ac \,.\, bd$,	$ac \,.\, bd \,.\, eg \,.\, fh$,	$aebf \,.\, cgdh$,
	$eg \,.\, fh$,	$ab \,.\, cd \,.\, ef \,.\, gh$,	$aedh \,.\, bfcg$,
		$ab \,.\, cd \,.\, eh \,.\, fg$,	$afdg \,.\, bech$,
		$ad \,.\, bc \,.\, ef \,.\, gh$,	$agbh \,.\, cedf$,
		$ad \,.\, bc \,.\, eh \,.\, fg$,	$afbe \,.\, chdg$,
			$ahde \,.\, bgcf$,
			$agdf \,.\, bhce$,
			$ahbg \,.\, cfde$.

$(aebfcgdh)$ cyc. $(1,\ ac\,.\,bd)$. The substitutions are **Eight letters. Ord. 16.**

$$\begin{array}{lllll}
1, & ac\,.\,bd, & ac\,.\,bd\,.\,eg\,.\,fh, & abcd\,.\,efgh, & aebfcgdh,\\
 & eg\,.\,fh, & & abcd\,.\,ehgf, & afdechbg,\\
 & & & adcb\,.\,efgh, & agbhcedf,\\
 & & & adcb\,.\,ehgf, & ahdgcfbe,\\
 & & & & aedhcgbf,\\
 & & & & afbgchde,\\
 & & & & agdfcebh,\\
 & & & & ahbecfdg.
\end{array}$$

ord. 18. $B\,(abcdefgh)_{18}$. This group, communicated to me by Mr Askwith, might be written **Ord. 18.**

$$[((ad\,.\,be\,.\,cf)\,\{(abc)\text{ cyc. }(def)\text{ cyc.}\})\,gh]\text{ pos.},$$

or for shortness

$$[(abcdef)_{18}\,gh]\text{ pos.}$$

ord. 24. $(abcdefgh)_{24}$. The substitutions are **Ord. 24.**

$$\begin{array}{lllll}
1, & ac\,.\,dg, & ace\,.\,bdf, & ac\,.\,bh\,.\,df\,.\,eg, & aceg\,.\,bhdf,\\
 & ae\,.\,bd, & aec\,.\,bfd, & ae\,.\,bd\,.\,cg\,.\,fh, & agec\,.\,bfdh,\\
 & ag\,.\,bh, & acg\,.\,dfh, & ag\,.\,bf\,.\,ce\,.\,dh, & acge\,.\,bdfh,\\
 & bf\,.\,ce, & agc\,.\,dhf, & & aegc\,.\,bhfd,\\
 & cg\,.\,fh, & aeg\,.\,bhd, & & aecg\,.\,bfhd,\\
 & eg\,.\,bh, & age\,.\,bdh, & & agce\,.\,bdhf.\\
 & & bfh\,.\,cge, & & \\
 & & bhf\,.\,ceg, & &
\end{array}$$

ord. 30. $(abcdefgh)_{30}$. The substitutions are **Ord. 30.**

$$\begin{array}{lllll}
1, & abcde, & fgh, & abcde\,.\,fgh, & ab\,.\,ce\,.\,fg,\\
 & acebd, & fhg, & abcde\,.\,fhg, & ac\,.\,de\,.\,fg,\\
 & adbec, & & acebd\,.\,fgh, & ad\,.\,bc\,.\,fg,\\
 & aedcb, & & acebd\,.\,fhg, & ae\,.\,bd\,.\,fg,\\
 & & & adbec\,.\,fgh, & be\,.\,cd\,.\,fg,\\
 & & & adbec\,.\,fhg, & ab\,.\,ce\,.\,fh,\\
 & & & aedcb\,.\,fgh, & ac\,.\,de\,.\,fh,\\
 & & & aedcb\,.\,fhg, & ad\,.\,bc\,.\,fh,\\
 & & & & ae\,.\,bd\,.\,fh,\\
 & & & & be\,.\,cd\,.\,fh,\\
 & & & & ab\,.\,ce\,.\,gh,\\
 & & & & ac\,.\,de\,.\,gh,\\
 & & & & ad\,.\,bc\,.\,gh,\\
 & & & & ae\,.\,bd\,.\,gh,\\
 & & & & be\,.\,cd\,.\,gh.
\end{array}$$

Eight letters. Ord. 32.

ord. 32. Six groups $L, M, N, P, Q, R\{(abcd)_8(efgh)_8\}$ dim., the two groups $(\quad)_8$ may be dimidiated

$(abcd)_8$ into	$(efgh)_8$ into
$A,\ A',$	$E,\ E',$
$B,\ B',$	$F,\ F',$
$C,\ C',$	$G,\ G'$;

where

$$A=(1,\ ac\ ,\ bd\ ,\ ac\,.\,bd),\quad E=(1,\ eg\ ,\ fh\ ,\ eg\,.\,fh),$$

$$B=(1,\ abcd\ ,\ ac\,.\,bd,\ adcb\ ),\quad F=(1,\ efgh\ ,\ eg\,.\,fh,\ ehgf\ ),$$

$$C=(1,\ ab\,.\,cd,\ ac\,.\,bd,\ ad\,.\,bc),\quad G=(1,\ ef\,.\,gh,\ eg\,.\,fh,\ eh\,.\,fg\ ),$$

and A', B', C', E', F', G' are the tails of the groups $(abcd)_8$ and $(efgh)_8$ respectively.

We have then

$$L\{(abcd)_8(efgh)_8\}\text{ dim.}=AE+A'E',$$

viz. the substitutions are

$$(1,\ ac,\ bd,\ ac\,.\,bd)(1,\ eg,\ fh,\ eg\,.\,fh)$$

$$+(ab\,.\,cd,\ ad\,.\,bc,\ abcd,\ adbc)(ef\,.\,gh,\ eh\,.\,fg,\ efgh,\ ehfg),$$

$$M \quad ,, \quad =BF+B'F',$$

$$N \quad ,, \quad =CG+C'G',$$

$$P \quad ,, \quad =BG+B'G'\ (\text{or } CF+C'F'),$$

$$Q \quad ,, \quad =CE+C'E'\ (\text{or } AG+A'G'),$$

$$R \quad ,, \quad =AF+A'F'\ (\text{or } BE+B'E').$$

For the remaining groups, we have

$(ae\,.\,bf\,.\,cg\,.\,dh)(abcd)$ cyc. $(efgh)$ cyc. The substitutions are

+	+	−	−	+	+	−
1,	$ac\,.\,bd$,	$abcd$,	$ac\,.\,bd\,.\,efgh$,	$ac\,.\,bd\,.\,ef\,.\,gh$,	$abcd\,.\,efgh$,	$aebfcgdh$,
	$ef\,.\,gh$;	$adcb$,	$ac\,.\,bd\,.\,ehgf$,	$ae\,.\,bf\,.\,cg\,.\,dh$,	$abcd\,.\,ehgf$,	$aedhcgbf$,
		$efgh$,	$eg\,.\,fh\,.\,abcd$,	$af\,.\,bg\,.\,ch\,.\,de$,	$adcb\,.\,efgh$,	$afbgchde$,
		$ehfg$;	$eg\,.\,fh\,.\,adcb$;	$ag\,.\,bh\,.\,ce\,.\,df$,	$adcb\,.\,ehgf$,	$afdechbg$,
				$ah\,.\,be\,.\,cf\,.\,dg$;	$aecg\,.\,bfdh$,	$agbhcedf$,
					$agce\,.\,bhdf$,	$agdfcebh$,
					$afch\,.\,bgde$,	$ahbecfdg$,
					$ahcf\,.\,bedg$;	$ahdgcfbe$.

$(ae\,.\,bf\,.\,cg\,.\,dh)(abcd)_4(efgh)_4$. The substitutions are

Eight letters. Ord. 32.

+	+	+	+
1,	ab . cd,	ab . cd . ef . gh,	aebf . cdgh,
	ac . bd,	ab . cd . eg . fh,	aecg . bfdh,
	ad . bc,	ab . cd . eh . fg,	aedh . bfcg,
	ef . gh,	ac . bd . ef . gh,	afbe . chdg,
	eg . fh,	ac . bd . eg . fh,	afch . bedg,
	eh . fg;	ac . bd . eh . fg,	afdg . bech,
		ad . bc . ef . gh,	agbh . cedf,
		ad . bc . eg . fh,	agce . bhdf,
		ad . bc . eh . fg,	agdf . bhce,
		ae . bf . cg . dh,	ahbg . cfde,
		af . be . ch . dg,	ahcf . bgde,
		ag . bh . ce . df,	ahde . bgcf.
		ah . bg . cf . de;	

ord. 48. $\{(abcdef)_{48}(gh)\}$ dim. The substitutions are

Ord. 48.

1,	abcd,	ac . bd,	abe . cdf,	ab . cd . ef,	+ gh	ac,	ab . cd,	ac . bedf,	ac . bd . ef;	abfcde,
	adcb,	ac . ef,	aeb . cfd,	ac . be . df,		bd,	ad . bc,	ac . bfde,		abecdf,
	aecf,	bd . ef;	abf . cde,	ac . bf . de,		ef;	ae . cf,	bd . aecf,		adecbf,
	afce,		afb . ced,	ad . bc . ef,			af . ce,	bd . afce,		adfcbe,
	bedf,		ade . bfc,	ae . bd . cf,			be . df,	ef . abcd,		aebcfd,
	bfde;		aed . bcf,	af . bd . ce;			bf . de;	ef . adcb;		aedcfb,
			adf . bec,							afbced,
			afd . bce;							afdceb;

$\{(abcdef)_{48}(gh)\}$ pos. The substitutions are

1,	ab . cd,	ac . bedf,	abe . cdf,	+ gh	ac,	abcd,	ab . cd . ef,	abecdf,
	ac . bd,	ac . bfde,	abf . cde,		bd,	adcb,	ac . bd . ef,	abfcde,
	ac . ef,	bd . aecf,	ade . bfc,		ef;	aecf,	ac . be . df,	adecbf,
	ad . bc,	bd . afce,	adf . cbe,			afce,	ac . bf . de,	adfcbe,
	ae . cf,	ef . abcd,	aeb . cfd,			bedf,	ad . bc . ef,	aebcfd,
	af . ce,	ef . adcb;	aed . bcf,			bfde;	ae . bd . cf,	aedcfb,
	bd . ef,		afb . ced,				af . bd . ce,	afbced,
	be . df,		afd . ceb;					afdceb;
	bf . de;							

Eight letters. Ord. 48. $(abcdefgh)_{48}$. The substitutions are

1,	ac . df,	ace . bdf,	ab . cf . de . gh,	ad . bchefg,	abch . defg,
	ae . bd,	acg . bdh,	ab . ch . de . fg,	ad . bgfehc,	abgf . cdeh,
	ag . bh,	aec . bfd,	ac . df . eg . bh,	be . afgdch,	aceg . bhdf,
	bf . ce,	aeg . bhd,	ad . bc . ef . gh,	be . ahcdgf,	acge . bdfh,
	bh . eg,	agc . bhd,	ad . be . cf . gh,	cf . abgdeh,	aecg . bfhd,
	cg . fh,	age . bdh,	ad . be . ch . fg,	cf . ahedgb,	aegc . bhfd,
		bhf . ceg,	ad . bg . ch . ef,	gh . abcdef,	afeh . bgdc,
		bfh . cge,	ae . bd . cg . fh,	gh . afedcb,	afgb . ched,
			af . be . cd . gh,		agce . bdhf,
			af . bg . cd . eh,		agec . bfdh,
			ag . bf . ce . dh,		ahcb . dgfe,
			ah . bc . dg . ef,		ahef . bcdg.
			ah . be . cf . dg,		

$(abcd)_{12}(efgh)_{12}$ tris. The substitutions are

1	1,	+ abc	efg,	+ acb	egf,
ab . cd	ef . gh,	adb	ehf,	abd	efh,
ac . bd	eg . fh,	bdc	fhg,	bcd	fgh,
ad . bc	eh . fg,	acd	egh,	adc	ehg.

Ord. 64. ord. 64. $(ae.bf.cg.dh)\,M\,\{(abcd)_8(efgh)_8\}$ dim. The substitutions are

+	−	+	+	−	−	+	−
1,	abcd,	ac . bd,	abcd . efgh,	ac . ef . gh,	ac . bd . efgh,	ac . bd . ef . gh,	aebfcgdh,
	adbc,	eg . fh,	adbc . efgh,	ac . eh . fg,	ac . bd . ehfg,	ab . cd . ef . gh,	aebhcgdf,
	efgh,	ac . eg,	abcd . ehgf,	bd . ef . gh,	eg . fh . abcd,	ab . cd . eh . fg,	aedfcgbh,
	ehgf;	ac . fh,	adbc . ehgf,	bd . eh . fg,	eg . fh . adbc;	ad . bc . ef . gh,	aedhcgbf,
		bd . eg,	aecg . bhdf,	ab . cd . eg,		ad . bc . eh . fg,	afbechdg,
		bd . fh;	agce . bhdf,	ab . cd . fh,		ae . bf . cg . dh,	afbgchde,
			aecg . bfdh,	ad . bc . eg,		ae . bh . cg . df,	afdechbg,
			agce . bfdh,	ad . bc . fh;		af . be . ch . dg,	afdgchbe,
			afch . bgde,			af . bg . ch . de,	agbfcedh,
			ahcf . bgde,			ag . bf . ce . dh,	agbhcedf,
			afch . bedg,			ag . bh . ce . df,	agdfcebh,
			ahcf . bedg;			ah . be . cf . dg,	agdhcebf,
						ah . bg . cf . de;	ahbecfdg,
							ahbgcfde,
							ahdecfbg,
							ahdgcfbe;

$(ae\,.\,bf\,.\,cg\,.\,dh)\; N\{(abcd)_8\,(efgh)_8\}$ dim. Eight letters. Ord. 64.

The substitutions (all positive) are

1,	$ab\,.\,cd$,	$ab\,.\,cd\,.\,ef\,.\,gh$,	$ac\,.\,efgh$,	$abcd\,.\,efgh$,
	$ac\,.\,bd$,	$ab\,.\,cd\,.\,eg\,.\,fh$,	$ac\,.\,ehgf$,	$abcd\,.\,ehgf$,
	$ad\,.\,bc$,	$ab\,.\,cd\,.\,eh\,.\,fg$,	$bd\,.\,efgh$,	$adcb\,.\,efgh$,
	$ef\,.\,gh$,	$ac\,.\,bd\,.\,ef\,.\,gh$,	$bd\,.\,ehgf$,	$adcb\,.\,ehgf$,
	$eg\,.\,fh$,	$ac\,.\,bd\,.\,eg\,.\,fh$,	$eg\,.\,abcd$,	$aebf\,.\,cgdh$,
	$eh\,.\,fg$,	$ac\,.\,bd\,.\,eh\,.\,fg$,	$eg\,.\,adbc$,	$aebh\,.\,cgdf$,
	$ac\,.\,eg$,	$ad\,.\,bc\,.\,ef\,.\,gh$,	$fh\,.\,abcd$,	$aecg\,.\,bfdh$,
	$ac\,.\,fh$,	$ad\,.\,bc\,.\,eg\,.\,fh$,	$fh\,.\,adcb$,	$aecg\,.\,bhdf$,
	$bd\,.\,eg$,	$ad\,.\,bc\,.\,eh\,.\,fg$,		$aedf\,.\,bhcg$,
	$bd\,.\,fh$,	$ae\,.\,bf\,.\,cg\,.\,dh$,		$aedh\,.\,bfcg$,
		$ae\,.\,bh\,.\,cg\,.\,df$,		$afbe\,.\,chdg$,
		$af\,.\,be\,.\,ch\,.\,dg$,		$afbg\,.\,chde$,
		$af\,.\,bg\,.\,ch\,.\,de$,		$afch\,.\,bedg$,
		$ag\,.\,bf\,.\,ce\,.\,dh$,		$afch\,.\,bgde$,
		$ag\,.\,bh\,.\,ce\,.\,df$,		$afde\,.\,bgch$,
		$ah\,.\,be\,.\,cf\,.\,dg$,		$afdg\,.\,bech$,
		$ah\,.\,bg\,.\,cf\,.\,de$,		$agbf\,.\,cedh$,
				$agbh\,.\,cedf$,
				$agce\,.\,bfdh$,
				$agce\,.\,bhdf$,
				$agdf\,.\,bhce$,
				$agdh\,.\,bfce$,
				$ahbe\,.\,cfdg$,
				$ahbg\,.\,cfde$,
				$ahcf\,.\,begd$,
				$ahcf\,.\,bgde$,
				$ahde\,.\,bgcf$,
				$ahdg\,.\,becf$.

ord. 72. $(abcdefgh)_{72}$. This group, communicated to me by Mr Askwith, may be written Ord. 72.

$$\{(abcdef)_{72}\, gh\}\ \text{pos.}$$

ord. 96. $\{(abcd)$ all $(efgh)_8$ com.$\}$ dim. This means Ord. 96.

$$(abcd)\ \text{pos.}\ (1,\ eg,\ fh,\ eg\,.\,fh) + (abcd)\ \text{neg. residue.}$$

Eight letters. Ord. 96. $\{(abcd)$ all $(efgh)_8$ cyc.$\}$ dim. This means

$$(abcd) \text{ pos. } (1,\ efgh,\ eg\,.\,fh,\ ehgf) + (abcd) \text{ neg. residue.}$$

$\{(abcd)$ all $(efgh)_8$ pos.$\}$ dim. This means

$$(abcd) \text{ pos. } (1,\ ef\,.\,gh,\ eg\,.\,fh,\ eh\,.\,fg) + (abcd) \text{ neg. residue;}$$

viz. each form $(efgh)_8$ is divided into two sets, which are combined with $(abcd)$ pos. and $(abcd)$ neg. respectively.

ord. 96. $(ae\,.\,bf\,.\,cg\,.\,dh)\,\{(abcd)_{12}\,(efgh)_{12}\}$ tris.

The substitutions (all positive) are those of

$\{(abcd)_{12}\,(efgh)_{12}\}$ tris., viz.

1	1 ,	+ *abc*	*efg* ,	+ *acb*	*egf* ,
ab . cd	*ef . gh*,	*adb*	*ehf* ,	*abd*	*efh*,
ac . bd	*eg . fh*,	*bdc*	*fhg*,	*bcd*	*fgh*,
ad . bc	*eh . fg*,	*acd*	*egh* ,	*adc*	*ehg*,

together with the following 48 substitutions

ae . bf . cg . dh,	*aebf . cgdh*,	*ae . bgdfch*,
af . be . ch . dg,	*aecg . bfdh*,	*ae . bhcfdg*,
ag . bh . ce . df,	*aedh . bfcg*,	*af . bgcedh*,
ah . bg . cf . de;	*afbe . chdg*,	*af . bhdecg*,
	afch . bedg,	*ag . bedhcf*,
	afdg . bech,	*ag . bfchde*,
	agbh . cedf,	*ah . becgdf*,
	agce . bhdf,	*ah . bfdgce*,
	agdf . bhce,	*be . agcfdh*,
	ahbg . cfde,	*be . ahdfcg*,
	ahcf . bgde,	*bf . agdech*,
	ahde . bgcf;	*bf . ahcedg*,
		bg . aechdf,
		bg . afdhce,
		bh . aedgcf,
		bh . afcgde,
		ce . afbgdh,
		ce . ahdgbf,
		cf . aebhdg,

Eight letters. Ord. 96.

$cf \, . \, agdhbe,$
$cg \, . \, afdebh,$
$cg \, . \, ahbedf,$
$ch \, . \, aedfbg,$
$ch \, . \, agbfde,$
$de \, . \, afbhcg,$
$de \, . \, agchbf,$
$df \, . \, aebgch,$
$df \, . \, ahcgeb,$
$dg \, . \, aecfbh,$
$dg \, . \, ahbfce,$
$dh \, . \, afcebg,$
$dh \, . \, agbecf.$

$(ae \, . \, bg \, . \, cf \, . \, dh)\,\{(abcd)_{12}\,(efgh)_{12}\}$ tris. The substitutions (all positive) are those of $\{(abcd)_{12}\,(efgh)_{12}\}$ tris., viz.

1	1 ,	$+\,abc$	efg,	$+\,acb$	egf,
$ab \, . \, cd$	$ef \, . \, gh$,	adb	ehf,	abd	efh,
$ac \, . \, bd$	$eg \, . \, fh$,	bdc	fhg,	bcd	fgh,
$ad \, . \, bc$	$eh \, . \, fg$,	acd	egh,	adc	ehg,

together with the following 48 substitutions

$ae \, . \, bf \, . \, ch \, . \, dg,$	$aebf \, . \, chdg,$
$ae \, . \, bg \, . \, cf \, . \, dh,$	$aebg \, . \, cfdh,$
$ae \, . \, bh \, . \, cg \, . \, df,$	$aebh \, . \, cgdf,$
$af \, . \, be \, . \, cg \, . \, dh,$	$aecf \, . \, bgdh,$
$af \, . \, bg \, . \, ch \, . \, de,$	$aecg \, . \, bhdf,$
$af \, . \, bh \, . \, ce \, . \, dg,$	$aech \, . \, bfdg,$
$ag \, . \, be \, . \, ch \, . \, df,$	$aedf \, . \, bhcg,$
$ag \, . \, bf \, . \, ce \, . \, dh,$	$aedg \, . \, bfch,$
$ag \, . \, bh \, . \, cf \, . \, de,$	$aedh \, . \, bgcf,$
$ah \, . \, be \, . \, cf \, . \, dg,$	$afbe \, . \, cgdh,$
$ah \, . \, bf \, . \, cg \, . \, de,$	$afbg \, . \, chde,$
$ah \, . \, bg \, . \, ce \, . \, df;$	$afbh \, . \, cedg,$
	$afce \, . \, bhdg,$
	$afcg \, . \, bedh,$
	$afch \, . \, bgde,$

Eight letters. Ord. 96.

$afde \,.\, bgch$,
$afdg \,.\, bhce$,
$afdh \,.\, becg$,
$agbe \,.\, chdf$,
$agbf \,.\, cedh$,
$agbh \,.\, cfde$,
$agce \,.\, bfdh$,
$agcf \,.\, bhde$,
$agch \,.\, bedf$,
$agde \,.\, bhcf$,
$agdf \,.\, bech$,
$agdh \,.\, bfce$,
$ahbe \,.\, cfdg$,
$ahbf \,.\, cgde$,
$ahbg \,.\, cedf$,
$ahce \,.\, bgdf$,
$ahcf \,.\, bedg$,
$ahcg \,.\, bfde$,
$ahde \,.\, bfcg$,
$ahdf \,.\, bgce$,
$ahdg \,.\, becf$.

Ord. 168. ord. 168. $(abcdefgh)_{168}$.

This is $(1, S^2, S^4)(1, T, \ldots, T^6)(1, U, \ldots, U^7)$, where $S = bdcgef$, $T = abcdefg$, $U = ahbfgecd$; it is a derivative of $(abcdefg)_{21} = (1, S^2, S^4)(1, T, \ldots, T^6)$, see post $(abcdefgh)_{336}$.

Ord. 192. ord. 192. [Two forms not examined.]

Ord. 288. ord. 288. $(ae \,.\, bf \,.\, cg \,.\, dh)(abcd)$ pos. $(efgh)$ pos.

The substitutions are those of $(abcd)$ pos. $(efgh)$ pos., viz. the 144 substitutions

1	1,
$ab \,.\, cd$	$ef \,.\, gh$,
$ac \,.\, bd$	$eg \,.\, fh$,
$ad \,.\, bc$	$eh \,.\, fg$,
abc	efg,
acb	egf,
abd	efh,
adb	ehf,
acd	egh,
adc	ehg,
bcd	fgh,
bdc	fhg,

together with the following 144 substitutions, viz. Eight letters. Ord. 288.

ae . bf . cg . dh,	*aebf . cgdh*,	*afbe . chdg*,	*agbe . cfdh*,	*ahbe . cgdf*,
ae . bg . ch . df,	*aebg . chdf*,	*afbg . cedh*,	*agbf . chde*,	*ahbf . cedg*,
ae . bh . cf . dg,	*aebh . cfdg*,	*afbh . cgde*,	*agbh . cedf*,	*ahbg . cfde*,
af . be . ch . dg,	*aecf . bhdg*,	*afce . bgdh*,	*agce . bhdf*,	*ahce . bfdg*,
af . bg . ce . dh,	*aecg . bfdh*,	*afcg . bhde*,	*agcf . bedh*,	*ahcf . bgde*,
af . bh . cg . de,	*aech . bgdf*,	*afch . bedg*,	*agch . bfde*,	*ahcg . bedf*,
ag . bh . ce . df,	*aedf . bgch*,	*afde . bhcg*,	*agde . bfch*,	*ahde . bgcf*,
ag . be . cf . dh,	*aedg . bhcf*,	*afdg . bech*,	*agdf . bhce*,	*ahdf . becg*,
ag . bf . ch . de,	*aedh . bfcg*,	*afdh . bgce*,	*agdh . becf*,	*ahdg . bfce*;
ah . bg . cf . de,				
ah . be . cg . df,				
ah . bf . ce . dg;				

ae . bfcgdh,	*af . bechdg*,	*ag . becfdh*,	*ah . becgdf*	*ce . afbgdh*,	*cf . aebhdg*,	*cg . aebfdh*,	*ch . aebgdf*,
„ *bfdhcg*,	„ *bedgch*,	„ *bedhcf*,	„ *bedfcg*	„ *afdhbg*,	„ *aedgbh*,	„ *aedhbf*,	„ *aedfbg*,
„ *bgchdf*,	„ *bgcedh*,	„ *bfchde*,	„ *bfcedg*	„ *agbhdf*,	„ *agbedh*,	„ *afbhde*,	„ *afbedg*,
„ *bgdfch*,	„ *bgdhce*,	„ *bfdech*,	„ *bfdgce*	„ *agdfbh*,	„ *agdhbe*,	„ *afdebh*,	„ *afdgbe*,
„ *bhcfdg*,	„ *bhcdge*,	„ *bhcedf*,	„ *bgcfde*	„ *ahbfdg*,	„ *ahbgde*,	„ *ahbedf*,	„ *agbfde*,
„ *bhdgcf*;	„ *bhdecg*;	„ *bhdfce*;	„ *bgdecf*;	„ *ahdgbf*;	„ *ahdebg*;	„ *ahdfbe*;	„ *agdebf*;

be . afchdg,	*bf . aecgdh*,	*bg . aechdf*,	*bh . aecfdg*	*de . afbhcg*,	*df . aebgch*,	*dg . aebhcf*,	*dh . aebfcg*,
„ *afdgch*,	„ *aedhcg*,	„ *aedfch*,	„ *aedgcf*	„ *afcgbh*,	„ *aechbg*,	„ *aecfbh*,	„ *aecgbf*,
„ *agcfdh*,	„ *agchde*,	„ *afcedh*,	„ *afcgde*	„ *agbfch*,	„ *agbhce*,	„ *afbech*,	„ *afbgce*,
„ *agdhcf*,	„ *agdech*,	„ *afdhce*,	„ *afdecg*	„ *agchbf*,	„ *agcebh*,	„ *afchbe*,	„ *afcebg*,
„ *ahcgdf*,	„ *ahcedg*,	„ *ahcfde*,	„ *agcedf*	„ *ahbgcf*,	„ *ahbecg*,	„ *ahbfce*,	„ *agbecf*,
„ *ahdfcg*;	„ *ahdgce*;	„ *ahdecf*;	„ *agdfce*;	„ *ahcfbg*;	„ *ahcgbe*;	„ *ahcebf*;	„ *agcfbe*.

ord. 336. $(abcdefgh)_{336}$. This is Ord. 336.

$$(1,\ S,\ S^2,\ S^3,\ S^4,\ S^5)\,(1,\ T,\ \ldots,\ T^6)\,(1,\ U,\ \ldots,\ U^7),$$

where $S = bdcgef$, $T = abcdefg$, $U = ahbfgecd$; it is a derivative of

$$(abcdefg)_{42} = (1,\ S,\ \ldots,\ S^5)\,(1,\ T,\ \ldots,\ T^6).$$

It will be noticed that there are some alterations in the numbers of the groups of the several orders as stated in the Table, p. 118, and that the number here obtained for the total number of the groups of eight letters (instead of 155 as in the table) is 157. Some of the groups, in particular those of the order 192, require further explanation.