914.

ON A SOLUBLE QUINTIC EQUATION.

[From the *American Journal of Mathematics*, vol. XIII. (1891), pp. 53—58.]

MR YOUNG, in his paper, "Solvable Quintic Equations with Commensurable Coefficients," *American Journal of Mathematics*, X. (1888), pp. 99—130, has given, in illustration of his general theory of the solution of soluble quintic equations (founded upon a short note by Abel), no less than twenty instances of the solution of a quintic equation with purely numerical coefficients, having a solution of the form $\sqrt[5]{A}+\sqrt[5]{B}+\sqrt[5]{C}+\sqrt[5]{D}$, where A, B, C, D are numerical expressions involving only square roots. But the solutions are not presented in their most simple form: thus in example 1, $x^5+3x^2+2x-1=0$, the expression involves a radical

$$\sqrt{\tfrac{47}{8}(21125+9439\sqrt{5})}:$$

here

$$(21125+9439)\sqrt{5},\ =\sqrt{5}\,(9439+4225\sqrt{5}),\ =\sqrt{5}\,.\,\tfrac{1}{2}(18+5\sqrt{5})^2(1+\sqrt{5})^2(2+\sqrt{5}),$$

so that, taking out the roots of the squared factors, we have as the proper form of the radical the very much more simple form $\sqrt{47(2+\sqrt{5})\sqrt{5}}$; where observe that $(2+\sqrt{5})(2-\sqrt{5})=-1$, and thence $(2+\sqrt{5})\sqrt{-47(2-\sqrt{5})\sqrt{5}}=\sqrt{47(2+\sqrt{5})\sqrt{5}}$, viz. the conjugate radicals $\sqrt{-47(2-\sqrt{5})\sqrt{5}}$ and $\sqrt{47(2+\sqrt{5})\sqrt{5}}$ differ only by a factor $2+\sqrt{5}$ which is rational in 1 and $\sqrt{5}$. To avoid fractions I consider the foregoing equation under the form

$$x^5+3000x^2+20000x-100000=0,$$

and I will presently give the solution; but first I consider the general theory. Writing

$$\begin{aligned}
A&=\alpha^5, & A'&=\alpha^2\gamma, & A''&=\alpha^3\beta,\\
B&=\beta^5, & B'&=\alpha\beta^2, & B''&=\beta^3\delta,\\
C&=\gamma^5, & C'&=\gamma^2\delta, & C''&=\alpha\gamma^3,\\
D&=\delta^5, & D'&=\beta\delta^2, & D''&=\gamma\delta^3,
\end{aligned}$$

we have $A'D'=\alpha^2\delta^2\beta\gamma$, $B'C'=\alpha\delta\beta^2\gamma^2$. Also

$$A''=\frac{A'B'}{\beta\gamma},\quad B''=\frac{B'D'}{\alpha\delta},\quad C''=\frac{A'C'}{\alpha\delta},\quad D''=\frac{C'D'}{\beta\gamma},$$

which determine A'', B'', C'', D'' in terms of A', B', C', D', $\alpha\delta$, $\beta\gamma$; and then

$$A = \frac{A'A''}{\beta\gamma},\quad B = \frac{B'B''}{\alpha\delta},\quad C = \frac{C'C''}{\alpha\delta},\quad D = \frac{D'D''}{\beta\gamma},$$

which give A, B, C, D.

If now we assume $x = \alpha + \beta + \gamma + \delta$, and regard A, B, C, D, A', B', C', D', A'', B'', C'', D'', $\alpha\delta$, $\beta\gamma$ each as a rational function, we may express x, x^2, x^3, x^5 each of them by means of rational functions or of rational functions multiplied into α, β, γ, δ respectively: thus,

$$\begin{aligned}
x &= \alpha + \beta + \gamma + \delta &&= \alpha + \beta + \gamma + \delta,\\
x^2 &= \alpha^2 + \beta^2 + \gamma^2 + \delta^2 &&= \frac{A'\beta}{\beta\gamma} + \frac{B'\delta}{\alpha\delta} + \frac{C'\alpha}{\alpha\delta} + \frac{D'\gamma}{\beta\gamma},\\
&\quad + 2\alpha\beta + 2\alpha\gamma + 2\alpha\delta + 2\beta\gamma + 2\beta\delta + 2\gamma\delta &&\quad + \frac{2B'\gamma}{\beta\gamma} + \frac{2A'\delta}{\alpha\delta} + 2\alpha\delta + 2\beta\gamma + \frac{2D'\alpha}{\alpha\delta} + \frac{2C'\beta}{\beta\gamma},
\end{aligned}$$

&c.; and we thus obtain

$$\begin{aligned}
&x^5 + qx^3 + rx^2 + sx + t\\
&= A + B + C + D\\
&\quad + (20\alpha\delta + 30\beta\gamma)(A' + D') + 30(\alpha\delta + 20\beta\gamma)(B' + C')\\
&\qquad\qquad + 3q(A' + B' + C' + D') + 2r(\alpha\delta + \beta\gamma) + t\\
&\quad + \alpha\Big\{5A'' + 5\frac{C'^2}{\alpha\delta} + \frac{5B''\beta\gamma}{\alpha\delta} + 10C'' + \frac{10D'^2}{\alpha\delta}\\
&\qquad\qquad + 10\alpha^2\delta^2 + 20B'' + 20D'' + 30\beta^2\gamma^2 + 30D''\frac{\beta\gamma}{\alpha\delta} + 60\alpha\beta\gamma\delta\\
&\qquad\qquad + q\left(\frac{B''}{\alpha\delta} + \frac{3D''}{\alpha\delta} + 3\alpha\delta + 6\beta\gamma\right) + r\left(\frac{C'}{\alpha\delta} + \frac{2D'}{\alpha\delta}\right) + s\Big\}\\
&\quad + \beta\Big\{5B'' + \frac{5A'^2}{\beta\gamma} + 5\frac{D''\alpha\delta}{\beta\gamma} + 10A'' + \frac{10C'^2}{\beta\gamma}\\
&\qquad\qquad + 10\beta^2\gamma^2 + 20D'' + 20C'' + 30\alpha^2\delta^2 + 30C''\frac{\alpha\delta}{\beta\gamma} + 60\alpha\beta\gamma\delta\\
&\qquad\qquad + q\left(\frac{D''}{\beta\gamma} + \frac{3C''}{\beta\gamma} + 3\beta\gamma + 6\alpha\delta\right) + r\left(\frac{A'}{\beta\gamma} + \frac{2C'}{\beta\gamma}\right) + s\Big\}\\
&\quad + \gamma\Big\{5C'' + \frac{5D'^2}{\beta\gamma} + 5\frac{A''\alpha\delta}{\beta\gamma} + 10D'' + \frac{10B'^2}{\beta\gamma}\\
&\qquad\qquad + 10\beta^2\gamma^2 + 20A'' + 20B'' + 30\alpha^2\delta^2 + 30B''\frac{\alpha\delta}{\beta\gamma} + 60\alpha\beta\gamma\delta\\
&\qquad\qquad + q\left(\frac{A''}{\beta\gamma} + \frac{3B''}{\beta\gamma} + 3\beta\gamma + 6\alpha\delta\right) + r\left(\frac{D'}{\beta\gamma} + \frac{2B'}{\beta\gamma}\right) + s\Big\}\\
&\quad + \delta\Big\{5D'' + \frac{5B'^2}{\alpha\delta} + 5\frac{C''\beta\gamma}{\alpha\delta} + 10B'' + \frac{10A'^2}{\alpha\delta}\\
&\qquad\qquad + 10\alpha^2\delta^2 + 20C'' + 20A'' + 30\beta^2\gamma^2 + 30A''\frac{\beta\gamma}{\alpha\delta} + 60\alpha\beta\gamma\delta\\
&\qquad\qquad + q\left(\frac{C''}{\alpha\delta} + \frac{3A''}{\alpha\delta} + 3\alpha\delta + 6\beta\gamma\right) + r\left(\frac{B'}{\alpha\delta} + \frac{2A'}{\alpha\delta}\right) + s\Big\}.
\end{aligned}$$

If, then, $x^5+qx^3+rx^2+sx+t=0$, we have the rational term $=0$, and the coefficients of α, β, γ, δ each $=0$; in the class of equations under consideration, these last equations differ only in the signs of the radicals contained therein, so that one of them being satisfied identically, the others will be also satisfied. In particular, if $q=0$, then $\alpha\delta+\beta\gamma=0$: the rational term gives

$$A+B+C+D-10\alpha\delta(A'+D'-B'-C')+t=0,$$

and the term in α gives

$$5A''+15B''+10C''-10D''+\frac{5}{\alpha\delta}(C'^2+2D'^2)+\frac{r}{\alpha\delta}(C'+2D')-20\alpha^2\delta^2+s=0.$$

For the equation $x^5+3000x^2+20000x-100000=0$, the expression for the root is $x=\sqrt[5]{A}+\sqrt[5]{B}+\sqrt[5]{C}+\sqrt[5]{D}$, where

$$\begin{aligned}
A &= 39000+18200\sqrt{5}+(1720+920\sqrt{5})\sqrt{235+94\sqrt{5}},\\
D &= 39000+18200\sqrt{5}+(-1720-920\sqrt{5})\sqrt{235+94\sqrt{5}},\\
B &= 39000-18200\sqrt{5}+(-1720+920\sqrt{5})\sqrt{235-94\sqrt{5}},\\
C &= 39000-18200\sqrt{5}+(1720-920\sqrt{5})\sqrt{235-94\sqrt{5}},
\end{aligned}$$

and where also

$$\begin{aligned}
A' &= -150-70\sqrt{5}+(-10-2\sqrt{5})\sqrt{235+94\sqrt{5}},\\
D' &= -150-70\sqrt{5}+(10+2\sqrt{5})\sqrt{235+94\sqrt{5}},\\
B' &= -150+70\sqrt{5}+(10-2\sqrt{5})\sqrt{235-94\sqrt{5}},\\
C' &= -150+70\sqrt{5}+(-10+2\sqrt{5})\sqrt{235-94\sqrt{5}},
\end{aligned}$$

and

$$\begin{aligned}
A'' &= -940-100\sqrt{5}+(-100+20\sqrt{5})\sqrt{235+94\sqrt{5}},\\
D'' &= -940-100\sqrt{5}+(100-20\sqrt{5})\sqrt{235+94\sqrt{5}},\\
B'' &= -940+100\sqrt{5}+(100+20\sqrt{5})\sqrt{235-94\sqrt{5}},\\
C'' &= -940+100\sqrt{5}+(-100-20\sqrt{5})\sqrt{235-94\sqrt{5}}.
\end{aligned}$$

The foregoing forms are in some respects the most convenient; but it is to be observed that we have

$$\begin{aligned}
A &= 2600\sqrt{5}(1+\sqrt{5})(2+\sqrt{5})+40(1+\sqrt{5})(18+5\sqrt{5})\sqrt{47\sqrt{5}(2+\sqrt{5})}, \text{ \&c.,}\\
A' &= -10\sqrt{5}(1+\sqrt{5})(2+\sqrt{5})-2\sqrt{5}(1+\sqrt{5})\sqrt{47\sqrt{5}(2+\sqrt{5})}, \text{ \&c.,}\\
A'' &= 20(1-\sqrt{5})(18+13\sqrt{5})+20\sqrt{5}(1-\sqrt{5})\sqrt{47\sqrt{5}(2+\sqrt{5})}, \text{ \&c.,}
\end{aligned}$$

or putting for shortness

$$\sqrt{Q}=\sqrt{47\sqrt{5}(2+\sqrt{5})},\quad \sqrt{Q_1}=\sqrt{-47\sqrt{5}(2-\sqrt{5})},$$

(so that, according to a foregoing remark, we have $(2+\sqrt{5})\sqrt{Q}=\sqrt{Q_1}$), then we have

$$\begin{aligned} A\ &= 40(1+\sqrt{5})\{65\sqrt{5}(2+\sqrt{5})+(18+5\sqrt{5})\sqrt{Q}\}, \text{ \&c.,}\\ A' &= -2\sqrt{5}(1+\sqrt{5})\{5(2+\sqrt{5})+\sqrt{Q}\}, \text{ \&c.,}\\ A'' &= 20(1-\sqrt{5})\{18+13\sqrt{5}+\sqrt{5}\sqrt{Q}\}, \text{ \&c.,}\end{aligned}$$

where observe that the term $2+\sqrt{5}$ is a factor of Q.

Starting from the values of A', B', C', D', we have

$$A'=-2\sqrt{5}(1+\sqrt{5})\{5(2+\sqrt{5})+\sqrt{Q}\},$$

$$D'=-2\sqrt{5}(1+\sqrt{5})\{5(2+\sqrt{5})-\sqrt{Q}\},$$

and therefore

$$A'D'=20(1+\sqrt{5})^2(2+\sqrt{5})\{25(2+\sqrt{5})-47\sqrt{5}\},$$

where the last factor is

$$=50-22\sqrt{5},\ =-2\sqrt{5}(11-5\sqrt{5}),\ =-\sqrt{5}(1-\sqrt{5})^2(2-\sqrt{5}).$$

Hence

$$A'D'=-\sqrt{5}\,.\,20(-4)^2(-1)=320\sqrt{5},$$

that is,

$$A'D'=(\alpha\delta)^2\beta\gamma=320\sqrt{5}, \text{ and similarly } B'C'=\alpha\delta(\beta\gamma)^2=-320\sqrt{5},$$

whence

$$\alpha\delta=-4\sqrt{5},\quad \beta\gamma=4\sqrt{5}, \text{ and } \alpha\delta+\beta\gamma=0, \text{ as above.}$$

We have, moreover,

$$A'=-2\sqrt{5}(1+\sqrt{5})\{5(2+\sqrt{5})+\sqrt{Q}\},$$

$$B'=\ 2\sqrt{5}(1-\sqrt{5})\{5(2-\sqrt{5})-\sqrt{Q_1}\},$$

and thence

$$\begin{aligned} A'B' &= 80\{-25-\sqrt{QQ_1}+5(2-\sqrt{5})\sqrt{Q}-5(2+\sqrt{5})\sqrt{Q_1}\},\\ &= 80\{-25-47\sqrt{5}+(5(2-\sqrt{5})-5)\sqrt{Q}\},\end{aligned}$$

that is,

$$\begin{aligned} A'B'\div\beta\gamma &= 4\sqrt{5}\{-25-47\sqrt{5}+5(1-\sqrt{5})\sqrt{Q}\}\\ &= -20(47+5\sqrt{5})+20\sqrt{5}(1-\sqrt{5})\sqrt{Q},\ =A'';\end{aligned}$$

and similarly we verify the values of B'', C'' and D''.

We have next

$$A'A''=160\sqrt{5}\{(10+5\sqrt{5}+\sqrt{Q})(18+13\sqrt{5}+\sqrt{5}\sqrt{Q})\},$$

or observing that $Q\sqrt{5}$ is $=235(2+\sqrt{5})$, the whole term in $\{\ \}$ is

$$\begin{aligned} &= (505+220\sqrt{5})+(470+235\sqrt{5})+(18+13\sqrt{5}+25+10\sqrt{5})\sqrt{Q},\\ &= 975+455\sqrt{5}+(43+23\sqrt{5})\sqrt{Q}=65\sqrt{5}(7+3\sqrt{5})+(43+23\sqrt{5})\sqrt{Q};\end{aligned}$$

or we have

$$\begin{aligned} A'A'' &= 160\sqrt{5}\{65\sqrt{5}(7+3\sqrt{5})+(43+23\sqrt{5})\sqrt{Q}\},\\ &= 160\sqrt{5}(1+\sqrt{5})\{65\sqrt{5}(2+\sqrt{5})+(18+5\sqrt{5})\sqrt{Q}\},\end{aligned}$$

and consequently

$$A'A''\div\beta\gamma=40(1+\sqrt{5})\{65\sqrt{5}(2+\sqrt{5})+(18+5\sqrt{5})\sqrt{Q}\},\ =A;$$

and similarly we verify the values of B, C, D.

In the proposed equation $x^5+3000x^2+20000x-100000=0$, we have $r=3000$, $s=20000$, $t=-100000$, $\alpha\delta=-4\sqrt{5}$; the two equations to be verified thus are

$$A+B+C+D+40\sqrt{5}\,(A'+D'-B'-C')-100000=0,$$

and

$$5A''+15B''+10C''-10D''-\frac{\sqrt{5}}{4}(C'^2+2D'^2)-150\sqrt{5}\,(C'+2D')-1600+20000=0.$$

As to the first of these, we have $A+B+C+D=156000$, $A'+D'-B'-C'=-280\sqrt{5}$, and the equation thus is

$$156000+40\sqrt{5}\,(-280\sqrt{5})-100000=0,$$

which is right.

For the second equation, if in the calculation we keep the radicals in the first instance distinct, we have

$$5A''+15B''+10C''-10D''=-18800+3000\sqrt{5}+(-1500+300\sqrt{5})\sqrt{Q}+(500+100\sqrt{5})\sqrt{Q_1},$$

$$\begin{aligned}-150\sqrt{5}\,(C'+2D')&=\{-450-70\sqrt{5}\\&\qquad+(20+4+\sqrt{5})\sqrt{Q}+(-10+2\sqrt{5})\sqrt{Q_1}\}(-150\sqrt{5})\\-1600+20000\quad&=18400\\-\frac{\sqrt{5}}{4}(C'^2+2D'^2)\quad&=-\frac{\sqrt{5}}{4}\{282000+416800\sqrt{5}\\&\qquad+(-8800-4000\sqrt{5})\sqrt{Q}+(4400-2000\sqrt{5})\sqrt{Q_1}\}.\end{aligned}$$

Substituting in the equation, we ought to have

$$\begin{aligned}0=&-18800+\ \ 3000\sqrt{5}+(-1500+\ \ 300\sqrt{5})\sqrt{Q}+(\ \ \ 500+\ \ 100\sqrt{5})\sqrt{Q_1}\\&+52500+67500\sqrt{5}+(-3000-3000\sqrt{5})\sqrt{Q}+(-1500+1500\sqrt{5})\sqrt{Q_1}\\&+18400\\&-52100-70500\sqrt{5}+(\ \ \ 5000+2200\sqrt{5})\sqrt{Q}+(\ \ \ 2500-1100\sqrt{5})\sqrt{Q_1},\end{aligned}$$

that is,

$$0=(500-500\sqrt{5})\sqrt{Q}+(1500+500\sqrt{5})\sqrt{Q_1},$$

viz.

$$0=\ 500\{(1-\sqrt{5})\sqrt{Q}+(3+\sqrt{5})\sqrt{Q_1}\},$$

which is satisfied in virtue of $\sqrt{Q}=(2+\sqrt{5})\sqrt{Q_1}$: this completes the verification.