911.

ON AN ALGEBRAICAL IDENTITY RELATING TO THE SIX COORDINATES OF A LINE.

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THE following identity may be verified without difficulty; but it is interesting in regard to the analytical theory of the line.

The identity is

$$\begin{aligned}0 = &\begin{vmatrix} B, & C, & F \\ b, & c, & f \\ b', & c', & f' \end{vmatrix} \begin{vmatrix} A, & B, & C \\ a, & b, & c \\ a', & b', & c' \end{vmatrix} \\
&-(bc'-b'c)^2(AF+BG+CH) \\
&+(bc'-b'c)(bC-cB)(Af'+Bg'+Ch'+Fa'+Gb'+Hc') \\
&-(bc'-b'c)(b'C-c'B)(Af+Bg+Ch+Fa+Gb+Hc) \\
&-(bC-cB)^2(a'f'+b'g'+c'h') \\
&+(bC-cB)(b'C-c'B)(af'+a'f+bg'+b'g+ch'+c'h) \\
&-(b'C-c'B)^2(af+bg+ch),\end{aligned}$$

where all the letters have arbitrary values.

It follows that, if

$$\begin{aligned}&af+bg+ch=0,\\
&a'f'+b'g'+c'h'=0,\\
&af'+a'f+bg'+b'g+ch'+c'h=0,\\
&AF+BG+CH=0,\\
&Af+Bg+Ch+Fa+Gb+Hc=0,\\
&Af'+Bg'+Ch'+Fa'+Gb'+Hc'=0,\end{aligned}$$

then either

$$\begin{vmatrix} B, & C, & F \\ b\,, & c\,, & f \\ b', & c', & f' \end{vmatrix} = 0,$$

or else

$$\begin{vmatrix} A, & B, & C \\ a\,, & b\,, & c \\ a', & b', & c' \end{vmatrix} = 0.$$

Supposing the first three of the six equations are satisfied, then (a, b, c, f, g, h) and (a', b', c', f', g', h') are the coordinates of two intersecting lines; and supposing that the last three equations are also satisfied, then (A, B, C, F, G, H) will be the coordinates of a line meeting each of the intersecting lines. The third line is thus either in the plane of the intersecting lines, or else passes through their point of intersection; and in fact, the first of the two determinant equations is the condition in order that the line may be in the plane of the two intersecting lines, and the second determinant equation is the condition in order that it may pass through their point of intersection. Each equation is in fact one out of four equivalent forms, viz. we may have in the first equation (B, C, F), (C, A, G), (A, B, H), or (F, G, H); and in the second equation (A, B, C), (A, H, G), (H, B, F), or (G, F, C).

The analytical theory may be presented in a complete form by means of the formulæ of my memoir "On the six coordinates of a line," (1867), *Camb. Phil. Trans.*, t. XI. pp. 290—323, [435]. Considering the two intersecting lines (a, b, c, f, g, h) and (a', b', c', f', g', h'), the coordinates of the plane through these two lines (that is, the coefficients ξ, η, ζ, ω of the equation $\xi x + \eta y + \zeta z + \omega w = 0$ of the plane) are there given (see p. 295*) in a fourfold form; and if we thence form the condition in order that the line (A, B, C, F, G, H) may lie in this plane, we have

$$\left(\begin{array}{cccc} 0, & C, & -B, & F \\ -C, & 0, & A, & G \\ B, & -A, & 0, & H \\ -F, & -G, & -H, & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc} af'+b'g+c'h, & bf'-b'f\,, & cf'-c'f\,, & -(bc'-b'c) \\ ag'-a'g\,, & a'f+bg'+c'h, & cg'-c'g\,, & -(ca'-c'a) \\ ah'-a'h\,, & bh'-b'h\,, & a'f+b'g+ch', & -(ab'-a'b) \\ gh'-g'h\,, & hf'-h'f\,, & fg'-f'g\,, & af'+bg'+ch' \end{array}\right) = 0;$$

viz. the condition is expressible in any one of the 16 forms obtained by combining a line of the first matrix with a line of the second matrix; thus one form is

$$0\,(af'+b'g+c'h) + C\,(bf'-b'f) - B\,(cf'-c'f) - F\,(bc'-b'c) = 0,$$

[* This Collection, vol. VII., p. 71.]

that is,
$$\begin{vmatrix} B, & C, & F \\ b, & c, & f \\ b', & c', & f' \end{vmatrix} = 0,$$
the foregoing first determinant equation, which thus belongs to the case where the line lies in the plane of the two intersecting lines.

Again we have (see p. 296*) an expression, in a fourfold form, for the coordinates of the point of intersection of the two intersecting lines; and thence for the condition, in order that the line (A, B, C, F, G, H) may pass through this point, we have
$$\left(\begin{array}{cccc} 0, & H, & -G, & A \\ -F, & 0, & F, & B \\ G, & -H, & 0, & C \\ -A, & -B, & -C, & 0 \end{array}\right)$$
$$\left(\begin{array}{cccc} af'+b'g+c'h, & ag'-a'g\ , & ah'-a'h\ , & gh'-g'h \\ bf'-b'f\ , & a'f+bg'+c'h, & bh'-b'h\ , & hf'-h'f \\ cf'-c'f\ , & cg'-c'g\ , & a'f+b'g+ch', & fg'-f'g \\ -(bc'-b'c)\ , & -(ca'-c'a)\ , & -(ab'-a'b)\ , & af'+bg'+ch' \end{array}\right) = 0,$$
viz. the condition is expressible in any one of the 16 forms obtained by combining a line of the first matrix with a line of the second matrix; thus one form is
$$A\,(bc'-b'c)+B\,(ca'-c'a)+C\,(ab'-a'b)+0\,(af'+bg'+ch')=0,$$
that is,
$$\begin{vmatrix} A, & B, & C \\ a, & b, & c \\ a', & b', & c' \end{vmatrix} = 0,$$
the foregoing second determinant equation, which thus belongs to the case where the line passes through the point of intersection of the two intersecting lines.

I remark that the original identity may be written in the very compendious symbolical form
$$\begin{pmatrix} A, & a, & a' \\ F, & f, & f' \end{pmatrix}\begin{pmatrix} B, & b, & b' \\ C, & c, & c' \end{pmatrix}^2 = \begin{vmatrix} A, & a, & a' \\ B, & b, & b' \\ C, & c, & c' \end{vmatrix}\ \begin{vmatrix} F, & f, & f' \\ B, & b, & b' \\ C, & c, & c' \end{vmatrix},$$
viz. here on the left-hand side the second factor denotes the three determinants $bc'-b'c$, $b'C-c'B$, $Bc-Cb$: the whole is a quadric function of these, the coefficients being
$$AF+BG+CH,\quad af+bg+ch,$$
$$a'f'+b'g'+c'h',\quad af'+bg'+ch'+fa'+gb'+hc',$$
$$a'F+b'G+c'H+f'A+g'B+h'C,$$
$$Af+Bg+Ch+Fa+Gb+Hc,$$
respectively; and the right-hand side is simply the product of two determinants.

[* *Loc. cit.*, p. 72.]