902.

ON THE FOCALS OF A QUADRIC SURFACE.

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In plane geometry, the focus of a curve is the node of the circumscribed line-system of the curve and the circular points at infinity; and so, in solid geometry, the focal of a surface or curve is the nodal line of the circumscribed developable of the surface or curve and the circle at infinity. And as in plane geometry the circular points at infinity may be regarded as an indefinitely thin conic, so in solid geometry the circle at infinity may be regarded as an indefinitely thin quadric surface.

In plane geometry, let it be proposed to find the circumscribed line-system (common tangents) of the two quadrics

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=0,\ \ \frac{x^2}{a'}+\frac{y^2}{b'}+\frac{z^2}{c'}=0\,;$$

if a common tangent be

$$\xi x+\eta y+\zeta z=0,$$

then we have

$$a\xi^2+b\eta^2+c\zeta^2=0,$$
$$a'\xi^2+b'\eta^2+c'\zeta^2=0\,;$$

here writing for shortness

$$f,\ g,\ h=bc'-b'c,\ ca'-c'a,\ ab'-a'b,$$

we have

$$\xi^2:\eta^2:\zeta^2=f:g:h\,;$$

and thence the tangent is

$$x\surd(f)\pm y\surd(g)\pm z\surd(h)=0,$$

viz. we have thus four tangents, and the rationalised form is of course

$$f^2x^4+g^2y^4+h^2z^4-2ghy^2z^2-2hfz^2x^2-2fgx^2y^2=0.$$

In connexion with the corresponding question in solid geometry, I obtain this equation in a different manner. We investigate the envelope of the line $\xi x+\eta y+\zeta z=0$, considering ξ, η, ζ as parameters connected by the foregoing two equations; by the ordinary process of indeterminate multipliers, we have

$$x+(\lambda a+\mu a')\,\xi=0,\quad y+(\lambda b+\mu b')\,\eta=0,\quad z+(\lambda c+\mu c')\,\zeta=0,$$

and thence, eliminating ξ, η, ζ, we obtain

$$\frac{x^2}{\lambda a+\mu a'}+\frac{y^2}{\lambda b+\mu b'}+\frac{z^2}{\lambda c+\mu c'}=0,$$

$$\frac{ax^2}{(\lambda a+\mu a')^2}+\frac{by^2}{(\lambda b+\mu b')^2}+\frac{cz^2}{(\lambda c+\mu c')^2}=0,$$

$$\frac{a'x^2}{(\lambda a+\mu a')^2}+\frac{b'y^2}{(\lambda b+\mu b')^2}+\frac{c'z^2}{(\lambda c+\mu c')^2}=0,$$

equations equivalent to two equations, from which λ and μ are to be eliminated. The second and third equations are the derived functions of the first equation in regard to λ, μ respectively; and hence, expressing the first equation in an integral form, the result is

$$\text{Disct. }\{x^2(\lambda b+\mu b')(\lambda c+\mu c')+\&\text{c.}\}=0\,;$$

viz. this is

$$\{(bc'+b'c)\,x^2+(ca'+c'a)\,y^2+(ab'+a'b)\,z^2\}^2-4\,(bcx^2+cay^2+abz^2)\,(b'c'x^2+c'a'y^2+a'b'z^2)=0,$$

or developing and reducing, we have the foregoing result, the coefficients entering through the combinations

$$f,\ g,\ h=bc'-b'c,\ ca'-c'a,\ ab'-a'b.$$

Writing $c=-1$, also $a'=b'=1$, $c'=0$: the equation of the first conic is

$$\frac{x^2}{a}+\frac{y^2}{b}-z^2=0,$$

and that of the second may be replaced by the two equations $x^2+y^2=0$, $z=0$, viz. these give the circular points at infinity: we have f, g, $h=1$, -1, $a-b$, and the equation of the line-system is

$$x^4+2x^2y^2+y^4-2h\,(x^2-y^2)\,z^2+h^2z^4=0.$$

If finally, $z=1$, then for the tangents from the circular points at infinity to the quadric

$$\frac{x^2}{a}+\frac{y^2}{b}-1=0,$$

the equation is

$$x^4+2x^2y^2+y^4-2h\,(x^2-y^2)+h^2=0,$$

where, as before, $h=a-b$; the four tangents intersect in pairs in the two circular points at infinity, and in four other points which are the foci of the quadric.

Passing now to the problem in solid geometry, we consider the two quadric surfaces

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}+\frac{w^2}{d}=0,$$

$$\frac{x^2}{a'}+\frac{y^2}{b'}+\frac{z^2}{c'}+\frac{w^2}{d'}=0;$$

if a common tangent plane hereof be

$$\xi x+\eta y+\zeta z+\omega w=0,$$

then we have

$$a\xi^2+b\eta^2+c\zeta^2+d\omega^2=0,$$
$$a'\xi^2+b'\eta^2+c'\zeta^2+d'\omega^2=0;$$

and the circumscribed developable is the envelope of the plane, considering in the equation thereof ξ, η, ζ, ω as variable parameters connected by the last two equations. By what precedes, it is at once seen that the resulting equation is

$$\text{Disct. }\{x^2(b\lambda+b'\mu)(c\lambda+c'\mu)(d\lambda+d'\mu)+\&\text{c.}\}=0,$$

viz. this is a quadric equation in (x^2, y^2, z^2, w^2), the coefficients a, b, c, d, a', b', c', d', entering therein through the combinations

$$\mathrm{a},\ \mathrm{b},\ \mathrm{c}=bc'-b'c,\ \ ca'-c'a,\ \ ab'-a'b;$$
$$\mathrm{f},\ \mathrm{g},\ \mathrm{h}=ad'-a'd,\ \ bd'-b'd,\ \ cd'-c'd.$$

The developed result is given in my paper "On the developable surfaces which arise from two surfaces of the second order," *Camb. and Dub. Math. Jour.*, t. v. (1850), pp. 45—53, [84]. We require here, for the particular case, $d=-1$, $a'=b'=c'=1$, $d'=0$, viz. one of the surfaces is taken to be

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}-w^2=0,$$

and the other is the circle at infinity $x^2+y^2+z^2=0$, $w=0$; we thus have a, b, c = 1, 1, 1; f, g, h $=b-c$, $c-a$, $a-b$, or now using the italic letters f, g, h to signify these values, we have $f+g+h=0$. The equation is

$$\begin{aligned}
&f^2x^8+g^2y^8+h^2z^8+f^2g^2h^2w^8\\
&\quad+\ 2g(g-h)\,y^6z^2+2h(h-f)\,z^6x^2+2f(f-g)\,x^6y^2\\
&\quad-\ 2h(g-h)\,y^2z^6-2f(h-f)\,z^2x^6-2g(f-g)\,x^2y^6\\
&\quad+\ 2f^2(g-h)\,x^6w^2+2g^2(h-f)\,y^6w^2+2h^2(f-g)\,z^6w^2\\
&\quad-\ 2f^2gh(g-h)\,x^2w^6-2fg^2h(h-f)\,y^2w^6-2fgh^2(f-g)\,z^2w^6\\
&\quad+\ f^2(f^2-6gh)\,x^4w^4+g^2(g^2-6hf)\,y^4w^4+h^2(h^2-6fg)\,z^4w^4\\
&\quad+\ (f^2-6gh)\,y^4z^4+(g^2-6hf)\,z^4x^4+(h^2-6fg)\,x^4y^4\\
&\quad+\ 2gh(gh-3f^2)\,w^4y^2z^2+2hf(hf-3g^2)\,w^4z^2x^2+2fg(fg-3h^2)\,w^4x^2y^2\\
&\quad+\ 2h(gh-3f^2)\,z^4x^2w^2+2f(hf-3g^2)\,x^4y^2w^2+2g(fg-3h^2)\,y^4z^2w^2\\
&\quad-\ 2g(gh-3f^2)\,y^4x^2w^2-2h(hf-3g^2)\,z^4y^2w^2-2f(fg-3h^2)\,x^4z^2w^2\\
&\quad+\ 2(gh-3f^2)\,x^4y^2z^2-2(hf-3g^2)\,x^2y^4z^2-2(fg-3h^2)\,x^2y^2z^4\\
&\quad+\ 2(g-h)(h-f)(f-g)\,x^2y^2z^2w^2=0.
\end{aligned}$$

The equation may be written in the following four forms:

$$x^2\Theta_1 + (gy^2 - hz^2 + ghw^2)^2 \{y^4 + 2y^2z^2 + z^4 - 2f(y^2 - z^2)w^2 + f^2w^4\} = 0,$$

$$y^2\Theta_1 + (hz^2 - fx^2 + hfw^2)^2 \{z^4 + 2z^2x^2 + x^4 - 2g(z^2 - x^2)w^2 + g^2w^4\} = 0,$$

$$z^2\Theta_1 + (fx^2 - gy^2 + fgw^2)^2 \{x^4 + 2x^2y^2 + y^4 - 2h(x^2 - y^2)w^2 + h^2w^4\} = 0,$$

$$w^2\Theta_1 + (x^2 + y^2 + z^2)^2 \times \{f^2x^4 + g^2y^4 + h^2z^4 - 2ghy^2z^2 - 2hfz^2x^2 - 2fgx^2y^2\} = 0;$$

the last of these shows that the circle at infinity $x^2 + y^2 + z^2 = 0$, $w = 0$, is a nodal line on the surface; this, however, is not regarded as a focal. The other three show that we have also, as nodal lines, three conics, which are the focals of the given surface; viz. now writing $w = 1$, we have, for the quadric surface

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} - 1 = 0,$$

the three focal conics

$$x = 0, \quad -\frac{y^2}{h} + \frac{z^2}{g} - 1 = 0,$$

$$y = 0, \quad \frac{x^2}{h} \quad - \frac{z^2}{f} - 1 = 0,$$

$$z = 0, \quad -\frac{x^2}{g} + \frac{y^2}{f} \quad - 1 = 0,$$

where, as before, f, g, $h = b - c$, $c - a$, $a - b$ respectively. If, as usual, a, b, c are positive and in order of decreasing magnitude, then for the ellipsoid

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} - 1 = 0,$$

we have the focal conics

$$x = 0, \quad -\frac{y^2}{a-b} + \frac{z^2}{a-c} - 1 = 0,$$

$$y = 0, \quad \frac{x^2}{a-b} \quad - \frac{z^2}{b-c} - 1 = 0,$$

$$z = 0, \quad \frac{x^2}{a-c} + \frac{y^2}{b-c} \quad - 1 = 0;$$

viz. these are an imaginary conic, the focal hyperbola, and the focal ellipse, in the coordinate planes $x = 0$, $y = 0$, $z = 0$ respectively.