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Group judgement with ties. A position - based approach

H. Bury, D. Wagner

Instytut Badań Systemowych
Polska Akademia Nauk

Systems Research Institute
Polish Academy of Sciences

POLSKA AKADEMIA NAUK

Instytut Badań Systemowych

ul. Newelska 6

01-447 Warszawa

tel.: (+48) (22) 3810100

fax: (+48) (22) 3810105

Kierownik Pracowni zgłaszający pracę:
Prof. dr hab. inż. Janusz Kacprzyk

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GROUP JUDGEMENT WITH TIES. A POSITION-BASED APPROACH

1. INTRODUCTION

Problems of determining a group judgement have been investigated for over two centuries. Since there is no ideal method satisfying all the requirements formulated, new methods possessing desirable properties and avoiding the deficiencies of previous ones are being developed. To efficiently analyze and solve problems of determining group decisions, some simplifying assumptions are introduced. Usually, it is assumed that no tied alternatives can occur either in expert opinion or in group judgement. However, in real life problems experts are not always able to uniquely determine the order of alternatives with respect to a given criterion or set of criteria. In such situations the occurrence of tied alternatives should be taken into account.

Some methods of group judgement can be adapted to ties in experts' opinions. But determining a group decision with tied alternatives is more complicated.

Generally, it is assumed that no tied alternatives can occur in group judgement, even if there are tied alternatives in experts' opinions. This assumption seems to be rather restrictive and may affect the solution obtained.

Experts' opinions may take different forms. In this paper it is assumed that preference orders are used. Cook and Seiford [11] proposed a system for enumerating positions taken by the alternatives in preference orders that makes the problem of tied alternatives easier to handle. This approach has been applied to positional methods of group judgement, i.e. methods taking into account the positions of the alternatives in preference orders. It will be shown that within this framework some methods defined for the case of no ties can be extended to the case of ties in experts' opinions, as well as in group judgement.

A modification of the Borda count is proposed making it possible – in the case of ties – to obtain the same results for the classical definition, as well as when using an outranking matrix. Moreover, some rules for generating structures of preference orders to be searched for in problems of determining group judgement are also given.

2. POSITIONS OF ALTERNATIVES IN A PREFERENCE ORDER

Assume there is a set of n alternatives $\mathcal{O}=\{O_1, \ldots, O_n\}$ and a group of K experts who are asked to order this set according to a given criterion (set of criteria). It is assumed that the alternative regarded as the best one (in the sense of a criterion/ criteria adopted) takes first position and the one regarded as the worst one takes last position.

A preference order with ties is generally of the form $O_{i_1},\ldots,(O_{i_p},\ldots,O_{i_{p+r}}),\ldots,O_{i_n}$, where r tied alternatives ($r\geq 0$) placed in the same position p are given in brackets. This notation is referred to as classical. The positions taken by the alternatives are as follows

$$1, 2, \ldots, p-1, (\underbrace{p, p, \ldots, p}_{r \text{ times}}), p+1, \ldots, n-r+1. \tag{1}$$

Cook and Seiford [11] proposed assigning to a group of r tied alternatives a position t defined as the mean one

$$t=\frac{p+(p+1)+\ldots+(p+r-1)}{r}=\frac{2p+(r-1)}{2r}r=p+\frac{r-1}{2}. \tag{2}$$

The expression obtained is of the form $v+½$ for any even r and is an integer otherwise; where p, r, v are integer numbers. This notation is henceforth referred to as fractional.

For n alternatives, the positions to be considered are taken from the set

$$\mathcal{T}=\{1, 1½, 2, 2½, 3, 3½, \ldots, n-1, n-½, n\}. \tag{3}$$

The number of possible positions is equal to $2n$-1. It is evident that there may be positions with no alternatives assigned to.

Example 1.

Three preference orders for ten alternatives are given below. Tied alternatives are given in brackets.

P^1:	$\{O_4, O_2, O_9, O_6, O_7, O_{10}, O_8, O_5, O_3, O_1\}$	
P^2:	$\{O_6, (O_1, O_2, O_3, O_4, O_5, O_7, O_8, O_{10}), O_9\}$	(4)
P^3:	$\{(O_3, O_7), O_5, O_4, O_9, (O_2, O_6), (O_1, O_8, O_{10})\}$	

The positions of alternatives in the preference orders considered are as follows.

- Using classical notation

	O_1	O_2	O_3	O_4	O_5	O_6	O_7	O_8	O_9	O_{10}	sum of numbers of the positions taken	
P^1:	10	2	9	1	8	4	5	7	3	6	55	
P^2:	2	2	2	2	2	1	2	2	3	2	20	(5)
P^3:	6	5	1	3	2	5	1	6	4	6	39	

It can be seen that the sum of numbers of the positions taken by the alternatives varies. It takes values from n (when all the alternatives are tied and have been placed in first position) to $n(n+1)/2$ (when there are no tied alternatives). The positions are numbered one by one.

- Using fractional notation

	O_1	O_2	O_3	O_4	O_5	O_6	O_7	O_8	O_9	O_{10}	sum of numbers of the positions taken	
P^1:	10	2	9	1	8	4	5	7	3	6	55	
P^2:	5,5	5,5	5,5	5,5	5,5	1	5,5	5,5	10	5,5	55	(6)
P^3:	9	6,5	1,5	4	3	6,5	1,5	9	5	9	55	

In this case the sum of numbers of the positions taken by the alternatives is constant and equals $n(n+1)/2$.

As mentioned before, when ties can occur in preference orders, it can happen that some positions are not used. This holds true both for the classical, and the fractional notation. However, the positions are more varied using fractional notation. Therefore - in the authors' opinion - it better describes experts' true preferences.

It is worth noting that in the case of no ties the classical and fractional notations are equivalent. A detailed description of the latter is given in [1, 4, 5, 10, 11].

Both these notations have some advantages and disadvantages. The choice of a notation is up to the person responsible for obtaining the group judgement.

However, it should be noted that the fractional notation makes it possible to formulate a framework for an optimization model for determining group judgement (see e.g. [6]).

3. POSITIONAL METHODS OF DETERMINING GROUP JUDGEMENT

For the case under consideration, a group judgement is derived on the basis of the positions of alternatives in preference orders.

The vector of weights (also called the voting vector) is denoted as follows

$$\mathbf{w} = (w_1, \ldots, w_n) \in R_n, \tag{7}$$

where w_j is the number assigned to position j[1)] taken by an alternative in the preference order. Generally, it is assumed that $\underset{j}{\forall} w_j \geq w_{j+1}$ and $w_1 > w_n$.

A scoring function s_i is

$$s_i = \sum_{k=1}^{K} \sum_{j=1}^{n} \delta_j^{ik} w_j, \quad \delta_j^{ik} = \begin{cases} 1 & \text{if the alternative } O_i \text{ takes the } j\text{-th position} \\ & \text{in the preference order given by the } k\text{-th expert} \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

The winner is the alternative with the highest score.

The form of the weighting vector describes the character of the voting rule e.g.

[1)] The position taken by an alternative in a preference order is denoted as j for the classical and as t for the fractional notation, respectively.

$\mathbf{w}=(1, 0, ..., 0)$ corresponds to the plurality method

$\mathbf{w}=(\underbrace{1,1,\ldots 1}_{m},0,\ldots,0)$ corresponds to the case of voting for m candidates

$\mathbf{w}=(1, 1, ...1, 0)$ corresponds to the antiplurality method

$\mathbf{w}=(n-1, ..., n-j, ..., 1, 0)$ determines the Borda count.

In order to show that the choice of voting rule really matters, let us consider the following example.

Example 2.

The preference orders of twelve experts for four alternatives are given below.

number of preference orders	preference order
1	$O_1 \succ O_2 \succ O_3 \succ O_4$
2	$O_1 \succ O_2 \succ O_4 \succ O_3$
3	$O_1 \succ O_4 \succ O_3 \succ O_2$
1	$O_4 \succ O_3 \succ O_2 \succ O_1$
2	$O_3 \succ O_4 \succ O_2 \succ O_1$
3	$O_2 \succ O_3 \succ O_4 \succ O_1$

(9)

The preference orders obtained using three positional methods are as follows:

method / vector of weights		score				preference order
		O_1	O_2	O_3	O_4	
plurality	(1, 0, 0, 0)	6	3	2	1	$O_1 \succ O_2 \succ O_3 \succ O_4$
antiplurality	(1, 1, 1, 0)	6	9	10	11	$O_4 \succ O_3 \succ O_2 \succ O_1$
Borda	(3, 2, 1, 0)	18	18	18	18	$O_1 \approx O_2 \approx O_3 \approx O_4$

(10)

The outcomes from applying these different methods seem to be rather unexpected. The preference orders determined under two systems can be opposite to each other. Also, alternatives may be assessed to be equivalent. Hence, it is important to choose a suitable method for the problem to be solved.

3.1. The Borda count

The Borda count is one of two fundamental methods for determining a group judgement. The second one is the Condorcet method. There has been a lot of debate over the past two centuries as to which method is better. Both of them have advantages and disadvantages. Some authors regard the Borda count as a method burdened with a relatively small number of drawbacks compared to other ones (Saari [21, 22, 23], Nurmi [18]). The Borda count always determines a winning alternative/ alternatives and fully utilizes the information given by experts. It also satisfies - among other things – the monotonicity condition, as well as the Condorcet loser criterion, but it does not satisfy the Condorcet winner criterion. It is manipulable and not independent of irrelevant alternatives. Saari (see e.g. [24]) is the main advocate of the Borda method. Other authors e.g. Risse [20], do not share this opinion on the primacy of the Borda count. However, they admit that the Condorcet method is not better.

The Borda count initiated the development of a whole family of positional methods. Bury and Wagner [3] give a description and examples of the application of different positional methods.

It should be emphasized that in the opinion of some authors the Borda count cannot be applied in the case of ties in preference orders. However, it has been suggested that after some modifications it may also be used for the case of tied alternatives. Therefore, the application of the fractional notation to the Borda algorithm seems to be of interest.

Let us recall the definition of the Borda score in the case of no ties.

$$\mathrm{WB}_i = \sum_{j=1}^{n} (n-j)\vartheta_i^j \ , \ i, j = 1 \ldots n, \ , \ \sum_{j=1}^{n} \vartheta_i^j = K \qquad (11)$$

where i – is the number of an alternative,

j – denotes position,

ϑ_i^j – is the number of experts, who placed alternative O_i in position j,

$(n-j)$ – is the weight assigned to an alternative taking position j.

Let l_{ih} denote the number of experts who regarded alternative O_i as better than alternative O_h. For simplicity, this is denoted by $O_i \succ O_h$. The l_{ih} coefficients define the so called outranking matrix [17]

$$\begin{array}{c|c|c|c|c} & O_1 & O_2 & \cdots & O_n \\ \hline O_1 & - & l_{12} & \cdots & l_{1n} \\ \hline O_2 & l_{21} & - & \cdots & l_{2n} \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline O_n & l_{n1} & l_{n2} & \cdots & - \end{array}, \text{ where } l_{ih} + l_{hi} = K; \qquad i, h = 1,\ldots,n. \tag{12}$$

The Borda score for an alternative can also be determined as the sum of elements in the corresponding row of the outranking matrix [17]:

$$WB_i = \sum_{h=1}^{n} l_{ih}. \tag{13}$$

The Borda winner is the alternative O_i, such that $WB_i = WB_{max} = \max_h WB_h$.

It is evident that $WB_{max} \leq (n\text{-}1)K$. Equality holds in the case where all experts regard a given alternative as the best one.

It can be shown that in the case of ties the direct application of formulas (11) and (13) may result in different outcomes.

Example 3.

The preference orders determined by seven experts for a set of six alternatives are given below. Tied alternatives are given in brackets. The positions (the classical notation is applied) taken by alternatives in the preference orders under consideration are also presented.

		O_1	O_2	O_3	O_4	O_5	O_6	
P^1: $\{O_2, O_1, (O_3, O_4, O_5, O_6)\}$	P^1:	2	1	3	3	3	3	
P^2: $\{O_3, O_5, O_6, O_4, O_1, O_2\}$	P^2:	5	6	1	4	2	3	
P^3: $\{(O_1, O_5, O_6), (O_3, O_4), O_2\}$	P^3:	1	3	2	2	1	1	
P^4: $\{O_1, O_2, O_3, O_4, O_5, O_6\}$	P^4:	1	2	3	4	5	6	(14)
P^5: $\{(O_2, O_3), O_5, O_4, (O_1, O_6)\}$	P^5:	4	1	1	3	2	4	
P^6: $\{O_1, O_5, O_6, O_3, O_4, O_2\}$	P^6:	1	6	4	5	2	3	
P^7: $\{O_6, (O_2, O_3), (O_1, O_4), O_5\}$	P^7:	3	2	2	3	4	1	

The Borda scores obtained with the use of formula (11) are as follows:

alternative	O_1	O_2	O_3	O_4	O_5	O_6
WB_i	25	21	26	18	23	21

(15)

The winning alternative is O_3. The preference order obtained with respect to the Borda score is $\{O_3, O_1, O_5, (O_2, O_6), O_4\}$.

The outranking matrix (12) is of the form.

	O_1	O_2	O_3	O_4	O_5	O_6	WB_i
O_1	0	4	4	4	4	3	19
O_2	3	0	2	4	4	3	16
O_3	3	3	0	5	4	3	18
O_4	2	3	0	0	2	2	9
O_5	2	3	2	4	0	4	15
O_6	2	4	3	4	1	0	14

(16)

The winning alternative in the sense of (13) is O_1. The preference order obtained with respect to the Borda score is $\{O_1, O_3, O_2, O_5, O_6, O_4\}$. It is evident that the values of the Borda scores derived with the use of (11) and (13) differ.

To remove this discrepancy, the fractional notation is applied and the outranking matrix is modified. According to the Borda method, the weights assigned to alternatives taking the possible fractional positions are given as follows:

t	1	1½	2	2½	.....	n-1	n-½	n
w_t	n-1	n-1½	n-2	n-2½	.....	1	½	0

(17)

Generally, for a fractional position t, the corresponding weight is $w_t = n-t$.

Let ϑ_i^t denote the number of experts who placed alternative O_i in position t.

The Borda score for alternative O_i is

$$\mathrm{WB}_i = \sum_{t\in\mathcal{T}} \mathrm{w}_t \vartheta_i^t = \sum_{t\in\mathcal{T}} (n-t)\vartheta_i^t = \sum_{t\in\mathcal{T}} n\vartheta_i^t - \sum_{t\in\mathcal{T}} t\vartheta_i^t, \qquad i = 1...n,\ t \in \mathcal{T}. \tag{18}$$

Since $\sum_{t\in\mathcal{T}} \vartheta_i^t = K$, (19)

$$\mathrm{WB}_i = nK - \sum_{t\in\mathcal{T}} t\vartheta_i^t. \tag{20}$$

Let m_{ih} denote the number of experts who regarded alternatives O_i and O_h as being tied, i.e. $O_i \approx O_h$ and $\bar{l}_{ih} = l_{ih} + 0{,}5m_{ih}$. (21)

Formula (13) becomes: $\overline{\mathrm{WB}}_i = \sum_{h=1}^{n} \bar{l}_{ih} = \sum_{h=1}^{n} (l_{ih} + 0{,}5m_{ih})$. (22)

One also has $\bar{l}_{ih} + \bar{l}_{hi} = l_{ih} + 0{,}5m_{ih} + l_{hi} + 0{,}5m_{ih} = l_{ih} + l_{hi} + m_{ih} = K$.

From formulas (20) and (22), it follows that $\overline{\mathrm{WB}}_i = \mathrm{WB}_i$.

Example 4.

Let us again consider the preference orders given in Example 3. The positions – corresponding to the fractional notation – taken by the alternatives for the preference orders considered are also given.

				O1	O2	O3	O4	O5	O6	
P^1:	$\{O_2, O_1, (O_3, O_4, O_5, O_6)\}$		P^1:	2	1	4,5	4,5	4,5	4,5	
P^2:	$\{O_3, O_5, O_6, O_4, O_1, O_2\}$		P^2:	5	6	1	4	2	3	
P^3:	$\{(O_1, O_5, O_6), (O_3, O_4), O_2\}$		P^3:	2	6	4,5	4,5	2	2	
P^4:	$\{O_1, O_2, O_3, O_4, O_5, O_6\}$		P^4:	1	2	3	4	5	6	(23)
P^5:	$\{(O_2, O_3), O_5, O_4, (O_1, O_6)\}$		P^5:	5,5	1,5	1,5	4	3	5,5	
P^6:	$\{O_1, O_5, O_6, O_3, O_4, O_2\}$		P^6:	1	6	4	5	2	3	
P^7:	$\{O_6, (O_2, O_3), (O_1, O_4), O_5\}$		P^7:	4,5	2,5	2,5	4,5	6	1	

The outranking matrix determined according to (21) is of the form:

	O_1	O_2	O_3	O_4	O_5	O_6	$\overline{WB}_i$
O_1	0	4	4	4,5	4,5	4	21
O_2	3	0	3	4	4	3	17
O_3	3	4	0	6	4,5	3,5	21
O_4	2,5	3	1	0	2,5	2,5	11,5
O_5	2,5	3	2,5	4,5	0	5	17,5
O_6	3	4	3,5	4,5	2	0	17

(24)

The Borda scores determined according to (20) are as follows:

alternative	O_1	O_2	O_3	O_4	O_5	O_6
WB_i	21	17	21	11,5	17,5	17

(25)

The results derived with the use of (20) and (22) agree. The winning alternatives in the sense of Borda are O_1 and O_3. The preference order obtained with respect to the Borda score is of the form $\{(O_1, O_3), O_5, (O_2, O_6), O_4\}$.

It follows from Example 4 that as a result of the application of the Borda count, some alternatives may have the same score. However, it should be emphasized that even in the case of no ties in experts' opinions, tied alternatives can occur in a group judgement.

Example 5.

The preference orders determined by five experts for a set of five alternatives are given below. There are no tied alternatives. The positions of alternatives are the same for both the classical and fractional notation.

P^1: $\{O_3, O_1, O_5, O_4, O_2\}$

P^2: $\{O_3, O_4, O_1, O_5, O_2\}$

P^3: $\{O_1, O_2, O_3, O_4, O_5\}$

P^4: $\{O_3, O_4, O_1, O_2, O_5\}$

P^5: $\{O_4, O_1, O_5, O_2, O_3\}$

	O_1	O_2	O_3	O_4	O_5
P^1:	2	5	1	4	3
P^2:	3	5	1	2	4
P^3:	1	2	3	4	5
P^4:	3	4	1	2	5
P^5:	2	4	5	1	3

(26)

The Borda scores are as follows

alternative	O_1	O_2	O_3	O_4	O_5
WB_i	14	5	14	12	5

(27)

The winning alternatives (in the sense of Borda) are O_1 and O_3. The preference order obtained with respect to the Borda score is of the form $\{(O_1, O_3), O_4, (O_2, O_5)\}$.

3.2. Some modifications of the Borda method

One can find various modifications of the Borda method in the literature. Nurmi [18] analyses other forms of scoring rule, namely geometric average, median rule, maximin and Litvak's rule. However, efforts to modify the method in order to satisfy various criteria (e.g. the Condorcet winner criterion or independence of irrelevant alternatives) generally result in worsening other properties. A general conclusion is that for non-political decision making, the drawbacks of the Borda method mentioned above are of less importance. Various versions of the Borda count are used to determine awards in competitions, e.g. in the Eurovision song contest and for project evaluation. An interesting application of the Borda count in the case of a fixed structure of alternatives is given in a paper by Richards et al. [19].

4. DETERMINING A GROUP JUDGEMENT BY MEANS OF DISTANCE MINIMIZATION

A group judgement can also be determined as a preference order $\hat{P}$ which is the closest one – in the sense of the distance applied – to the set of preference orders $\{P^k\}$ given by experts. This problem can be formulated as follows:

$$\min_{P} \sum_{k=1}^{K} d(P^k, P) \rightarrow \hat{P}. \tag{28}$$

Such a problem can be solved e.g. by an exhaustive search over the set of all possible preference orders for a given *n*. However, this approach is limited, due to the fact that the number of preference orders to be searched through grows rapidly with *n*. Nevertheless, in the case where the structure of group opinion is subject to some restrictions, this difficulty is not so serious. Another approach consists of formulating and solving an optimization problem.

4.1 Distance defined on the basis of the alternatives' positions in preference orders

The distance between preference orders can be formulated in many ways. For the purpose of this paper, definitions making use of the positions of alternatives in preference orders are considered. It is usually assumed that

$$d(P, P^{(k)}) = \sum_{k=1}^{K} d(P, P^k) = \sum_{k=1}^{K} \sum_{i=1}^{n} f(q_i^k - q_i), \tag{29}$$

where

q_i^k denotes the position taken by alternative O_i in the preference order given by expert *k*,

q_i denotes the position taken by alternative O_i in the preference order P to be searched for,

$f(q_i^k - q_i) \geq 0$.

A simple, but frequently used, form of the function f is $f(q_i^k - q_i) = |q_i^k - q_i|$, applied e.g. in [11]. The distance formulated in such a way has a simple intuitive interpretation.

When there are no ties, problem (28) can be formulated as a linear assignment model [13]. Let us assume that alternative O_i takes position j in the preference order P. The distance (29) can be written as

$$d(P, P^{(k)}) = \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij} y_{ij}, \text{ where} \tag{30}$$

$$d_{ij} = \sum_{k=1}^{K} f(q_i^k - j), \tag{31}$$

$$y_{ij} = \begin{cases} 1 & \text{if } O_i \text{ takes position } j \text{ in the preference order P} \\ 0 & \text{otherwise,} \end{cases} \tag{32}$$

$$\underset{i=1,\ldots,n}{\forall} \sum_{j=1}^{n} y_{ij} = 1, \tag{33}$$

$$\underset{j=1,\ldots,n}{\forall} \sum_{i=1}^{n} y_{ij} = 1. \tag{34}$$

In the case of ties – in preference orders given by experts and/ or in group judgement – problem (28) can be solved as a linear integer optimization problem by imposing additional constraints [6].

Cook and Seiford [12] – referring to the well known book of Kendall [14] – suggested to assume that $f(q_i^k - q_i) = (q_i^k - q_i)^2$. Problem (28) is then of the form

$$\min_{P} \sum_{i=1}^{n} \sum_{k=1}^{K} (q_i^k - q_i)^2 \to \hat{P} \quad . \tag{35}$$

Kendall [14] proved that in the case of no ties the preference order $\hat{P}$ and the preference order determined using the Borda method are the same. The following example illustrates this property. However, using a counterexample, it can be shown that in the case of ties it is not always true [7].

Example 6.

The preference orders determined by eleven experts for a set of 5 alternatives are given below. There are no tied alternatives in experts' opinions. The positions taken by the alternatives are as follows (the classical and fractional notations are equivalent).

		O_1	O_2	O_3	O_4	O_5
P^1: $\{O_3, O_5, O_2, O_4, O_1\}$	P^1:	5	3	1	4	2
P^2: $\{O_1, O_5, O_4, O_3, O_2\}$	P^2:	1	5	4	3	2
P^3: $\{O_3, O_1, O_5, O_2, O_4\}$	P^3:	2	4	1	5	3
P^4: $\{O_2, O_1, O_3, O_4, O_5\}$	P^4:	2	1	3	4	5
P^5: $\{O_5, O_2, O_3, O_4, O_1\}$	P^5:	5	2	3	4	1
P^6: $\{O_3, O_5, O_2, O_1, O_4\}$	P^6:	4	3	1	5	2
P^7: $\{O_3, O_1, O_5, O_4, O_2\}$	P^7:	2	5	1	4	3
P^8: $\{O_5, O_3, O_4, O_1, O_2\}$	P^8:	4	5	2	3	1
P^9: $\{O_5, O_2, O_3, O_4, O_1\}$	P^9:	5	2	3	4	1
P^{10}: $\{O_3, O_5, O_1, O_4, O_2\}$	P^{10}:	3	5	1	4	2
P^{11}: $\{O_2, O_5, O_1, O_3, O_4\}$	P^{11}:	3	1	4	5	2

(36)

The Borda scores are as follows:

alternative	O_1	O_2	O_3	O_4	O_5
WB_i	19	19	31	10	31

(37)

The preference order obtained with respect to the Borda score is: $\{(O_3, O_5), (O_1, O_2), O_4\}$.

The solution of problem (35) is:

$$\{(O_3, O_5), (O_1, O_2), O_4)\}. \quad (38)$$

Problem (35) was solved as a generalized linear assignment problem with the use of a so called table of structures [4, 5, 6]; CPLEX software was applied to perform the calculations. It is worth noting that the formulation of optimization problem (35) makes it possible to take into account some additional constraints imposed on the group judgement. This is not possible with the use of classic positional methods.

4.2. The Kemeny median method

One distance minimization method which deserves special attention is the Kemeny median. It has desirable properties - it satisfies the Condorcet winner criterion and a weaker version of independence of irrelevant alternatives. It can be interpreted as the preference order that is the closest one (in the sense of the distance defined for a pairwise comparison matrix) to the set of the experts' opinions.

The Kemeny median method is not a positional one. However, in the case of no ties the pairwise comparison matrix can easily be derived on the basis of the positions taken by the alternatives in the experts' orderings.

Computing the Kemeny median is NP-hard. However, Davenport and Kalagnanam [9] and Conitzer et al. [8] showed that some greedy algorithms, as well as modified branch and bound algorithms, can be used to solve such problems.

The problem (28) of determining the Kemeny median in the case of ties is presented in [6].

5. THE NUMBER OF PREFERENCE ORDERS OF n ALTERNATIVES

Some problems of determining a group judgement subject to a given criterion can be solved by an exhaustive search over the whole set of preference orders. If some constraints are imposed on the structure of a group decision, then the number of preference orders to be analyzed may be significantly reduced. For example, it can be assumed that the first m

alternatives from the n considered cannot be tied, or the last m alternatives are tied. However, the problem of generating all the possible structures of preference orders will be considered first.

5.1. Structures of positions – determining subsets of elements

Consider a preference order with n positions. Assume that the relation between two elements in this order may only be of the form $\succ$ or $\approx$. Then the number of possible structures of the positions taken by the alternatives that may be created with the use of these relations is equal to 2^{n-1}. The number of positions taken by the relation symbols in this preference order is equal to (n-1). Only one form i.e. $\succ$ or $\approx$ of the relation considered can be placed in a given position. This is illustrated in Fig. 1.

Fig. 1

Hence, for a given n the numbers of the possible structures of positions L_n (in the classical sense) are as follows

n=2	n=3	n=4	n=5	n=6
L_2=2	L_3=4	L_4=8	L_5=16	L_6=32

(39)

It should be noted that the numbers given in (39) do not determine all the preference orders for the n alternatives. They only indicate the number of structures derived under the assumption that alternatives $O_{i_1},\ldots,O_{i_n}$ can be placed in n positions.

Example 7.

Let us assume that n=5 and the structure of the preference order is as follows:

$$O_{i_1} \approx O_{i_2} \succ O_{i_3} \approx O_{i_4} \succ O_{i_5}. \quad (40)$$

Using classical notation, this can be written as

$$(O_{i_1}, O_{i_2}), (O_{i_3}, O_{i_4}), O_{i_5} . \qquad (41)$$

The number of preference orders corresponding to structure (40) is equal to $\binom{5}{2}\cdot\binom{3}{2} = 10\cdot 3 = 30$, because the first pair of tied alternatives can be chosen in $\binom{5}{2}$ ways, the second one in $\binom{3}{2}$ ways. Once these four alternatives have been chosen, the fifth one is fixed.

In the authors' opinion, it is advantageous to describe preference orders by means of position numbers with tied alternatives being marked. Such an approach makes it easier to generate structures of preference orders.

Hence, preference orders considered should be written as follows:

(1, 2), (3, 4), 5, (42)

where 1, 2, 3, 4, 5 denote the numbers of the positions and – according to (41)– the positions taken by tied alternatives are given in brackets.

All the position structures possible for n=3, 4, 5, as well as the numbers of preference orders related to each of these structures, are given in Table 1. For a given n the sum of the latter is equal to the number of all preference orders for n alternatives.

Table 1. Structures of positions and the numbers of preference orders for n=3, 4, 5

n=3		
1°	1, 2, 3	6=3!
2°	(1, 2), 3	$3=\binom{3}{2}$
3°	1, (2, 3)	3
4°	(1, 2, 3)	1
		13

n=4		
1°	1, 2, 3, 4	24=4!
2°	(1, 2), 3, 4	$12=\binom{4}{2}2!$
3°	1, (2, 3), 4	12
4°	1, 2, (3, 4)	12
5°	(1, 2), (3, 4)	$6=\binom{4}{2}$
6°	1, (2, 3, 4)	$4=\binom{4}{3}$
7°	(1, 2, 3), 4	4
8°	(1, 2, 3, 4)	1
		75

n=5		
1°	1, 2, 3, 4, 5	120=5!
2°	(1, 2), 3, 4, 5	$60=\binom{5}{2}3!$
3°	1, (2, 3), 4, 5	60
4°	1, 2, (3, 4), 5	60
5°	1, 2, 3, (4, 5)	60
6°	(1, 2), (3, 4), 5	$30=\binom{5}{2}\cdot\binom{3}{2}$
7°	(1, 2), 3, (4, 5)	30
8°	1, (2, 3), (4, 5)	30
9°	(1, 2, 3), 4, 5	$20=\binom{5}{2}2!$
10°	1, (2, 3, 4), 5	20
11°	1, 2, (3, 4, 5)	20
12°	(1, 2, 3), (4, 5)	$10=\binom{5}{2}$
13°	(1, 2), (3, 4, 5)	10
14°	1, (2, 3, 4, 5)	$5=\binom{5}{4}$
15°	(1, 2, 3, 4), 5	5
16°	(1, 2, 3, 4, 5)	1
		541

Structures of a similar type are marked in the same colour (white or grey). The concept of similar type is understood in the sense of the subsets considered, e.g. structure no° 10 for n=5 consists of one subset of three elements and two subsets of one element, just as structures no° 9 and 11.

5.2. Structures of positions – determining partitions of a set of n elements into u non-empty subsets

Let $\mathcal{S}_n$ denote the number of all possible preference orders for a set of n alternatives. It can be shown [16] that

$$\mathcal{S}_n = \sum_{u=1}^{n} \mathcal{S}_{nu}, \text{ where } \mathcal{S}_{nu} = u!\,S(n,u), \tag{43}$$

and $S(n,u)$ denotes the Stirling number of the second kind. The number of preference orders $\mathcal{S}_n$ to be considered is given by the approximation [2]

$$\mathcal{S}_n \approx \frac{n!}{2(\log 2)^{n+1}} \tag{44}$$

Table 2. Number of preference orders with no ties or with ties allowed (determined using (44)) for n=3, …, 10.

Number of alternatives	Number of preference orders - no ties	$\mathcal{S}_n$ - total number of preference orders
3	6	13
4	24	75
5	120	541
6	720	4 683
7	5 040	47 293
8	40 320	545 835
9	362 880	7 087 261
10	3 628 800	102 247 563

The numbers of all preference orders for n=3, 4 and 5 determined in Table 1 are equal to those given in Table 2.

The problem of determining the set of possible preference orders for n alternatives can also be solved by generating partitions of a set of n elements into u nonempty subsets, u=1, ..., n. The number of such partitions is called the n-th Bell number, denoted B_n.

It can be defined as [15] $B_n = \sum_{u=0}^{n} S(n,u)$. (45)

The Bell numbers for n=1,...,7 are as follows:

n	1	2	3	4	5	6	7
B_n	1	2	5	15	52	203	877

Partitions of a set of n elements into u subsets, u=1, ...,n, are presented in Table 3.

Table 3. Types of preference orders (partitions of a set) and numbers of possible preference orders (shaded columns) for each partition for n=3, 4 and 5

n=3				n=4				n=5			
		u	$u!$			u	$u!$			u	$u!$
1°	1, 2, 3	3	6	1°	1, 2, 3, 4	4	24	1°	1, 2, 3, 4, 5	5	120
								2°	(1,5),2,3,4	4	24
								3°	1,(2,5),3,4	4	24
								4°	1,2,(3,5),4	4	24
								5°	1,2,3,(4,5)	4	24
2°	(1,3),2	2	2	2°	(1,4),2,3	3	6	6°	(1,4),2,3,5	4	24
								7°	(1,4,5),2,3	3	6
								8°	(1,4),(2,5),3	3	6
								9°	(1,4),2,(3,5)	3	6
3°	1,(2,3)	2	2	3°	1,(2,4),3	3	6	10°	1,(2,4),3,5	4	24
								11°	(1,5),(2,4),3	3	6

continued *n*=3				*n*=4				*n*=5			
								12°	1,(2,4,5),3	3	6
								13°	1,(2,4),(3,5)	3	6
4°	(1,2),3	2	2	4°	1,2,(3,4)	3	6	14°	1,2,(3,4),5	4	24
								15°	(1,5),2,(3,4)	3	6
								16°	1,(2,5),(3,4)	3	6
								17°	1,2,(3,4,5)	3	6
				5°	(1,3),2,4	3	6	18°	(1,3),2,4,5		24
								19°	(1,3,5),2,4	3	6
								20°	(1,3),(2,5),4	3	6
								21°	(1,3),2,(4,5)	3	6
				6°	1,(2,3),4	3	6	22°	(1,3,4),2,5	3	6
								23°	(1,3,4,5),2	2	2
								24°	(1,3,4),(2,5)	2	2
								25°	(1,3),(2,4),5	3	6
				7°	(1,2),3,4	3	6	26°	(1,3,5),(2,4)	2	2
								27°	(1,3),(2,4,5)	2	2
								28°	1,(2,3),4,5	4	24
								29°	(1,5),(2,3),4	3	6
				8°	(1,3),(2,4)	2	2	30°	1,(2,3,5),4	3	6
								31°	1,(2,3),(4,5)	3	6
								32°	(1,4),(2,3),5	3	6
				9°	(1,4),(2,3)	2	2	33°	(1,4,5),(2,3)	2	2
								34°	(1,4),(2,3,5)	2	2
								35°	1,(2,3,4),5	3	6
				10°	(1,2),(3,4)	2	2	36°	(1,5),(2,3,4)	2	2
								37°	1,(2,3,4,5)	2	2
								38°	(1,2),3,4,5	4	24
				11°	(1,3,4),2	2	2	39°	(1,2,5),3,4	3	6
								40°	(1,2),(3,5),4	3	6
								41°	(1,2),3,(4,5)	3	6

continued

$n=3$				$n=4$				$n=5$			
		u	$u!$			u	$u!$			u	$u!$
				12°	1,(2,3,4)	2	2	42°	(1,2,4),3,5	3	6
								43°	(1,2,4,5),3	2	2
								44°	(1,2,4),(3,5)	2	2
				13°	(1,2,4),3	2	2	45°	(1,2),(3,4),5	3	6
								46°	(1,2,5),(3,4)	2	2
								47°	(1,2),(3,4,5)	2	2
				14°	(1,2,3),4	2	2	48°	(1,2,3),4,5	3	6
								49°	(1,2,3,5),4	2	2
								50°	(1,2,3),(4,5)	2	2
								51°	(1,2,3,4),5	2	2
5°	(1,2,3)	1	1	15°	(1,2,3,4)	1	1	52°	(1,2,3,4,5)	1	1
$B_3=5$			$\mathcal{S}_3=13$	$B_4=15$			$\mathcal{S}_4=75$	$B_5=52$			$\mathcal{S}_5=541$

The number of preference orders of a given type (i.e. for a given partition into u subsets) is equal to the number of permutations of the subsets. The sum of the number of all the permutations (given in the last row of Table 3), for a given n, determines the number of all possible preference orders and is the same as obtained using formula (43). Simplified notation has been applied, i.e. (1, 2), 3 denotes the ordering {(1, 2), 3}, where the positions of tied alternatives are given in brackets.

6. CONCLUDING REMARKS

The paper has considered the problem of the occurrence of tied alternatives in the preference order presented by an expert, as well as in group judgement. It makes it possible to extend the area of applying positional methods and to generate new classes of solutions. The Borda method and its modification in the case of ties are discussed. Although it is considered to be a classical method, due to its desirable properties it is still in use, especially in non-political contexts. Some other methods of determining group decisions on the basis of distance minimization, where distance is defined by the use of the positions of alternatives in preference orders, have also been presented. Group judgement is significantly influenced by the so called "curse of dimensionality" (Saari [24]). Saari [25] has shown that the Arrow and Sen theorems result directly from these circumstances. Group judgement is simple in the case of 3 or 4 alternatives, having a direct geometric interpretation, but it becomes complex and numerically difficult for larger *n*. This problem is especially difficult in the case of ties. Therefore, any approach that reduces the number of preference orders to be taken into account is of interest. Generating the possible structures of preference orders enables one to select the preference orders of interest. Some rules regarding how to manage this problem are considered.

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