

164/2008

**Raport Badawczy**  
**Research Report**

**RB/44/2008**

**Quantitative stability analysis  
in vector problems  
of 0-1 programming**

**V. Emelichev, D. Podkopaev**

**Instytut Badań Systemowych**  
**Polska Akademia Nauk**

**Systems Research Institute**  
**Polish Academy of Sciences**



# **POLSKA AKADEMIA NAUK**

## **Instytut Badań Systemowych**

ul. Newelska 6

01-447 Warszawa

tel.: (+48) (22) 3810100

fax: (+48) (22) 3810105

Kierownik Pracowni zgłaszający pracę:  
Prof. dr hab. inż. Janusz Kacprzyk

Warszawa 2008

# Quantitative stability analysis in vector problems of 0-1 programming<sup>1</sup>

Vladimir Emelichev<sup>2</sup> and Dmitry Podkopaev<sup>3</sup>

## Abstract

We consider multiple objective 0-1 programming problems in the situation, where parameters of objective functions and parameters of linear constraints are exposed to independent perturbations. We study quantitative characteristics of stability (stability radii) of problem solutions. An approach to deriving formulae and estimations of stability radii is presented. This approach is applied to stability analysis of the linear 0-1 programming problem and problems with nonlinear objective functions, namely absolute value of linear function and quadratic objective function.

*Keywords:* 0-1 programming, vector optimization, stability radius.

## 0 Introduction

Discrete optimization models of decision making are widespread in design, control, economics and many other fields of applied research. One of research areas of discrete optimization problems motivated by real-life applications is analysis of solution stability under perturbations of initial data (of problem parameters). Various formulations of stability concept give rise to numerous directions of research. Not touching upon this wide spectrum of questions, we only refer to the extensive bibliography by Greenberg [9].

In this work we address problem of deriving quantitative characteristics of solution stability of vector 0-1 programming problems with linear constraints. A quantitative characteristic called *stability radius* is defined as the limit level of perturbations of the problem parameters, which preserve a given property of the solution set (or of a single solution). The perturbed parameters are usually coefficients of the scalar or vector objective function, and also parameters of constraints determining the feasible solution set.

Investigations of stability radius are aimed at deriving its formal expressions and building methods for its calculation or estimation. In the case of a single objective

---

<sup>1</sup>Submitted to "Discrete Optimization"

<sup>2</sup>Belarusian State University, emelichev@bsu.by

<sup>3</sup>System Research Institute, dmitry.podkopaev@ibspan.waw.pl

function, formulae of stability radius are obtained for problems of 0-1 programming, problems on systems of subsets and on graphs (see the survey by Sotskov, Leontiev and Gordeev [12]), and also for some scheduling problems (see the survey by Sotskov, Tanaev and Werner [13]). In the case of multiple objectives, analogous results are obtained for few types of problems (we refer to a short survey in Emelichev et al. [3]). Algorithms of calculating or estimating stability radii are built so far for some scalar problems only. For example, Libura et al. [11] elaborated methods for estimating stability radius of an optimal solution of the traveling salesman problem based on information about  $k$ -best solution. Interrelation between solving a combinatorial problem and calculating its stability radius is investigated by Chakravarti, Van Hoesel and Wagelmans [1], [15]. In particular, they developed an approach to building polynomial algorithm of calculating stability radius for some classes of polynomially solvable problems.

All investigations mentioned above are conducted in the case, where perturbations affect parameters of the objective function(s) only. The situation where parameters of both objective function and constraints are perturbed is studied for 0-1 programming problems. Leontiev and Mamutov [10] obtained a formula of stability radius for the linear single objective problem in the case of a unique optimal solution. Emelichev et al. [4]–[7] derived some formulas and estimations of stability radii for the linear 0-1 programming problem with multiple objective functions on the basis of technique proposed in [10].

In this work we present an approach to deriving formulae and estimations of stability radii of vector 0-1 programming problems. This approach allows to obtain results known before [4]–[7] and to characterize stability of problems with two types of nonlinear objective functions. The paper is organized as follows. A formulation of the vector 0-1 programming problem with linear constraints and linear objective functions under uncertainty conditions is given in sub-section 0.1. Stability radii are defined in sub-section 0.2. Section 1 contains supplementary statements which are used lately for deriving formulae and estimations of stability radii. These formulae and estimations are obtained in Section 2 for the problem with linear objective functions. In Section 3 we show how to extend the obtained results to problems with

nonlinear objectives. And finally in Section 4 we discuss possibility of constructing algorithm of calculating stability radii on the basis of our formulae.

### 0.1 Vector problem of 0-1 programming with perturbed parameters

Consider the  $k$ -objective linear problem of 0-1 programming with  $m$  constraints:

$$Cx \rightarrow \max, \quad (1)$$

$$Ax \leq b, \quad x \in \mathbf{E}^n, \quad (2)$$

where  $k, m, n \in \mathbf{N}$ ,  $n \geq 2$ ,  $C \in \mathbf{R}^{k \times n}$ ,  $\mathbf{E} = \{0, 1\}$ ,  $A = [a_{ij}]_{m \times n} \in \mathbf{R}^{m \times n}$ ,  $b = (b_1, b_2, \dots, b_m)^T \in \mathbf{R}^m$ ,  $x = (x_1, x_2, \dots, x_n)^T$ .

Denote by  $X$  the set of *feasible solutions* of the problem, i. e. the set of vectors satisfying (2). We assume that the problem is *solvable* i. e.  $X \neq \emptyset$ .

Vector  $x \in \mathbf{E}^n$  is called *Pareto optimal solution* (or *Pareto optimum* for short) of problem (1)–(2), if this vector is its feasible solution and no other feasible solution *dominates* it, i. e. there does not exist  $x' \in X$  such that

$$Cx' \geq Cx, \quad Cx' \neq Cx.$$

The set of all Pareto optima (called *Pareto set*) of problem (1)–(2) is denoted by  $P$ . It is evident that  $X \neq \emptyset$  implies  $P \neq \emptyset$ . In the case of a single objective ( $k = 1$ ),  $P$  denotes the set of optimal solutions of the problem.

If  $P = \{x\}$  then  $x$  is called *ideal solution* of problem (1)–(2).

The *perturbation* of problem parameters is understood as arbitrary independent change of coefficients of objective functions (1) and also coefficients and right-hand sides of constraints (2). It is modeled by adding *perturbing* number arrays  $A' \in \mathbf{R}^{m \times n}$ ,  $C' \in \mathbf{R}^{k \times n}$  and  $b' \in \mathbf{R}^m$  to matrices  $A$ ,  $C$  and vector  $b$  respectively. Thus a *perturbed problem* is formulated as follows:

$$(C + C')x \rightarrow \max, \quad (3)$$

$$(A + A')x \leq (b + b'), \quad x \in \mathbf{E}^n. \quad (4)$$

Denote by  $X(A', b')$  the set of feasible solutions of problem (3)–(4) and by  $P(A', b', C')$  the set of its Pareto optimal solutions.

The notation presented above is used to formulate a decision making problem under uncertainty. Let (1)–(2) be a model of a real-life problem. But in fact the real-life problem is described precisely by model (3)–(4), whose parameters are unknown and are different from the parameters of (1)–(2). This difference called perturbation may be caused by inaccuracy of initial data, inadequacy of the model specification, errors of rounding off and other factors. In this situation it is important not only to solve problem (1)–(2), but also to estimate a quantitative characteristic of "maximal allowable uncertainty of parameters", for which the solution of (1)–(2) relates to the solution of real-life problem (3)–(4). Such a characteristic called *stability radius* is defined as the limit level of perturbations, such than for any perturbation below this level a given relation between solutions of problems (1)–(2) and (3)–(4) is preserved. If level of uncertainty in problem parameters is not greater than stability radius, then we are guaranteed that solving problem (1)–(2) we obtain practically relevant result.

The notion of stability radius can also be useful in the case, where one needs to solve a series of instances of a computationally hard problem. Consider two consequent instances, first of which is already solved and the second instance is unsolved yet. Let (1)–(2) be formulation of the solved instance. Formulate the unsolved instance as a problem with perturbed parameters (3)–(4), where perturbations are differences between parameters of the unsolved and solved instances. If these differences are small enough, then the latter instance may have the same solution as the previous instance. So it makes sense to find stability radius of (1)–(2) and before solving subsequent problem instances to check if they have the same solution as (1)–(2). The described scheme was studied by Leontiev and Gordeev [8] by the example of solving a series of traveling salesman problems.

## 0.2 Definition of stability radii

We define norms  $l_\infty$  and  $l_1$  in space  $\mathbf{R}^d$  for any finite dimension  $d \in \mathbf{N}$  :

$$\|y\| = \max\{|y_i| : i \in N_d\}, \quad \|y\|_* = \sum_{i \in N_d} |y_i|.$$

where  $y = (y_1, y_2, \dots, y_d)^T \in \mathbf{R}^d$ ,

$$N_d = \{1, 2, \dots, d\}.$$

Under a norm of a matrix we understand the norm of the vector composed from all elements of the matrix.

The number

$$r(A', b', C') = \max \{ \|A'\|, \|b'\|, \|C'\| \}$$

is called *distance between problems* (1)–(2) and (3)–(4).

Put

$$\Omega = \mathbf{R}^{m \times n} \times \mathbf{R}^m \times \mathbf{R}^{k \times n}.$$

We assume that

$$\inf \emptyset = +\infty. \quad (5)$$

**Definition 1** *Let  $x$  be a Pareto optimum of problem (1)–(2). The number*

$$\rho^k(x, A, b, C) = \inf \{ r(A', b', C') : x \notin P(A', b', C'), (A', b', C') \in \Omega \}$$

*is called stability radius of  $x$ .*

In other words, the stability radius of  $x$  is the maximum level of parameter perturbations such that  $x$  remains Pareto optimal. If  $x$  remains Pareto optimal for any perturbations, then its stability radius is assumed to be infinite.

**Definition 2** *The number*

$$\rho_1^k(A, b, C) = \inf \{ r(A', b', C') : P(A', b', C') \not\subseteq P \vee P(A', b', C') = \emptyset, (A', b', C') \in \Omega \}$$

*is called stability radius of problem (1)–(2).*

By Definition 2, the stability radius of problem (1)–(2) is the maximum level of parameter perturbations such that new Pareto optima do not appear and the problem remains solvable.

**Definition 3** *The number*

$$\rho_2^k(A, b, C) = \inf \{ r(A', b', C') : P \not\subseteq P(A', b', C'), (A', b', C') \in \Omega \}$$

*is called quasi-stability radius of problem (1)–(2).*

By Definition 3, the quasi-stability radius of problem (1)–(2) is the maximum level of perturbations of its parameters, at which all Pareto optimal solutions stay Pareto optimal.

**Remark 1** *The problem is called stable (quasi-stable), if  $\rho_1^k(A, b, C) > 0$  ( $\rho_2^k(A, b, C) > 0$ ). It is easy to see that stability and quasi-stability are discrete analogues of upper and lower Hausdorff semicontinuity respectively at point  $(A, b, C)$  of the optimal mapping*

$$P : \Omega \rightarrow 2^{\mathbf{E}^n},$$

which puts in correspondence the set of Pareto optima to each point of the space of problem parameters.

We refer to the book by Tanino and Sawaragi [14] for more information about notion of semicontinuity in stability analysis.

## 1 Supplementary statements

The proofs of our statements concerning stability radii are build on the basis of supplementary statements presented in Section 1. In the beginning of this section we state three simple lemmas about limit levels of perturbations of linear inequality parameters under which the inequalities stay true. These lemmas help us to characterize some aspects of behavior of problem solutions under perturbations of problem parameters in Sub-sections 1.1-1.3.

For any  $p, q \in \mathbf{N}$ ,  $y, y' \in \mathbf{E}^q$ ,  $y \neq y'$ , we define two numbers

$$\varphi^{(1)}(y, y') = \max \left\{ \frac{G_i(y - y')}{\|y - y'\|_*} : i \in N_p \right\},$$

$$\varphi^{(2)}(y, y') = \min \left\{ \frac{G_i(y - y')}{\|y - y'\|_*} : i \in N_p \right\},$$

where  $G = [g_{ij}]_{p \times q}$ ,  $G_i$  is the  $i$ -th row of matrix  $G$ , i. e.  $G_i = (g_{i1}, g_{i2}, \dots, g_{iq})$ .

For any number  $d \in \mathbf{N}$ , put

$$\mathbf{0}_{(d)} = (0, 0, \dots, 0)^T \in \mathbf{R}^d.$$

**Lemma 1** *If  $\varphi^{(1)}(y, y') \geq 0$ , then*

$$\inf \{ \|G'\| : G' \in \mathbf{R}^{p \times q}, (G + G')(y - y') \leq \mathbf{0}_{(p)} \} = \varphi^{(1)}(y, y'). \quad (6)$$



The proof of Lemma 1 is given in Appendix 1.

The next lemma is proved analogously to Lemma 1.

**Lemma 2** *If  $\varphi^{(2)}(y, y') \geq 0$ , then*

$$\inf \{ \| G' \| : G' \in \mathbf{R}^{p \times q}, \exists i \in N_p ((G_i + G'_i)(y - y') < 0) \} = \varphi^{(2)}(y, y'). \quad (7)$$

**Lemma 3** *Let  $y' \in \mathbf{E}^q$ ,  $Y \subseteq \mathbf{E}^q \setminus \{y'\}$  and*

$$\varphi := \max \{ \varphi^{(2)}(y, y') : y \in Y \} \geq 0.$$

*Then*

$$\inf \{ \| G' \| : G' \in \mathbf{R}^{p \times q}, \forall y \in Y \exists i \in N_p ((G_i + G'_i)(y - y') < 0) \} = \varphi. \quad (8)$$

The proof of Lemma 3 is given in Appendix 1.

**Remark 2** *It is easy to see that if we replace the inequality*

$$(G + G')(y - y') \leq \mathbf{0}_{(p)}$$

*by the condition*

$$(G + G')(y - y') \leq \mathbf{0}_{(p)} \ \& \ (G + G')(y - y') \neq \mathbf{0}_{(p)}$$

*in formula (6) and replace the condition*

$$\exists i \in N_p ((G_i + G'_i)(y - y') < 0)$$

*by*

$$\exists i \in N_p ((G_i + G'_i)(y - y') < 0) \vee (G + G')(y - y') = \mathbf{0}_{(p)}$$

*in formulae (7) and (8), then statements of Lemmas 1-3 will remain true.*

### 1.1 Feasible and unfeasible solutions under perturbations of parameters of the problem constraints

In this section based on Lemmas 1-3 we characterize limit levels of perturbations of the parameters of constraints (2) such that relations of membership and non-membership of 0-1 vectors to the set of feasible solutions are preserved.

Let us use following notation:

$$\Omega^* = \mathbf{R}^{m \times n} \times \mathbf{R}^m,$$

$$r(A', b') = \max \{ \| A' \|, \| b' \| \}.$$

For any  $x \in \mathbf{E}^n$ , put

$$\alpha(x) = \min \left\{ \frac{b_i - A_i x}{\|x\|_* + 1} : i \in N_m \right\}. \quad (9)$$

It is evident that  $\alpha(x) \geq 0$  if and only if  $x \in X$ .

We will show that for any  $x \in X$  number  $\alpha(x)$  is the maximum level of perturbations of parameters of (2) such that  $x$  remains feasible solution. We will also prove that if  $x \in \mathbf{E}^n \setminus X$ , then  $-\alpha(x)$  is the maximum level of mentioned perturbations such that  $x$  remains unfeasible.

**Lemma 4** *For any  $x \in X$  we have*

$$\inf \{ r(A', b') : x \notin X(A', b'), (A', b') \in \Omega^* \} = \alpha(x).$$

**Proof.** Set  $p = m$ ,  $q = n + 1$ ,  $y' = (0, 0, \dots, 0, 1)^T \in \mathbf{E}^q$ . To each vector  $x = (x_1, x_2, \dots, x_n)^T \in \mathbf{E}^n$  we put in correspondence vector  $y := (x_1, x_2, \dots, x_n, 0)^T \in \mathbf{E}^q$ . Consider matrices  $G = [g_{ij}]_{p \times q}$  and  $G' = [g'_{ij}]_{p \times q}$  with the elements defined by

$$g_{ij} = \begin{cases} -a_{ij}, & \text{if } (i, j) \in N_m \times N_n, \\ -b_i, & \text{if } (i, j) \in N_m \times \{n+1\}, \end{cases} \quad g'_{ij} = \begin{cases} -a'_{ij}, & \text{if } (i, j) \in N_m \times N_n, \\ -b'_i, & \text{if } (i, j) \in N_m \times \{n+1\}. \end{cases}$$

From this notation we have

$$\varphi^{(2)}(y, y') = \alpha(x),$$

$$\exists i \in N_p ((G_i + G'_i)(y - y') < 0) \iff x \notin X(A', b').$$

Therefore Lemma 4 follows directly from Lemma 2.  $\square$

**Lemma 5** *If  $x \in \mathbf{E}^n \setminus X$ , then*

$$\inf \{ r(A', b') : x \in X(A', b'), (A', b') \in \Omega^* \} = -\alpha(x).$$

Lemma 5 is proved analogously to Lemma 4. The difference is that in the proof of Lemma 5, elements of  $G$  and  $G'$  are defined by

$$g_{ij} = \begin{cases} a_{ij}, & \text{if } (i, j) \in N_m \times N_n, \\ b_i, & \text{if } (i, j) \in N_m \times \{n+1\}, \end{cases} \quad g'_{ij} = \begin{cases} a'_{ij}, & \text{if } (i, j) \in N_m \times N_n, \\ b'_i, & \text{if } (i, j) \in N_m \times \{n+1\}. \end{cases}$$

Then

$$\begin{aligned} \varphi^{(1)}(y, y') &= -\alpha(x), \\ (G + G')(y - y') \leq \mathbf{0}_{(m)} &\iff x \in X(A', b') \end{aligned}$$

and Lemma 5 follows directly from Lemma 1.

### Lemma 6

$$\inf\{r(A', b') : X(A', b') = \emptyset, (A', b') \in \Omega^*\} = \max\{\alpha(x) : x \in \mathbf{E}^n\} \geq 0. \quad (10)$$

To prove Lemma 6 we use the same notation as in the proof of Lemma 4. In addition we put  $Y = \mathbf{E}^n \times \{0\}$ . Then we have  $Y \subseteq \mathbf{E}^q \setminus \{y'\}$ ,

$$\begin{aligned} \forall y \in Y \exists i \in N_p ((G_i + G'_i)(y - y') < 0) &\iff X = \emptyset, \\ \max\{\varphi^{(2)}(y, y') : y \in Y\} &= \max\{\alpha(x) : x \in \mathbf{E}^n\} \geq 0. \end{aligned}$$

Now we see that Lemma 6 follows directly from Lemma 3.

## 1.2 Domination relation under perturbations of objective function parameters

Let us define the binary relation of Pareto domination on set  $\mathbf{E}^n$  for any matrix  $C \in \mathbf{R}^{k \times n}$ :

$$x \succ_C x' \iff Cx \geq Cx' \ \& \ Cx \neq Cx'.$$

In this sub-section we characterize the limit levels of perturbations of the parameters of (1) which preserve the domination relation and non-domination relation between a given pair of vectors.

For two different vectors  $x, x' \in \mathbf{E}^n$ , denote

$$\beta^{(1)}(x, x') = \max \left\{ \frac{C_i(x - x')}{\|x - x'\|_*} : i \in N_k \right\},$$

$$\beta^{(2)}(x, x') = \min \left\{ \frac{C_i(x - x')}{\|x - x'\|_*} : i \in N_k \right\}. \quad (11)$$

The following implications are evident:

$$\beta^{(1)}(x, x') > 0 \Rightarrow x' \underset{C}{\prec} x, \quad x' \underset{C}{\prec} x \Rightarrow \beta^{(1)}(x, x') \geq 0, \quad (12)$$

$$\beta^{(2)}(x, x') > 0 \Rightarrow x \underset{C}{\succ} x', \quad x \underset{C}{\succ} x' \Rightarrow \beta^{(2)}(x, x') \geq 0, \quad (13)$$

where  $\underset{C}{\prec}$  is the negation of  $\underset{C}{\succ}$ .

Taking into account Remark 2 we can easily check that Lemmas 1-3 imply correspondingly Lemmas 7-9 stated below.

**Lemma 7** *If  $x \neq x'$ ,  $x' \underset{C}{\prec} x$ , then*

$$\inf \{ \|C'\| : C' \in \mathbf{R}^{k \times n}, x' \underset{C+C'}{\prec} x \} = \beta^{(1)}(x, x').$$

**Lemma 8** *If  $x \underset{C}{\succ} x'$ , then*

$$\inf \{ \|C'\| : C' \in \mathbf{R}^{k \times n}, x \underset{C+C'}{\prec} x' \} = \beta^{(2)}(x, x').$$

**Lemma 9** *Let  $x' \in \mathbf{E}^n$  and*

$$\beta(x') := \max \{ \beta^{(2)}(x, x') : x \in \mathbf{E}^n \setminus \{x'\} \} \geq 0. \quad (14)$$

*Then*

$$\inf \left\{ \|C'\| : C' \in \mathbf{R}^{k \times n}, \forall x \in \mathbf{E}^n \setminus \{x'\} \left( x \underset{C+C'}{\prec} x' \right) \right\} = \beta(x').$$

### 1.3 Pareto optimal and ideal solutions under perturbation of constraint and objective function parameters

In this sub-section we consider situations where a given Pareto optimal (ideal) solution of problem (1)–(2) loses its Pareto optimality (ideality) as a result of parameters perturbation. Formulae for calculating limit levels of such perturbations are derived from results of two previous sub-sections.

For any  $x \in P$  and any  $s \in N_2$  denote

$$\gamma^{(s)}(x) = \begin{cases} \min_{x' \in \mathbf{E}^n \setminus X} \max \{ \beta^{(s)}(x, x'), -\alpha(x') \}, & \text{if } X \neq \mathbf{E}^n. \\ +\infty, & \text{if } X = \mathbf{E}^n. \end{cases} \quad (15)$$

**Lemma 10** *Let  $x \in P$ . Then*

$$\inf \left\{ r(A', b', C') : \exists x' \in \mathbf{E}^n \setminus X \left( x' \in X(A', b') \& x' \underset{C'+C'}{>} x \right), (A', b', C') \in \Omega \right\} = \gamma^{(1)}(x).$$

The proof of Lemma 10 is given in Appendix 1.

Lemma 10 states that  $\gamma^{(1)}(x)$  is the limit level of perturbations of the problem parameters such that solution  $x \in P$  loses its Pareto optimality when another vector  $x' \in \mathbf{E}^n \setminus X$  becomes a feasible solution dominating  $x$  in the perturbed problem.

The case of ideal solution is considered in the next lemma.

**Lemma 11** *Let  $P = \{x\}$ . Then*

$$\inf \left\{ r(A', b', C') : \exists x' \in \mathbf{E}^n \setminus X \left( x' \in X(A', b') \& x \underset{C'+C'}{\bar{>}} x' \right), (A', b', C') \in \Omega \right\} = \gamma^{(2)}(x).$$

The proof of this lemma is given in Appendix 1.

According to Lemma 11,  $\gamma^{(2)}(x)$  is the limit level of perturbations of the problem parameters, at which ideal solution  $x$  becomes not ideal when another vector  $x' \in \mathbf{E}^n \setminus X$  becomes a feasible solution not dominated by  $x$  in the perturbed problem.

For any  $x \in P$  and  $s \in N_2$  denote

$$\delta^{(s)}(x) = \begin{cases} \min\{\beta^{(s)}(x, x') : x' \in X, x' \neq x\}, & \text{if } X \neq \{x\}, \\ +\infty, & \text{if } X = \{x\}. \end{cases} \quad (16)$$

It is easy to see that  $\delta^{(1)}(x) \geq 0$  for any  $x \in P$ , and  $\delta^{(2)}(x) \geq 0$  if  $P = \{x\}$ .

Lemmas 12 and 13 stated below are easy to prove resting upon Lemmas 7 and 8 respectively.

**Lemma 12** *Let  $x \in P$ . Then*

$$\inf \{ \| C' \| : x \notin P(\mathbf{0}_{(m \times n)}, \mathbf{0}_{(m)}, C'), C' \in \mathbf{R}^{k \times n} \} = \delta^{(1)}(x).$$

**Lemma 13** *Let  $P = \{x\}$ . Then*

$$\inf \{ \| C' \| : P(\mathbf{0}_{(m \times n)}, \mathbf{0}_{(m)}, C') \neq \{x\}, C' \in \mathbf{R}^{k \times n} \} = \delta^{(2)}(x).$$

Lemma 12 (Lemma 13) says that  $\delta^{(1)}(x)$  ( $\delta^{(2)}(x)$ ) is the maximum level of perturbations of objective function parameters such that solution  $x$  remains to be Pareto optimal (ideal).

## 2 Stability radii the problem with linear objective functions

Now we are in a position to state results concerning quantitative characterization of stability of problem (1)–(2). The assertions presented in this section have been previously published by Emelichev, Krichko and Podkopaev in journals issued in CIS countries. We present these results to broader audience and use them to demonstrate how technique of stability analysis developed in the previous section works.

**Theorem 1** ([5]) *Stability radius of any Pareto optimum  $x$  of  $k$ -objective problem (1)–(2) is expressed by*

$$\rho^k(x, A, b, C) = \min\{\alpha(x), \gamma^{(1)}(x), \delta^{(1)}(x)\}, \quad (17)$$

where  $\alpha(x)$ ,  $\gamma^{(1)}(x)$  and  $\delta^{(1)}(x)$  are defined by (9), (15) and (16) respectively.

**Proof.** Denote by  $\psi$  the right-hand side of (17). From definitions of  $\alpha(x)$ ,  $\gamma^{(1)}(x)$  and  $\delta^{(1)}(x)$  we have  $\psi \geq 0$ .

It follows from Lemmas 4, 10 and 12 that the stability radius of  $x$  does not exceed numbers  $\alpha(x)$ ,  $\gamma^{(1)}(x)$  and  $\delta^{(1)}(x)$  respectively. Hence inequality  $\rho^k(x, A, b, C) \leq \psi$  holds.

It remains to prove inequality  $\rho^k(x, A, b, C) \geq \psi$  in the case  $\psi > 0$ . Let  $(A', b', C') \in \Omega$  be an arbitrary triple such that  $r(A', b', C') < \psi$ . Then  $r(A', b') < \alpha(x)$ ,  $\|C'\| < \delta(x)$  and  $r(A', b', C') < \gamma^{(1)}(x)$ . Lemma 4 implies  $x \in X(A', b')$ ; Lemmas 12 and 10 yield that there does not exist vector  $x' \in \mathbf{E}^n$  such that  $x' \in X(A', b')$  and  $x' \succ_{C'} x$ . It follows that  $x \in P(A', b', C')$  for any  $(A', b', C') \in \Omega$ ,  $r(A', b', C') < \psi$ . Hence  $\rho^k(x, A, b, C) \geq \psi$ .  $\square$

Observe that stability radius of any Pareto optimum  $x$  is finite since  $\alpha(x) < \infty$ .

**Corollary 1** *The quasi-stability radius of  $k$ -objective problem (1)–(2),  $k \geq 1$ , is expressed by*

$$\rho_2^k(A, b, C) = \min_{r \in I'} \min\{\alpha(x), \delta^{(1)}(r), \gamma^{(1)}(r)\} \quad (18)$$

**Theorem 2** ([7]) *Let  $\bar{P} := \mathbf{E}^n \setminus P \neq \emptyset$ . Set*

$$\underline{\psi} = \max \left\{ 0, \min_{x' \in \bar{P}} \max_{x \in X \setminus \{x'\}} \min\{\alpha(x), \beta^{(2)}(x, x')\} \right\},$$

$$\bar{\psi} = \min_{x' \in \bar{P}} \max_{x \in \mathbf{E}^n \setminus \{x'\}} \max\{-\alpha(x'), \beta^{(2)}(x, x')\},$$

where  $\alpha(x')$  and  $\beta^{(2)}(x, x')$  are defined by (9) and (11) respectively.

The stability radius of  $k$ -objective problem (1)–(2) has following bounds:

$$\underline{\psi} \leq \rho_1^k(A, b, C) \leq \bar{\psi}. \quad (19)$$

**Proof.** First, let us prove the inequality

$$\underline{\psi} \leq \rho_1^k(A, b, C). \quad (20)$$

We assume that  $\underline{\psi} > 0$  (otherwise (20) is trivial). Let  $(A', b', C') \in \Omega$  be an arbitrary triple such that  $r(A', b', C') < \underline{\psi}$ . Then for any  $x' \in \bar{P}$  there exists  $x \in X \setminus \{x'\}$  such that  $r(A', b') < \alpha(x)$  and  $\|C'\| < \beta^{(2)}(x, x')$ . Combining these two inequalities with Lemmas 4 and 8 respectively, we obtain  $x \in X(A', b')$  and  $x \succ_{C+C'} x'$ . Thus we have proved that  $P(A', b', C') \neq \emptyset$  and no one vector  $x' \in \bar{P}$  becomes a Pareto optimal solution of perturbed problem (3)–(4), if  $r(A', b', C') < \underline{\psi}$ . This implies inequality (20).

Further, let us prove that

$$\rho_1^k(A, b, C) \leq \bar{\psi}. \quad (21)$$

Let  $\varepsilon > \bar{\psi}$ . Then there exists  $x' \in \bar{P}$  such that

$$\varepsilon > -\alpha(x'), \quad (22)$$

$$\varepsilon > \beta(x'), \quad (23)$$

where  $\beta(x')$  is defined by (14).

Inequality (22) and Lemma 5 imply that there exists a pair  $(A', b') \in \Omega^*$  such that

$$r(A', b') < \varepsilon,$$

$$x' \in X(A', b').$$

Inequality (23) and Lemma 9 imply that there exists a matrix  $C' \in \mathbf{R}^{n \times n}$  satisfying

$$\|C'\| < \varepsilon,$$

$$\forall x \in \mathbf{E}^n \setminus \{x'\} \quad (x \succ_{C+C'} x').$$

Summarizing the above we obtain that for any  $\varepsilon > \overline{\psi}$  there exists  $(A', b', C') \in \Omega$  such that

$$\begin{aligned} r(A', b', C') &< \varepsilon, \\ x' &\in P(A', b', C'). \end{aligned}$$

Hence inequality (21) holds.  $\square$

It is easy to build examples showing that the upper and lower bounds of the problem stability radius stated by Theorem 2 are attainable.

The next two theorems provide formulas for the problem stability radius in two particular cases.

**Theorem 3 ([7])** *Let  $P = \mathbf{E}^n$ . Then the stability radius of  $k$ -objective problem (1)–(2) is expressed by*

$$\rho_1^k(A, b, C) = \max\{\alpha(x) : x \in \mathbf{E}^n\}. \quad (24)$$

where  $\alpha(x)$  is defined by (9).

Indeed, if  $P = \mathbf{E}^n$ , then no one perturbation of the problem parameters can cause appearance of new Pareto optima. Hence the definition of stability radius is reduced to the following:

$$\rho_1^k(A, b, C) = \inf\{r(A', b') : X(A', b') = \emptyset, (A', b') \in \Omega^*\}.$$

Applying Lemma 6 we obtain the assertion of Theorem 3.

**Theorem 4 ([7])** *If  $P = \{x^0\}$ , then the stability radius of  $k$ -objective problem (1)–(2) is expressed by*

$$\rho_1^k(A, b, C) = \min\{\alpha(x^0), \gamma^{(2)}(x^0), \delta^{(2)}(x^0)\}, \quad (25)$$

where  $\alpha(x^0)$ ,  $\gamma^{(2)}(x^0)$  and  $\delta^{(2)}(x^0)$  are defined by (9), (15) and (16) respectively.

**Proof.** Denote by  $\psi$  the right-hand side of (25). From the definitions of  $\alpha(x^0)$ ,  $\delta^{(2)}(x^0)$  and  $\gamma^{(2)}(x^0)$  we have  $\psi \geq 0$ .

It follows from Lemmas 4, 11 and 13 that the stability radius does not exceed numbers  $\alpha(x^0)$ ,  $\gamma^{(2)}(x^0)$  and  $\delta^{(2)}(x^0)$  respectively. Therefore  $\rho_1^k(A, b, C) \leq \psi$ .

It remains to prove the inequality  $\rho_1^k(A, b, C) \geq \psi$  in the case  $\psi > 0$ . Consider a triple  $(A', b', C') \in \Omega$  such that  $r(A', b', C') < \psi$ . Then  $r(A', b') < \alpha(x^0)$ .



$r(A', b', C') < \gamma^{(2)}(x^0)$  and  $\|C'\| < \delta^{(2)}(x^0)$ . Combining each of these inequalities with Lemmas 4, 11 and 13 respectively, we obtain

$$\begin{aligned} x^0 &\in X(A', b'), \\ \forall x \in \mathbf{E}^n \setminus X &\left( x \notin X(A', b') \vee x^0 \underset{C+C'}{\succ} x \right), \\ \forall x \in X \setminus \{x^0\} &\left( x^0 \underset{C+C'}{\succ} x \right). \end{aligned}$$

Thus we have

$$\forall(A', b', C') \in \Omega \quad (r(A', b', C') < \psi \Rightarrow P(A', b', C') = \{x^0\})$$

which implies  $\rho^k(A, b, C) \geq \psi$ .  $\square$

Note that Leontiev and Mamutov [10] derived a formula for the stability radius of problem (1)–(2) with single objective function ( $k = 1$ ) in the case of a unique optimal solution. But that formula differs from our formula obtained in Theorem 4. Namely, the number defined as below is used in [10] instead of  $\gamma^{(2)}(x^0)$ :

$$t(x^0) := \begin{cases} \min\{-\alpha(x) : x \in D\}, & \text{if } D \neq \emptyset, \\ +\infty, & \text{if } D = \emptyset, \end{cases}$$

where  $D = \{x \in \mathbf{E}^n : \beta^{(2)}(x^0, x) + \alpha(x) < 0\}$  and numbers  $\alpha(x^0)$  and  $\delta^{(2)}(x^0)$  are the same as in (25). The next example illustrates consequences of replacing  $\gamma^{(2)}(x^0)$  by  $t(x^0)$ .

**Example 1** Consider the following single objective problem of 0-1 programming

$$x_1 + x_2 - 0.05x_3 \rightarrow \max.$$

$$x_1 + x_2 + x_3 \leq 2.9, \quad x \in \mathbf{E}^3.$$

It has a unique optimal solution  $x^0 = (1, 1, 0)$ . Applying the formula from [10], we obtain

$$\alpha(x^0) = 0.3, \quad \delta^{(2)}(x^0) = 0.525, \quad t(x^0) = \infty$$

which means that the problem stability radius should be equal to 0.3. But the perturbed problem

$$x_1 + x_2 + 0.05x_3 \rightarrow \max,$$

$$x_1 + x_2 + x_3 \leq 3, \quad x \in \mathbf{E}^3,$$

which is located on the distance 0.1 from the initial problem, has a unique optimal solution (1,1,1). It follows that the stability radius can not exceed 0.1. From Theorem 4 taking into account  $\gamma^{(2)}(x^0) = 0.05$  we obtain that the stability radius of the problem is equal to 0.05.

The following evident theorem gives a trivial upper bound of stability radii.

**Theorem 5** *The number  $\|C\|$  is an upper bound for*

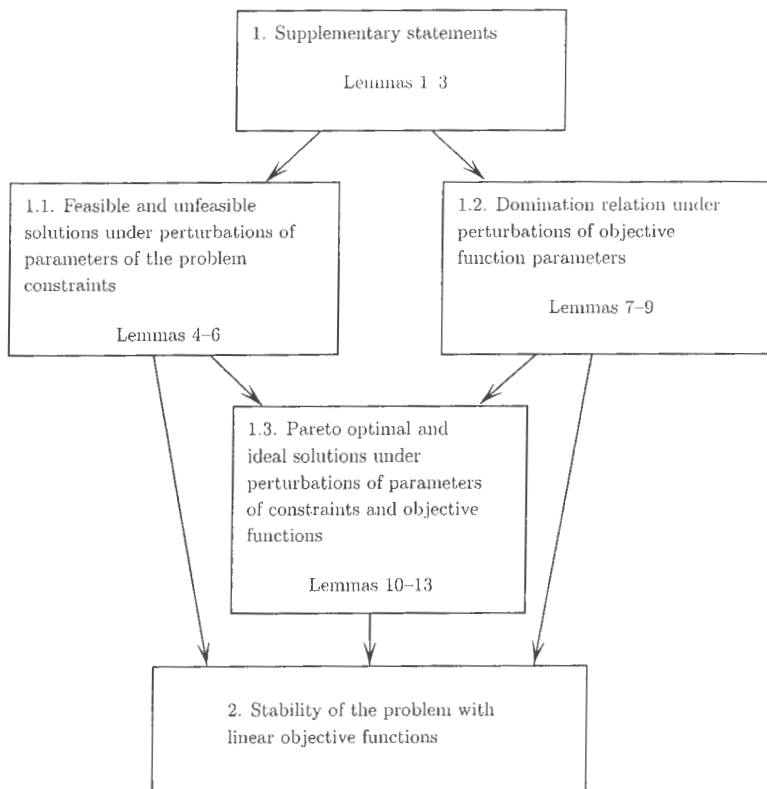
- *the stability radius of Pareto optimum  $x$  in the case  $p \neq \{r\}$ ;*
- *the stability radius of problem (1)-(2) in the case  $P \neq X$ ;*
- *the quasi-stability radius of problem (1)-(2) in the case  $|X| > 1$ .*

We conclude this section with a brief scheme of the framework for deducing formulae and bounds of stability radii built in Sections 1-2.

### 3 Problems with nonlinear objectives

In this section we demonstrate how the apparatus developed in Sections 1-2 can be applied to quantitative stability analysis of 0-1 programming problems with nonlinear objective functions. The system of supplementary statements is constructed in such a way that only lemmas in Sub-section 1.2 depend on objective function specification. In all the subsequent assertions relying on lemmas from Sub-section 1.2 (see Picture 1), objective function specification is taken into account not directly but via "interface" represented by these lemmas. To adapt our results to a problem with nonlinear objective functions it is enough to modify the contents of Sub-section 1.2. Namely, we need to change the definition of Pareto domination relation according to the objective function specification, to re-define  $\beta^{(1)}(x, x')$  and  $\beta^{(2)}(x, x')$ , and to prove statements which are analogous to Lemmas 7-9.

Below we implement described modification for the case of absolute value of linear objective functions and for quadratic objective functions.



Picture 1. Scheme of deducing formulae for stability radii

### 3.1 Problem with absolute value objective functions

Consider  $k$ -objective problem

$$f(x, C) = (|C_1x|, |C_2x|, \dots, |C_kx|) \rightarrow \max, \quad (26)$$

$$Ax \leq b, \quad x \in \mathbf{E}^n, \quad (27)$$

where  $C_i$  is the  $i$ -th row of matrix  $C \in \mathbf{R}^{k \times n}$ ;  $A \in \mathbf{R}^{m \times n}$ ;  $b \in \mathbf{R}^m$ ;  $k, m \geq 1$ ,  $n \geq 2$ .

The binary relation of Pareto domination is defined by

$$x \succ_C x' \iff f(x, C) \geq f(x', C) \ \& \ f(x, C) \neq f(x', C).$$

Further we define numbers  $\tilde{\beta}^{(1)}(x, x')$  and  $\tilde{\beta}^{(2)}(x, x')$  which will be used instead of  $\beta^{(1)}(x, x')$  and  $\beta^{(2)}(x, x')$  in analogues of Lemmas 7-9.

Let

$$K(x, x') = \{i \in N_k : |C_i x| \geq |C_i x'|\}.$$

It is evident that  $K(x, x') \neq \emptyset$  if  $x' \succ_C x$ .

For any  $x, x' \in \mathbf{E}^n$  such that

$$x \neq x', \quad K(x, x') \neq \emptyset,$$

set

$$\tilde{\beta}^{(1)}(x, x') = \begin{cases} \xi^{(1)}(x, x'), & \text{if } x' \neq \mathbf{0}_{(n)}, \\ +\infty, & \text{if } x' = \mathbf{0}_{(n)}, \end{cases} \quad (28)$$

$$\tilde{\beta}^{(2)}(x, x') = \begin{cases} \xi^{(2)}(x, x'), & \text{if } x' \neq \mathbf{0}_{(n)}, \\ \xi^{(1)}(x, x'), & \text{if } x' = \mathbf{0}_{(n)}, \end{cases} \quad (29)$$

where

$$\xi^{(1)}(x, x') = \max \{ \xi_i(x, x') : i \in K(x, x') \},$$

$$\xi^{(2)}(x, x') = \min \{ \xi_i(x, x') : i \in K(x, x') \},$$

$$\xi_i(x, x') = \min \{ \nu_i(x, x', h) : h \in \{-1; 1\} \},$$

$$\nu_i(x, x', h) = \frac{|C_i(x + hx')|}{\|x + hx'\|_*}.$$

If  $K(x, x') = \emptyset$  then by definition we assume

$$\tilde{\beta}^{(1)}(x, x') = \tilde{\beta}^{(2)}(x, x') = 0.$$

Inequalities  $\tilde{\beta}^{(s)}(x, x') \geq 0$ ,  $s \in N_2$  are evident. It is easy to check that implications (12) and (13) remain true when replacing  $\beta$  by  $\tilde{\beta}$ .

The next two lemmas applied to problem (26)–(27) are substitutes for Lemmas 7 and 8 respectively.

**Lemma 14** *If  $x \neq x'$ ,  $x' \succ_C x$ , then*

$$\inf \{ \| C' \| : C' \in \mathbf{R}^{k \times n}, x' \succ_{C+C'} x \} = \tilde{\beta}^{(1)}(x, x').$$

**Lemma 15** *If  $x \succ_C x'$ , then*

$$\inf \{ \| C' \| : C' \in \mathbf{R}^{k \times n}, x \succ_{C+C'} x' \} = \tilde{\beta}^{(2)}(x, x').$$

The proofs of Lemmas 14 and 15 are given in Appendix 2.

The next lemma replaces Lemma 9 in the case of objective functions (26). It follows from the evident fact that if  $C' = -C$  then  $x \succ_{C+C'} x'$  for any  $x, x' \in \mathbf{E}^n$ .

**Lemma 16** *For any  $x' \in \mathbf{E}^n$  we have*

$$\inf \left\{ \| C' \| : C' \in \mathbf{R}^{k \times n}, \forall x \in \mathbf{E}^n \setminus \{x'\} \left( x \succ_{C+C'} x' \right) \right\} \leq \| C \|.$$

Summing up, Theorems 1–4 and Corollary 1 are valid for problem (26)–(27), if we replace  $\beta^{(1)}(x, x')$  and  $\beta^{(2)}(x, x')$  with  $\tilde{\beta}^{(1)}(x, x')$  and  $\tilde{\beta}^{(2)}(x, x')$  defined by (28) and (29) respectively, and also in Theorem 2 replace the upper bound of stability radius with

$$\min_{x' \in \bar{P}} \max \{ -\alpha(x'), \| C \| \}.$$

The latter replacement is caused by difference between Lemma 16 and Lemma 9 statements.

### 3.2 Problem with quadratic objective functions

Consider  $k$ -objective problem

$$g(x, D) = (\langle D_1 x, x \rangle, \langle D_2 x, x \rangle, \dots, \langle D_k x, x \rangle) \rightarrow \max, \quad (30)$$

$$Ax \leq b, \quad x \in \mathbf{E}^n, \quad (31)$$

where  $D_i = [d_{ij}]_{n \times n} \in \mathbf{R}^{n \times n}$ ;  $D = [d_{ij}]_{k \times n \times n}$ ;  $\langle \cdot, \cdot \rangle$  is the scalar product of vectors.

The set of parameters of objective function is given as three-index matrix  $D$ . Perturbations of these parameters are defined by perturbing matrix  $D' \in \mathbf{R}^{k \times n \times n}$ , whose norm is defined as the maximum of absolute values of its elements (by analogy with norms of perturbing arrays  $A'$  and  $b'$ ).

To obtain definitions and statements for problem (30)–(31), analogous to those formulated in Sections 0.1–2 for linear problem (1)–(2), it is enough to replace  $C$  by  $D$ ,  $C'$  by  $D'$  and to redefine

$$\Omega = \mathbf{R}^{m \times n} \times \mathbf{R}^m \times \mathbf{R}^{k \times n \times n}.$$

By this technical manipulation we get definitions of perturbed problem and distance between initial and perturbed problems; definitions of stability radii analogous to Definitions 1–3; formulations and proofs of all assertions in Sections 1.1, 1.3 and 2 in terms of quadratic problem (which are independent on objective function specification). As it was explained in the beginning of Section 3, we have to modify contents of Section 1.2 which depends on function specification, in particular to formulate and prove lemmas for the quadratic problem, which are analogous to Lemmas 7–9. This building block inserted into our framework makes the main results presented in Section 2 valid for problem (30)–(31).

The relation of Pareto domination is now defined as follows:

$$x \succ_D x' \iff g(x, D) \geq g(x', D) \ \& \ g(x, D) \neq g(x', D).$$

The following values play role of analogues for  $\beta^{(1)}(x, x')$  and  $\beta^{(2)}(x, x')$  respectively:

$$\hat{\beta}^{(1)}(x, x') = \max \left\{ \frac{\langle D_i(x - x'), x - x' \rangle}{(\|x\|_*^2 + \|x'\|_*^2) - 2\langle x, x' \rangle} : i \in N_k \right\}, \quad (32)$$

$$\hat{\beta}^{(2)}(x, x') = \min \left\{ \frac{\langle D_i(x - x'), x - x' \rangle}{(\|x\|_*^2 + \|x'\|_*^2) - 2\langle x, x' \rangle} : i \in N_k \right\}. \quad (33)$$

The next three lemmas are analogs for Lemmas 7–9 respectively.

**Lemma 17** *If  $x \neq x'$  and  $x' \succ_D x$ , then*

$$\inf \{ \|D'\| : D' \in \mathbf{R}^{k \times n \times n}, x' \succ_{D+D'} x \} = \hat{\beta}^{(1)}(x, x').$$

**Lemma 18** If  $x \succ_D x'$ , then

$$\inf \{ \| D' \| : D' \in \mathbf{R}^{k \times n}, x \succ_{D+D'} x' \} = \hat{\beta}^{(2)}(x, x').$$

**Lemma 19** Let  $x' \in \mathbf{E}^n$  and

$$\hat{\beta}(x') := \max \{ \hat{\beta}^{(2)}(x, x') : x \in \mathbf{E}^n \setminus \{x'\} \} \geq 0.$$

Then

$$\inf \{ \| D' \| : D' \in \mathbf{R}^{k \times n}, \forall x \in \mathbf{E}^n \setminus \{x'\} \left( x \succ_{D+D'} x' \right) \} = \hat{\beta}(x').$$

To prove Lemmas 17–19, we use following simple manipulation. To each vector  $x \in \mathbf{E}^n$  we put in correspondence vector  $\hat{x} = (\hat{x}_{11}, \hat{x}_{12}, \dots, \hat{x}_{nn}) \in \mathbf{E}^{n \times n}$  with elements

$$\hat{x}_{ij} = \begin{cases} 1, & \text{if } x_i x_j = 1, \\ 0, & \text{if } x_i x_j = 0; \end{cases}$$

to each matrix  $D_i = [d_{ij}]_{n \times n} \in \mathbf{R}^{n \times n}$ ,  $i \in N_k$ , we put in correspondence row vector

$$\hat{D}_i = (d_{i11}, d_{i12}, \dots, d_{inn}) \in \mathbf{R}^{n^n}.$$

Then we have

$$\langle D_i x, x \rangle = \hat{D}_i \hat{x},$$

$$(\| x \|_\bullet)^2 + (\| x' \|_\bullet)^2 - 2 \langle x, x' \rangle = \| \hat{x} - \hat{x}' \|_\bullet.$$

Therefore Lemmas 17–19 directly follow from Lemmas 7–9 correspondingly.

Thus replacing  $\beta^{(1)}(x, x')$  and  $\beta^{(2)}(x, x')$  with  $\hat{\beta}^{(1)}(x, x')$  and  $\hat{\beta}^{(2)}(x, x')$  defined by (32) and (33) respectively, we transform Theorems 1–4 and Corollary 1 into analogous statements which are valid for problem (30)–(31).

## 4 Algorithmic aspects of quantitative stability analysis.

### Discussion

Formulae and estimations of stability radii obtained in Section 2 imply full enumeration of subsets of  $\mathbf{E}^n$  whose cardinality may depend exponentially on  $n$ . So far, no polynomial algorithms of calculating or estimating stability radii for multiple objective problems have been built. The question whether such algorithms exist for

any class of multiple objective problems is still open. In this section we describe an approach which, in our opinion, can be used for developing polynomial algorithms of calculating and estimating stability radii of problems (1)-(2), (26)-(27) and (30)-(31). We conclude this section outlining a roadmap for future research in this direction.

The approach of calculating stability radius of a  $\epsilon$ -optimal solution of a linear scalar 0-1 programming problem in polynomial time is presented by Chakravarti and Wagelmans [1]. In the case  $\epsilon = 0$ , the problem from [1] takes the form

$$\check{c}x \rightarrow \min_{x \in X}, \quad (34)$$

where,  $X \subseteq \mathbf{E}^n$ ,  $n \in \mathbf{N}$ ,  $n \geq 2$ ;  $\check{c} = (\check{c}_1, \check{c}_2, \dots, \check{c}_n) \in \mathbf{R}^n$ .

It is more convenient for us to consider (34) as maximization problem which brings it into accordance with terminology of the present work:

$$cx \rightarrow \max_{x \in X}, \quad (35)$$

where  $c = -\check{c}$ . The difference between problems (35) and (1)-(2) consists in the fact that (35) is scalar problem and its feasible solution set is fixed.

Chakravarti and Wagelmans studied the stability radius of optimal solution to perturbations of a given sub-set of objective function coefficients. We restrict our consideration to the case where all the coefficients of objective function are perturbed. Then the definition of stability radius of optimal solution of (35) presented in [1] is reduced to the following:

$$\rho(x, c) = \inf \{ \|c'\| : \exists x' \in X((c+c')x' > (c+c')x), c' \in \mathbf{R}^n \}.$$

where  $x$  is an optimal solution of (35). Observe that the above is a simplified Definition 1 when there is only one objective function and only its coefficients are perturbed.

The approach to calculating stability radius presented in [1] relies on the following theorem.

Let  $x$  be an optimal solution of (35). For any  $i \in N_n$  put

$$d_i = \begin{cases} 1, & x_i = 0, \\ -1, & x_i = 1. \end{cases} \quad (36)$$



**Theorem 6 ([1])** *The stability radius of optimal  $x$  is the maximum number  $\rho$ , for which the following inequality holds:*

$$\min_{x' \in X \setminus \{x\}} \left\{ \sum_{i \in N_n} (-c_i - \psi d_i) x'_i \right\} \geq \sum_{i \in N_n} (-c_i + \psi) x_i. \quad (37)$$

It was shown in [1] that the function in left-hand side of (37) is a concave piecewise linear function with the number of pieces polynomial to  $n$ . This yields polynomial algorithm of its construction on the segment  $[0; \rho^*]$ , where  $\rho^* = \max\{c_i \mid i \in N_n\}$  is the upper bound of stability radius. When the function is constructed it is easy to find the maximal  $\psi$  for which (37) holds.

Let us show that Theorem 6 can be obtained as a corollary of our results. For this purpose we modify formula of stability radius of a Pareto optimum 1 so that it will be applied to problem (34), i. e. to the case where  $k = 1$  and the parameters of constraints are not perturbed. When  $X$  is fixed, optimal solution  $x$  loses its optimality only if another feasible solution starts dominating it as a result of perturbation of objective function coefficients. According to Lemma 12 the stability radius of optimal solution  $x$  is equal to  $\delta^{(1)}(x)$  in this case. Taking into account  $k = 1$  we write down this in the following theorem.

**Theorem 7** *Let  $x$  be an optimal solution of problem (34),  $X \neq \{x\}$ . Then stability radius of  $x$  is expressed by*

$$\rho(x, c) = \min_{x' \in X \setminus \{x\}} \left\{ \frac{\sum_{i \in N_k} c_i (x_i - x'_i)}{\|x - x'\|_*} \right\}.$$

We will need the following evident lemma.

**Lemma 20** *Let  $x, x' \in \mathbf{E}^n$ ,  $x \neq x'$ . Then*

$$\|x - x'\|_* = \sum_{i \in N_n} (x_i + d_i x'_i),$$

Where  $d_i$  is defined by (36).

It follows from Theorem 7 that stability radius of  $x$  is the maximal  $\psi$  satisfying inequality

$$\psi \leq \min_{x' \in X \setminus \{x\}} \left\{ \frac{\sum_{i \in N_k} c_i (x_i - x'_i)}{\|x - x'\|_*} \right\}.$$

Let us rewrite this inequality in the following form:

$$\min_{x' \in X \setminus \{x\}} \left\{ \sum_{i \in N_k} c_i(x_i - x'_i) - \psi \|x - x'\|_* \right\} \geq 0.$$

Using Lemma 20 we obtain

$$\min_{x' \in X \setminus \{x\}} \left\{ \sum_{i \in N_k} c_i(x_i - x'_i) - \psi \sum_{i \in N_n} x_i - \psi \sum_{i \in N_n} d_i x'_i \right\} \leq 0$$

which is equivalent to (37).

Thus Theorem 7 yields Theorem 6.

We demonstrated that formula of stability radius can be easily transformed into inequality which determines stability radius as proposed by Chakravarti and Wagelmans. By analogous way such inequalities can be derived for multiple objective 0-1 programming problems.

Let us describe a possible scheme of research aimed at building algorithm of calculating a stability radius for multiple objective problem (1)–(2), (26)–(27) or (30)–(31).

1) To derive inequality analogous to (37) such that the stability radius is defined as the maximal value of a parameter for which the inequality holds. This can be done by transforming a formula of stability radius in the way demonstrated above.

2) To study properties of the function in left-hand side of the inequality for answering questions "is this function concave?"; "is the number of function segments polynomial with respect to  $n$ ?".

3) To develop an algorithm of constructing the function on segment  $[0, \rho^*]$  where  $\rho^*$  is a trivial upper bound of stability radius (see for example Theorem 5). Having this function described explicitly, one can easily find the maximum value of parameter mentioned above which provides the value of stability radius.

## References

- [1] N. Chakravarti, A. Wagelmans, Calculation of stability radius for combinatorial optimization problems, *Operations Research Letters* 23 (1998) 1–7.

- [2] M. Ehrgott, X.A. Gandibleux, Survey and annotated bibliography of multiobjective combinatorial optimization, *OR Spectrum* 22 (2000) 425–460.
- [3] V.A. Emelichev, E. Girlich, Yu.V. Nikulin, D.P. Podkopaev, Stability and regularization of vector problems of integer linear programming, *Optimization* 51 (2002) 645–676.
- [4] V.A. Emelichev, V.N. Krichko, On the stability radius of a Pareto optimum of the vector problem of Boolean programming, *Izv. AN Respubliki Moldova. Matematika*. No. 1 (1999) 79–84.
- [5] V.A. Emelichev, V.N. Krichko, On stability of a Pareto optimum of vector Boolean programming, *Discrete Math. and Appl.* 11 (1999) 607–613.
- [6] V.A. Emelichev, V.N. Krichko, Stability radius of a Pareto optimal solution to a vector quadratic problem of Boolean programming, *Comput. Math. and Mathem. Physics* 41 (2001) 322–326.
- [7] V.A. Emelichev, V.N. Krichko, D.P. Podkopaev, On the radius of stability of a vector problem of linear Boolean programming, *Discrete Math. and Appl.* 10 (2000) 1–4.
- [8] E. N. Gordeev, V. K. Leontiev, Applying stability analysis to solving choice problems, in "Mathematical Methods of Image Recognition and Discrete Optimization", Moscow: VC AN SSSR (1987). P. 36–51. (in Russian)
- [9] H.J. Greenberg, An annotated bibliography for post-solution analysis in mixed integer programming and combinatorial optimization, in: D.L. Woodruff (ed.), *Advances in Computational and Stochastic Optimization, Logic Programming, and Heuristic Search* (Kluwer Academic Publishers, Boston, MA, 1998) 97–148.
- [10] V.K. Leontiev, K.X. Mamutov, Stability of solutions in problems of linear Boolean programming, *Jurn. Vychisl. Matematiki i Matem. Fiziki* 28 (1988) 1475–1481.
- [11] M. Libura, E.S. Van der Poort, G. Sierksma, J.A.A. Van der Veen, Stability aspects of the traveling salesman problem based on  $k$ -best solutions. *Discrete Appl. Math.* 87 (1998) 159–185.

- [12] Yu.N. Sotskov, V.K. Leontiev, E.N. Gordeev, Some concepts of stability analysis in combinatorial optimization, *Discrete Appl. Math.* 58 (1995) 169–190.
- [13] Yu.N. Sotskov, V.S. Tanaev, F. Werner, Stability radius of an optimal schedule: A survey and recent developments, in: G. Yu (ed.), *Industrial Applications of Combinatorial Optimization* (Kluwer Academic Press, Boston, MA, 1998) 72–108.
- [14] T. Tanino, Y. Sawaragi, Stability of nondominated solutions in multicriteria decision-making, *J. Opt. Theory and Appl.* V. 30 (1980) 229–253.
- [15] S. Van Hoesel, A. Wagelmans, On the complexity of postoptimality analysis of 0-1 programs, *Discrete Appl. Math.* 91 (1999) 251–263.

## Appendix 1. Proofs of some supplementary statements

**Proof of Lemma 1.** First, let us prove inequality

$$\omega \geq \varphi^{(1)}(y, y'), \quad (38)$$

where  $\omega$  is the left-hand side of (6). Without loss of generality, assume  $\varphi^{(1)}(y, y') > 0$  (inequality (38) is evident in the case  $\varphi^{(1)}(y, y') = 0$ ).

Consider a matrix  $G' \in \mathbf{R}^{p \times q}$  such that  $\sigma := \|G'\| < \varphi^{(1)}(y, y')$ . By definition of  $\varphi^{(1)}(y, y')$ , we have  $\sigma < G_i(y - y') / \|y - y'\|_*$  for some  $i \in N_p$ . Then we derive

$$(G_i + G'_i)(y - y') = G_i(y - y') + G'_i(y - y') \geq G_i(y - y') - \sigma \|y - y'\|_* > G_i(y - y') - G_i(y - y') = 0. \quad (39)$$

Thus for any  $G' \in \mathbf{R}^{p \times q}$ ,  $\|G'\| < \varphi^{(1)}(y, y')$ , we have

$$\exists i \in N_p ((G_i + G'_i)(y - y') > 0).$$

This implies inequality (38).

To prove inequality  $\omega \leq \varphi^{(1)}(y, y')$ , it suffices to prove that  $\omega \leq \sigma$  for any number  $\sigma > \varphi^{(1)}(y, y')$ . This can be done by building a matrix  $G' = [g'_{ij}] \in \mathbf{R}^{p \times q}$  such that

$$\|G'\| = \sigma \quad \text{and} \quad \forall i \in N_p ((G_i + G'_i)(y - y') < 0).$$

Inequality  $\sigma > \varphi^{(1)}(y, y')$  implies

$$\forall i \in N_p \left( \frac{\sigma > G_i(y - y')}{\|y - y'\|_*} \right).$$

It follows that when for any  $i \in N_p$  elements of  $G'$  are defined by

$$g'_{ij} = \begin{cases} \sigma, & \text{if } y'_j = 1, \\ -\sigma, & \text{if } y'_j = 0, \end{cases} \quad (40)$$

then for any  $i \in N_p$  we have

$$(G_i + G'_i)(y - y') = G_i(y - y') + G'_i(y - y') = G_i(y - y') - \sigma \|y - y'\|_* < G_i(y - y') - G_i(y - y') = 0.$$

Thus required matrix  $G'$  is obtained.  $\square$

**Proof of Lemma 3.** Denote the left-hand side of (8) by  $\omega$ . The definition of  $\varphi$  implies that for any matrix  $G' \in \mathbf{R}^{p \times q}$  such that  $\|G'\| \leq \varphi$  there exists  $y \in Y$  such that  $\|G'\| \leq \varphi^{(2)}(y, y')$ . Using Lemma 2 we have  $(G + G')(y - y') \geq \mathbf{0}_{(p)}$ . Recalling the definition of  $\omega$  we obtain  $\varphi \geq \omega$ .

Further let us prove that  $\varphi \leq \omega$ . For any  $\sigma > \varphi$  and any  $y \in Y$  there exists  $i \in N_p$  such that  $\sigma > G_i(y - y') / \|y - y'\|_*$ . Then consider perturbing matrix  $G' \in \mathbf{R}^{p \times q}$  with the elements defined by (40), repeat the reasoning below (40) and conclude

$$\forall \sigma > \varphi \exists G' \in \mathbf{R}^{p \times q} \quad (\|G'\| = \sigma \ \& \ \forall y \in Y \exists i \in N_p ((G_i + G'_i)(y - y') < 0)).$$

This yields  $\varphi \leq \omega$ .  $\square$

**Proof of Lemma 10.** If  $X = \mathbf{E}^n$ , then infimum in the lemma statement is taken over the empty set. The assertion of Lemma 10 follows from (5) in this case.

Further we assume  $X \neq \mathbf{E}^n$ . Observe that  $\alpha(x') \leq 0$  for any  $x' \in \mathbf{E}^n \setminus X$  which implies  $\gamma^{(1)}(x) \geq 0$ .

Denote by  $\omega$  the left-hand side of equality in the lemma statement.

First, let us prove that

$$\omega \geq \gamma^{(1)}(x). \quad (41)$$

Suppose that  $\gamma^{(1)}(x) > 0$  (inequality (41) is trivial if  $\gamma^{(1)}(x) = 0$ ). Consider any triple  $(A', b', C') \in \Omega$  such that

$$r(A', b', C') < \gamma^{(1)}(x). \quad (42)$$

It follows from (15) that for any  $x' \in \mathbf{E}^n \setminus X$  at least one of the next two conditions holds:

$$r(A', b') \leq r(A', b', C'') < -\alpha(x'), \quad (43)$$

$$\|C'\| \leq r(A', b', C') < \beta^{(1)}(x, x'). \quad (44)$$

If (43) takes place, then  $x' \notin X(A', b')$  by Lemma 5. If (44) takes place, then (12) yields  $x' \underset{C'}{\succ} x$ , which allows us to apply Lemma 7 to get  $x' \underset{C'+C''}{\succ} x$ .

Thus for any  $(A', b', C') \in \Omega$  satisfying (42) we have

$$\exists x' \in \mathbf{E}^n \setminus X \left( x' \in X(A', b') \& x' \underset{C'+C''}{\succ} x \right).$$

Hence (41) is true.

Further, we prove that  $\omega \leq \gamma^{(1)}(x)$ . Let  $\varepsilon > \gamma^{(1)}(x)$ . Then there exists  $x' \in \mathbf{E}^n \setminus X$  such that

$$\varepsilon > -\alpha(x'), \quad (45)$$

$$\varepsilon > \beta^{(1)}(x, x'). \quad (46)$$

It follows from (45) and Lemma 5 that there exists a pair  $(A^0, b^0) \in \Omega^*$  such that

$$r(A^0, b^0) < \varepsilon,$$

$$x' \in X(A^0, b^0).$$

Inequality (46) and Lemma 7 imply that there exists a matrix  $C^0 \in \mathbf{R}^{k \times n}$  satisfying conditions

$$\|C^0\| < \varepsilon,$$

$$x' \underset{C+C^0}{\succ} x.$$

Thus we obtain that for any  $\varepsilon > \gamma^{(1)}(x)$  there exists  $(A^0, b^0, C^0) \in \Omega$  such that  $r(A^0, b^0, C^0) < \varepsilon$  and

$$\exists x' \in \mathbf{E}^n \setminus X \left( x' \in X(A^0, b^0) \& x' \underset{C+C^0}{\succ} x \right).$$

Hence  $\omega \leq \gamma^{(1)}(x)$ .  $\square$

**Proof of Lemma 11.** The assertion of the lemma is trivial in the case  $X = \mathbf{E}^n$ . Therefore we assume  $X \neq \mathbf{E}^n$ . It is easy to check that  $\gamma^{(2)}(x) \geq 0$  in this case.

Denote by  $\omega$  the left-hand side of equality in the lemma statement.

First, let us prove the inequality  $\omega \geq \gamma^{(2)}(x)$  in the case  $\gamma^{(2)}(x) > 0$  (the inequality is trivial when  $\gamma^{(2)}(x) = 0$ ). Let  $(A', b', C') \in \Omega$  be a triple such that

$$r(A', b', C') < \gamma^{(2)}(x). \quad (47)$$

According to (15), for any  $x' \in \mathbf{E}^n \setminus X$  at least one of the following inequalities holds:

$$r(A', b') < -\alpha(x'), \quad (48)$$

$$\|C'\| < \beta^{(2)}(x, x'). \quad (49)$$

If (48) takes place, then  $x' \notin X(A', b')$  follows from Lemma 5.

If (49) is true, then  $\beta^{(2)}(x, x') > 0$ . From (13) we obtain  $x \underset{C'}{\succ} x'$ . This allows us to apply Lemma 8 to get  $x \underset{C'+C'}{\succ} x'$ .

Thus for any  $(A', b', C') \in \Omega$  satisfying (47) we have

$$\exists x' \in \mathbf{E}^n \setminus X \left( x' \in X(A', b') \ \& \ x \underset{C'+C'}{\succ} x' \right).$$

Hence  $\omega \geq \gamma^{(2)}(x)$ .

Further, we prove that  $\omega \leq \gamma^{(2)}(x)$ . Let  $\varepsilon > \gamma^{(2)}(x)$ . Then there exists  $x' \in \mathbf{E}^n \setminus X$  such that

$$\varepsilon > -\alpha(x'), \quad (50)$$

$$\varepsilon > \beta^{(2)}(x, x'). \quad (51)$$

It follows from (50) and Lemma 5 that there exists a pair  $(A^0, b^0) \in \Omega^*$  such that

$$r(A^0, b^0) < \varepsilon,$$

$$x' \in X(A^0, b^0).$$

Inequality (51) and Lemma 8 imply that there exists a matrix  $C^0 \in \mathbf{R}^{k \times n}$  satisfying conditions

$$\|C^0\| < \varepsilon, \quad x \underset{C+C^0}{\succ} x'.$$

Thus we obtain that for any  $\varepsilon > \gamma^{(2)}(x)$  there exists  $(A^0, b^0, C^0) \in \Omega$ ,  $r(A^0, b^0, C^0) < \varepsilon$ , such that

$$\exists x' \in \mathbf{E}^n \setminus X \left( x' \in X(A^0, b^0) \ \& \ x \underset{C+C^0}{\succ} x' \right).$$

Hence  $\omega \leq \gamma^{(2)}(x)$ .  $\square$

## Appendix 2. Proofs of two lemmas for the problem with absolute value of linear objective functions

Hereinafter we use the following evident equivalence valid for any  $z, z' \in \mathbf{R}$ :

$$|z| > |z'| \iff \exists h \in \{-1, 1\} \forall h' \in \{-1, 1\} (hz > h'z'), \quad (52)$$

For any  $z \in \mathbf{R}$ , set

$$\text{sg } z = \begin{cases} 1, & \text{if } z \geq 0, \\ -1, & \text{if } z < 0. \end{cases}$$

**Proof of Lemma 14.** Let  $x' \neq \mathbf{0}_{(n)}$  (the lemma is trivial in the contrary case in view of (5)). Then  $\tilde{\beta}^{(1)}(x, x') = \xi^{(1)}(x, x')$ . Denote by  $\omega$  the left-hand side of the equality in lemma statement.

First, we prove inequality  $\omega \geq \xi^{(1)}(x, x')$  in the case  $\xi^{(1)}(x, x') > 0$  (this inequality is trivial if  $\xi^{(1)}(x, x') = 0$ ). Let  $C' \in \mathbf{R}^{k \times n}$ ,  $\|C'\| < \xi^{(1)}(x, x')$ . By the definition of  $\xi^{(1)}(x, x')$ , there exists  $i \in N_k$  such that

$$\forall h \in \{-1, 1\} (\|C'\| < \nu_i(x, x', h)). \quad (53)$$

Therefore  $\xi_i(x, x') > 0$  which implies

$$|C_i x| > |C_i x'|. \quad (54)$$

Denote  $\sigma = \text{sg } C_i x$ . Taking into account (54), it is easy to check that for any  $h \in \{-1, 1\}$  the following equality holds:

$$C_i(\sigma x + hx') = |C_i(x + \sigma hx')|.$$

Applying (53), for any  $h \in \{-1, 1\}$  we deduce

$$\begin{aligned} (C_i + C'_i)(\sigma x + hx') &= |C_i(x + \sigma hx')| + \sigma C'_i(x + \sigma hx') \geq |C_i(x + \sigma hx')| - \\ &\|C'\| \|\cdot\| \|x + \sigma hx'\|_* > |C_i(x + \sigma hx')| - |C_i(x + \sigma hx')| = 0. \end{aligned} \quad (55)$$

Thus for any  $h \in \{-1, 1\}$  we have  $\sigma(C_i + C'_i)x > h(C_i + C'_i)x'$ . Taking into account (52) we obtain

$$|(C_i + C'_i)x| > |(C_i + C'_i)x'| \quad (56)$$

which implies  $x' \stackrel{\text{sg}}{<}_{C+C'} x$ .



Summarizing the above we conclude that  $x' \in \sum_{C+C'} x$  for any  $C' \in \mathbf{R}^{k \times n}$  such that  $\|C'\| < \xi^{(1)}(x, x')$ . Hence  $\omega \geq \xi^{(1)}(x, x')$ .

It remains to prove that  $\omega \leq \xi^{(1)}(x, x')$ .

Denote

$$\sigma = \text{sg } C_i x, \quad \sigma' = \text{sg } C_i x'.$$

Set

$$N(x, x') = |\{j \in N_n : x_j = 1 \& x'_j = 0\}|.$$

Observe that since  $x' \neq \mathbf{0}_{(n)}$ , the following inequality holds:

$$N(x, x') < \|x + x'\|_* . \quad (57)$$

Take an arbitrary number  $\varepsilon > \xi^{(1)}(x, x')$ . From the definition of  $\xi^{(1)}(x, x')$  we have

$$\forall i \in N(x, x') \quad (\varepsilon > \xi_i(x, x')). \quad (58)$$

For any  $i \in N_k$  we will consider four possible cases:

$$i \in K(x, x'), \quad \nu_i(x, x', -1) \leq \nu_i(x, x', 1), \quad |C_i x| + |C_i x'| > 0;$$

$$i \in K(x, x'), \quad \nu_i(x, x', -1) \leq \nu_i(x, x', 1), \quad |C_i x| + |C_i x'| = 0;$$

$$i \in K(x, x'), \quad \nu_i(x, x', -1) > \nu_i(x, x', 1);$$

$$i \notin K(x, x')$$

and in each of these cases we will build row  $C'_i = (c'_{i1}, c'_{i2}, \dots, c'_{in})$  of perturbing matrix  $C'$  such that  $\max\{|c'_{ij}| : j_n \in N_n\} < \varepsilon$  and

$$|(C_i + C'_i)x'| > |(C_i + C'_i)x|. \quad (59)$$

Case 1:  $i \in K(x, x'), \nu_i(x, x', -1) \leq \nu_i(x, x', 1), |C_i x| + |C_i x'| > 0$ . Taking into account (58) we obtain that there exists number  $\delta$  such that

$$\nu_i(x, x', -1) < \delta < \varepsilon. \quad (60)$$

In addition, we impose the following condition on  $\delta$ :

$$\delta N(x, x') < |C_i x| + |C_i x'|. \quad (61)$$

Note that in the case  $N(x, x') = 0$  inequality (61) follows; if  $N(x, x') > 0$  then (61) does not contradict to (60) since taking into account (57) we have

$$\nu(x, x', -1) \leq \nu(x, x', -1) < \frac{|C_i x| + |C_i x'|}{N(x, x')}.$$

Put

$$c'_{ij} = \begin{cases} -\sigma\delta, & \text{if } x_j = 1 \text{ \& } x'_j = 0, \\ \sigma'\delta, & \text{if } x_j = 0 \text{ \& } x'_j = 1, \\ 0 & \text{in the rest of cases.} \end{cases}$$

Using (60) we derive

$$\sigma'(C_i + C'_i)x' - \sigma(C_i + C'_i)x = |C_i x| - |C_i x'| + \delta N(x', x) + \delta N(x, x') \geq -|C_i(x - x')| + \delta \|x - x'\|_* \geq -|C_i(x - x')| + \nu_i(x, x', -1) \|x - x'\|_* = 0.$$

Using (61) and taking into account  $N(x, x') \geq 0$  we derive

$$\sigma'(C_i + C'_i)x' + \sigma(C_i + C'_i)x = |C_i x| + |C_i x'| + \delta N(x', x) - \delta N(x, x') \geq |C_i x| + |C_i x'| - \delta \|x + x'\|_* \geq |C_i x| + |C_i x'| - \nu_i(x, x', 1) \|x + x'\|_* \geq 0. \text{ Taking into account (52) we get (59).}$$

Case 2:  $i \in K(x, x')$ ,  $\nu_i(x, x', -1) \leq \nu_i(x, x', 1)$ ,  $|C_i x| + |C_i x'| = 0$ . Put  $0 < \delta < \varepsilon$  and define the elements of  $C'_i$  by the following way.

If  $x \geq x'$  then take  $u, v \in N_n$  such that  $x_u = x'_u = 1$ ,  $x_v = 1$ ,  $x'_v = 0$  and put  $c'_{iu} = \delta$ ,  $c'_{iv} = -\delta/2$ ,  $c'_{ij} = 0$  for any  $j \in N_n \setminus \{u, v\}$ . Otherwise take  $u \in N_n$  such that  $x_u = 0$ ,  $x'_u = 1$  and put  $c'_{iu} = \delta$ ,  $c'_{ij} = 0$  for any  $j \in N_n \setminus \{u\}$ .

Using evident inequalities  $|(C_i + C'_i)x| = |C'_i x|$  and  $|(C_i + C'_i)x'| = |C'_i x'|$  it is easy to verify that (59) holds.

Case 3:  $i \in K(x, x')$ ,  $\nu_i(x, x', -1) > \nu_i(x, x', 1)$ . Then by (58) there exists a number  $\delta$  satisfying inequalities

$$\nu(x, x', 1) < \delta < \varepsilon, \tag{62}$$

$$\delta < \nu(x, x', -1). \tag{63}$$

Observe that since  $i \in K(x, x')$  we have  $\sigma C_i x \geq \sigma C_i x'$  which implies

$$\sigma C_i(x - x') = |C_i(x - x')|. \tag{64}$$

Put  $c'_{ij} = -\sigma\delta$  for any  $j \in N_n$ . Using (62) we derive

$$-\sigma(C_i + C'_i)x' - \sigma(C_i + C'_i)x = -\sigma C_i(x + x') + \delta(\|x\|_* + \|x'\|_*) \geq -|C_i(x + x')| + \nu(x, x', 1)(\|x\|_* + \|x'\|_*) \geq 0.$$

Using (63) and (64) we derive

$$-\sigma(C_i + C'_i)x' + \sigma(C_i + C'_i)x = \sigma C_i(x - x') - \delta(\|x\|_* - \|x'\|_*) \geq |C_i(x - x')|$$

$$-\nu(x, x', -1)(\|x - x'\|_*) = 0.$$

Applying (52) we obtain (59).

Case 4:  $i \in N_k \setminus K(x, x')$ . Then assuming  $C'_i = \mathbf{0}_{(n)}$  we have (59).

Thus for any  $\varepsilon > \xi^{(1)}(x, x')$  we have a matrix  $C' \in \mathbf{R}^{k \times n}$  such that  $\|C'\| < \varepsilon$  and  $x' \underset{C+C'}{>} x$ . Hence  $\omega \leq \xi^{(1)}(x, x')$ .  $\square$

**Proof of Lemma 15.** Denote by  $\omega$  the left-hand side of the equality in lemma statement. First, consider the case  $x' \neq \mathbf{0}_{(n)}$ . Then  $\tilde{\beta}^{(2)}(x, x') = \xi^{(2)}(x, x')$ .

To prove inequality  $\omega \geq \xi^{(2)}(x, x')$  in the case  $\xi^{(2)}(x, x') > 0$  (the inequality is trivial in the contrary case) we consider an arbitrary matrix  $C' \in \mathbf{R}^{k \times n}$  such that  $\|C'\| < \xi^{(2)}(x, x')$ . Taking into account the definition of  $\xi^{(2)}(x, x')$  and the relation  $x \underset{C}{>} x'$  it is easy to see that inequalities (54) and (53) hold for any  $i \in N_k$ . Using the argumentation below inequalities (54) and (53) in proof of Lemma 14, we get inequality (56). Since it holds for any  $i \in N_k$ , we obtain  $x \underset{C+C'}{>} x'$ . Thus we conclude  $\omega \geq \tilde{\beta}^{(2)}(x, x')$ .

Further let us prove that  $\omega \leq \xi^{(2)}(x, x')$ . Let  $\varepsilon > \xi^{(2)}(x, x')$ . By definition of  $\xi^{(2)}(x, x')$ , there exists  $i \in K(x, x')$  such that  $\varepsilon > \xi_i(x, x')$ . Then one of cases 1, 2 or 3 considered in the proof of Lemma 14 is possible. Build  $i$ -th row of matrix  $C' \in \mathbf{R}^{k \times n}$  using the same argumentation as when considering these three cases in the proof of Lemma 14 and set the rest of rows to be zero. Then we have  $\|C'\| < \varepsilon$  and  $|(C_i + C'_i)x'| > |(C_i + C'_i)x|$ . Hence  $x \underset{C+C'}{>} x'$ . Thus we have proved inequality  $\omega \leq \tilde{\beta}^{(2)}(x, x')$ .

Now consider the case  $x' = \mathbf{0}_{(n)}$ . Then  $\tilde{\beta}^{(2)}(x, x') = \xi^{(1)}(x, x')$ . It is evident that  $x \underset{C+C'}{>} x'$  if and only if  $(C + C')x = \mathbf{0}_{(k)}$ .

If  $\xi^{(1)}(x, x') > 0$ , then for any matrix  $C' \in \mathbf{R}^{k \times n}$  such that  $\|C'\| < \xi^{(1)}(x, x')$  there exists  $i \in K(x, x')$  satisfying (54) and (53). From here using the same argumentation as in the proof of Lemma 14 we derive  $|(C_i + C'_i)x| > 0$  which means that  $x \underset{C+C'}{>} x'$ . Hence  $\omega \geq \tilde{\beta}^{(1)}(x, x')$ .

On the other hand, if each element  $c'_{ij}$  of perturbing matrix  $C' = (c'_{ij})_{k, n}$  is equal to  $-C_i x / \|x\|_*$ , then  $\|C'\| = \xi^{(1)}(x, x')$  and  $(C + C')x = \mathbf{0}_{(k)}$ . Hence  $\omega \leq \xi^{(1)}(x, x')$ .

$\square$









