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ABSTRACT: The paper deals with the topology optimization of the elastic contact problems using the level set approach. A piecewise constant level set method is used to follow the evolution of design domain interfaces rather than the standard level set method. The piecewise constant level set function takes distinct constant values in each subdomain of a whole design domain. Using a two-phase approximation the original optimization problem is reformulated as an equivalent constrained optimization problem in terms of the piecewise constant level set function. Necessary optimality condition is formulated. Finite difference and finite element methods are applied as the approximation methods. Numerical examples are provided and discussed.

1 INTRODUCTION

The paper deals with the numerical solution of a structural optimization problem for contact problems between an elastic body and a rigid foundation. This contact phenomenon is governed by an elliptic variational inequality. The structural optimization problem for the elastic body in unilateral contact consists in finding such topology of the domain occupied by the body and/or the shape of its boundary that the normal contact stress along the boundary of the body is minimized. The volume of the body is assumed bounded.

The standard level set method (Allaire et al. 2004, Osher et al. 2003) is employed in structural optimization for numerical tracking the evolution of the domain boundary on a fixed mesh and finding an optimal domain. This method is based on an implicit representation of the boundaries of the optimized structure. Recently, different modifications (De Cezaro et al. 2012, Yamada et al. 2010) of the standard level set method are developed to increase its effectiveness. Among others an arbitrary number of subdomains can be identified using only one discontinuous piecewise constant level set function taking distinct constant values on each subdomain (De Cezaro et al. 2012, Lie et al. 2005, Myśliński 2015, Wei et al. 2009, Zhu et al. 2011).

In the paper the original structural optimization problem is approximated by a two-phase optimization problem using weak and strong phases (Allaire et al. 2004). Using the piecewise constant level set method (De Cezaro et al. 2012) this approximated problem is reformulated as an equivalent constrained optimization problem in terms of the piecewise constant level set function only. Therefore neither shape nor topological sensitivity analysis is required. During the evolution of the piecewise constant level set function small holes can be created without use of the topological derivatives. The paper extends results contained in (Myśliński 2015). Necessary optimality condition is formulated. The finite difference and finite element methods are used as the approximation methods. This discretized optimization problem is solved numerically using the augmented Lagrangian method. Numerical examples are provided and discussed.

2 PROBLEM FORMULATION

Consider deformations of an elastic body occupying two-dimensional domain Ω with the smooth boundary Γ. The elastic body obeying Hooke's law is subject to body forces $f(x) = (f_1(x), f_2(x))$, $x \in \Omega$. Moreover, surface tractions $p(x) = (p_1(x), p_2(x))$, $x \in \Gamma$, are applied to a portion Γ_1 of the boundary Γ. Assume the body is clamped along the portion Γ_0 of the boundary Γ, and that the contact conditions are prescribed on the portion Γ_2, where $\Gamma_i \cap \Gamma_j = \emptyset$, $i \neq j$, $i,j = 0,1,2$, $\Gamma = \bar{\Gamma}_0 \cup \bar{\Gamma}_1 \cup \bar{\Gamma}_2$. We denote by $u = (u_1, u_2)$, $u = u(x)$, $x \in \Omega$, the displacement of the body and by $e(x) = \{e_{ij}(u(x))\}$ as well as by $\sigma(x) = \{\sigma_{ij}(u(x))\}$, $i,j = 1,2$, the strain field and stress field in the body, respectively.
Let us formulate a contact problem in variational form. Denote by V_{sp} and K the space and set of kinematically admissible displacements given by $V_{sp} = \{z \in [H^1(\Omega)]^2 = H^1(\Omega) \times H^1(\Omega) : z_i = 0 \text{ on } \Gamma_0,\ i = 1,2\}$ and $K = \{z \in V_{sp} : z_N \leq 0 \text{ on } \Gamma_2\}$. Let Λ

denotes the set of Lagrange multipliers $\Lambda = \{\zeta \in L^2(\Gamma_2) : |\zeta| \leq 1\}$. Variational formulation of contact problem has the form: *find a pair* $(u, \lambda) \in K \times \Lambda$ *satisfying*

$$\int_\Omega a_{ijkl}e_{ij}(u)e_{kl}(\varphi - u)dx - \int_\Omega f_i(\varphi_i - u_i)dx - \quad (1)$$

$$\int_{\Gamma_1} p_i(\varphi_i - u_i)ds + \int_{\Gamma_2} \lambda(\varphi_T - u_T)ds \geq 0,$$

$$\int_{\Gamma_2} (\zeta - \lambda)u_T ds \leq 0, \quad (2)$$

for all $(\varphi, \zeta) \in K \times \Lambda$ $i, j, k, l = 1, 2$. The elasticity tensor satisfying usual requirements is denoted by $\{a_{ijkl}\}$ and the tangential (normal) displacement by $u_T(u_N)$. We use here and throughout the paper the summation convention over repeated indices (Myśliński 2015).

Before formulating a structural optimization problem for (1)-(2) let us introduce first the set U_{ad} of admissible domains in the form $U_{ad} = \{\Omega : \Omega \text{ is Lipschitz continuous}, \ Vol(\Omega) - Vol^{giv} \leq 0,\ Per(\Omega) \leq const_1\}$ where $Vol(\Omega) \stackrel{def}{=} \int_\Omega dx$ and $Per(\Omega) \stackrel{def}{=} \int_\Gamma dx$. The set U_{ad} is assumed to be nonempty. In order to define a cost functional we shall also need the following set M^{st} of auxiliary functions $M^{st} = \{\eta = (\eta_1, \eta_2) \in [H^1(D)]^2 : \eta_i \leq 0 \text{ on } D,\ i = 1, 2,\ \| \eta \|_{[H^1(D)]^2} \leq 1\}$ where the norm $\| \eta \|_{[H^1(D)]^2} = (\sum_{i=1}^2 \| \eta_i \|^2_{H^1(D)})^{1/2}$. Recall from (Myśliński 2015) the cost functional approximating the normal contact stress on the contact boundary

$$J_\eta(u(\Omega)) = \int_{\Gamma_2} \sigma_N(u)\eta_N(x)ds, \quad (3)$$

depending on the auxiliary given bounded function $\eta(x) \in M^{st}$. σ_N and ϕ_N are the normal components of the stress field σ corresponding to a solution u satisfying system (1)-(2) and the function η, respectively.

We shall consider the following structural optimization problem: *for a given function* $\eta \in M^{st}$, *find a domain* $\Omega^\star \in U_{ad}$ *such that*

$$J_\eta(u(\Omega^\star)) = \min_{\Omega \in U_{ad}} J_\eta(u(\Omega)) \quad (4)$$

3 PIECEWISE CONSTANT LEVEL SET APPROACH

In (Allaire et al. 2004) the standard level set method (Osher et al. 2003) is employed to govern the evolution of domains and to solve numerically problem (4). Denote by $t > 0$ the artificial time variable and consider the evolution of a domain Ω under a velocity field V. Under the suitable regular mapping $T(t, V)$ we have $\Omega_t = T(t, V)(\Omega) = (I + tV)(\Omega),\ \ t > 0$. By Ω_t^- (resp. Ω_t^+) we denote the interior (resp. outside) of the domain Ω_t. The domain Ω_t and its boundary $\partial\Omega_t$ are determined by a function $\phi = \phi(x, t)$: $R^2 \times [0, t_0) \to R$ satisfying: $\phi(x, t) = 0$, if $x \in \partial\Omega_t$, $\phi(x, t) < 0$, if $x \in \Omega_t^-$, $\phi(x, t) > 0$, if $x \in \Omega_t^+$. Function ϕ is called the level set function (Osher et al. 2003).

Let us reformulate problem (4) in terms of a piecewise constant level set function. For hold-all domain $D \subset R^2$ partitioned into N subdomains $\{\Omega_i\}_{i=1}^N$ such that $D = \bigcup_{i=1}^N (\Omega_i \cup \partial\Omega_i)$ where N is a given integer and $\partial\Omega_i$ denotes the boundary of the subdomain Ω_i a piecewise constant level set function $\phi : D \to R$ is defined as (Yamada et al. 2010, Wei et al. 2009)

$$\phi = i \ \text{ in } \Omega_i, \quad i = 1, 2, ..., N. \quad (5)$$

Consider piecewise constant density function $\rho : D \to R^2$ defined as

$$\rho(x) = \begin{cases} \epsilon & \text{if } x \in D \setminus \bar{\Omega}, \\ 1 & \text{if } x \in \Omega, \end{cases} \quad (6)$$

where $\epsilon > 0$ is a small constant. We confine to consider a two-phase problem in the domain D where the characteristic functions of the subdomains are $\chi_1(x) = 2 - \phi(x)$ and $\chi_2(x) = \phi(x) - 1$. Therefore $\rho(x) = \rho_1\chi_1(x) + \rho_2\chi_2(x) = (1 - \epsilon)\phi(x) + 2\epsilon - 1$.

Using it as well as (5) the structural optimization problem (4) can be transformed into the following one: *find* $\phi \in U_{ad}^\phi$ *such that*

$$\min_{\phi \in U_{ad}^\phi} J_\eta(\phi) = \int_{\Gamma_2} \rho(\phi)\sigma_N(u_\epsilon)\eta_N ds \quad (7)$$

where the set U_{ad}^ϕ of the admissible functions is given as

$$U_{ad}^\phi = \{\phi \in H^1(D) \ : \ Vol(\phi) - Vol^{giv} \leq 0, \quad (8)$$

$$Vol(\phi) \stackrel{def}{=} \int_\Omega \rho(\phi)dx,$$

$$W(\phi) \stackrel{def}{=} (\phi - 1)(\phi - 2) = 0,$$

$$Per(\phi) \stackrel{def}{=} \int_\Omega |\nabla\phi| dx \leq const_1\}. \quad (9)$$

The element $(u_\epsilon, \lambda_\epsilon) \in K \times \Lambda$ satisfies the state system (1)-(2) in the domain D rather than Ω.

4 NECESSARY OPTIMALITY CONDITION

Using a two-phase approximation the original structural optimization problem (4) is reformulated as an equivalent constrained optimization problem (7) in terms of the piecewise constant level set function $\phi(x)$. Using the Augmented Lagrangian associated to the problem (7) the derivative of the cost functional is calculated and a necessary optimality condition is formulated. Let us formulate the necessary optimality condition for the optimization problem (7)-(9). We denote by $\tilde{\mu} = \{\tilde{\mu}_i\}_{i=1}^3$ Lagrange multiplier associated with constraints in the set (8). Let us introduce the Augmented Lagrangian $L(\phi, \tilde{\mu})$ associated with this optimization problem:

$$L(\phi, \tilde{\mu}) = L(\phi, u_\epsilon, \lambda_\epsilon, p^a, q^a, \tilde{\mu}) =$$

$$J_\eta(\phi) + \int_D \rho(\phi) a_{ijkl} e_{ij}(u_\epsilon) e_{kl}(p^a) dx - \qquad (10)$$

$$\int_D \rho(\phi) f_i p_i^a dx - \int_{\Gamma_1} p_i p_i^a ds + \int_{\Gamma_2} \lambda_\epsilon p_T^a ds +$$

$$\int_{\Gamma_2} q^a u_{\epsilon T} ds + \tilde{\mu} c(\phi) + \sum_{i=1}^{3} \frac{1}{2\beta_i} c_i^2(\phi),$$

where $i, j, k, l = 1, 2$. Moreover $c(\phi) \stackrel{def}{=} \{c_i(\phi)\}_{i=1}^3 = [Vol(\phi), Per(\phi), W(\phi)]^T$, $c^T(\phi)$ denotes a transpose of $c(\phi)$ and $\beta_m > 0$, $m = 1, 2, 3$, are a given real numbers. The pair $(p^a, q^a) \in K_1 \times \Lambda_1$ denotes an adjoint state defined as the solution to the following variational inequality (Myśliński 2015):

$$\int_D \rho(\phi) a_{ijkl} e_{ij}(\eta + p^a) e_{kl}(\varphi) dx +$$

$$\int_{\Gamma_2} q^a \varphi_T ds = 0 \quad \forall \varphi \in K_1, \qquad (11)$$

$$\int_{\Gamma_2} \zeta (p_T^a + \eta_T) ds = 0 \ \ \forall \zeta \in \Lambda_1. \qquad (12)$$

The sets K_1 and Λ_1 are given by

$$K_1 = \{\xi \in V_{sp} \ : \ \xi_N = 0 \ \text{ on } A^{st} \}, \qquad (13)$$

and by

$$\Lambda_1 = \{\zeta \in \Lambda \ : \ \zeta(x) = 0 \text{ on } B^{st} \}, \qquad (14)$$

while the coincidence set $A^{st} = \{x \in \Gamma_2 \ : \ u_N + v = 0\}$ and $B^{st} = B_1 \cup B_2 \cup B_1^+ \cup B_2^+$. Moreover $B_1 = \{x \in \Gamma_2 : \lambda(x) = -1\}$, $B_2 = \{x \in \Gamma_2 : \lambda(x) = +1\}$, $\tilde{B}_i = \{x \in B_i \ : \ u_N(x) + v = 0\}$, $i = 1, 2$, $B_i^+ = B_i \setminus \tilde{B}_i$, $i = 1, 2$. For interpretation of these sets see (Myśliński 2015).
The derivatives of functions $\rho(\phi)$, $c(\phi)$ with respect to ϕ are equal to (Myśliński 2015, Zhu et al. 2011) $\rho'(\phi) = 1 - \epsilon$, $c'(\phi) = [Vol'(\phi), W'(\phi), Per'(\phi)]$, respectively. Moreover $Vol'(\phi) = 1$, $W'(\phi) = 2\phi - 3$ and

$$Per'(\phi) = \chi_{\{\partial\Omega = const_0\}} \max\{0, \qquad (15)$$

$$-\nabla \cdot (\frac{\nabla \phi}{|\nabla \phi|})\} - \chi_{\{\partial\Omega > const_0\}} \nabla \cdot (\frac{\nabla \phi}{|\nabla \phi|}). \qquad (16)$$

Therefore the derivative of the Lagrangian L with respect to ϕ has the form:

$$\frac{\partial L}{\partial \phi}(\phi, \tilde{\lambda}) = \int_D \rho'(\phi) [a_{ijkl} e_{ij}(u_\epsilon) e_{kl}(p^a + \eta) -$$

$$f_i(p_i^a + \eta)] dx + \tilde{\mu} c'(\phi) + \sum_{i=1}^{3} \frac{1}{\beta_i} c(\phi) c'(\phi). \qquad (17)$$

Using (17)-(16) we can formulate the necessary optimality condition for topology optimization problem (7)-(9). It takes the form (Myśliński 2015): if $\hat{\phi} \in U_{ad}^\phi$ is an optimal solution to the problem (7)-(9) than there exists Lagrange multiplier $\tilde{\mu}^\star \in R^3$ such that $\tilde{\mu}_1^\star, \tilde{\mu}_2^\star \geq 0$ and satisfying for all $\phi \in U_{ad}^\phi$ and $\tilde{\mu} \in R^3$ the inequalities

$$L(\hat{\phi}, \tilde{\mu}) \leq L(\hat{\phi}, \tilde{\mu}^\star) \leq L(\phi, \tilde{\mu}^\star). \qquad (18)$$

Condition (18) implies (Myśliński 2015) that for all $\phi \in U_{ad}^\phi$ and $\tilde{\mu} \in R^3$

$$\frac{\partial L(\hat{\phi}, \tilde{\mu}^\star)}{\partial \phi} \geq 0 \ \text{ and } \ \frac{\partial L(\hat{\phi}, \tilde{\mu}^\star)}{\partial \tilde{\mu}} \leq 0, \qquad (19)$$

hold at the optimal point $(\hat{\phi}, \tilde{\mu}^\star) \in U_{ad}^\phi \times R^3$.

5 NUMERICAL IMPLEMENTATION

The optimization problem (7) is discretized using the finite difference and the finite element methods. The discretized optimization problem is numerically solved using Uzawa type iterative algorithm. The minimization of the Augmented Lagrangian with respect to function ϕ is realized using the gradient flow equation. For details see (Myśliński 2015).

6 NUMERICAL EXPERIMENTS

The discretized topology optimization problem (7)-(9) has been solved numerically in Matlab environment. The elastic body in unilateral contact with the rigid foundation is assumed to occupy two-dimensional domain Ω given by

$$\Omega = \{(x_1, x_2) \in R^2 : 0 \leq x_1 \leq 8 \ \wedge$$

$$0 < v(x_1) \leq x_2 \leq 4\}, \qquad (20)$$

with the function $v(x_1) = 0.125 * (x_1 - 4)^2$. The boundary Γ of the domain Ω is divided into three pieces

$$\Gamma_0 = \{(x_1, x_2) \in R^2 : x_1 = 0, 8 \ \wedge$$

$$0 < v(x_1) \leq x_2 \leq 4\}, \qquad (21)$$

$$\Gamma_1 = \{(x_1, x_2) \in R^2 : 0 \leq x_1 \leq 8 \ \wedge \ x_2 = 4\},$$

$$\Gamma_2 = \{(x_1, x_2) \in R^2 : 0 \leq x_1 \leq 8 \ \wedge \ x_2 = v(x_1)\}.$$

The computations are carried out for the elastic body characterized by the Poisson's ratio $\nu = 0.29$ and strong material Young modulus $E = 2.1 \cdot 10^{11} N/m^2$. The weak material phase parameter $\epsilon = 10^{-3}$. The body is loaded by the boundary traction $p_1 = 0$, $p_2 = -5.6 \cdot 10^6 N$ along the boundary Γ_1. The body forces $f_i = 0$, $i = 1, 2$. The computational domain

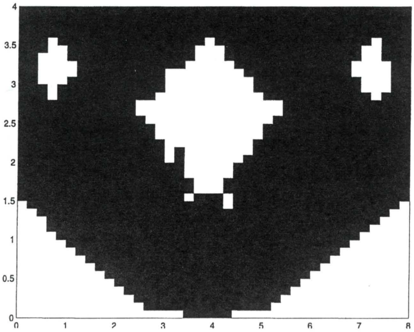

Figure 1: Optimal topology domain $\Omega^{\star}$. $\phi^0 = 1.5$.

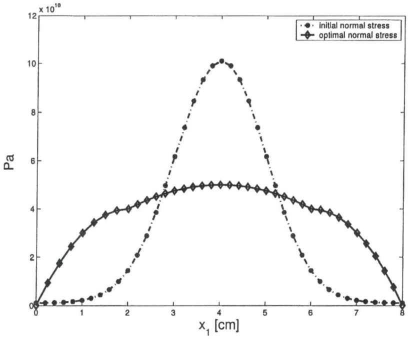

Figure 2: Normal contact stress distributions for initial and optimal domains. $\phi^0 = 1.5$.

$D = [0,8] \times [0,4]$ is selected. Domain D is discretized with a fixed rectangular mesh into 3200 elements. Auxiliary function η in (3) is selected as a piecewise linear on computational domain D. Material volume fraction $r_{fr} = 0.5$ is prescribed. The other computational parameters are equal to: the tolerance parameter $\varepsilon_1 = 10^{-4}$, the smoothness parameter $\varepsilon_2 = 10^{-6}$, the penalty parameter $\beta_i = 10^{-6}, i = 1,2,3$. The computations have been performed for the initial level set function $\phi^0 = 1.5$.

Figure 1 presents the optimal topology domain of structural optimization problem (7)-(9). The big area with low values of density function, i.e., filled with the weaker material, appear in the central part of the domain Ω and is symmetrically distributed. Two smaller such areas appear near the fixed edges. They are slightly unsymmetrically distributed due to numeric errors (Myśliński 2015). Figure 2 presents the distribution of the normal stress along the contact boundary for the initial and the optimal topology domains. The peak of the normal contact stress at the initial topology domain has been smeared out and significantly reduced. For the optimal topology domain the obtained normal contact stress along the contact boundary is almost constant.

7 CONCLUSIONS

New method for solving topology optimization problems for elastic unilateral contact problems with a given friction based on piecewise constant level set functions has been proposed. The original topology optimization problem is approximated by the two-phase optimization problem and is transformed into the constrained optimization problem in terms of the piecewise constant level set function. The proposed method does not require to solve Hamilton - Jacobi equation and to perform the reinitialization process of the signed distance function as in standard level set approach. Moreover the proposed method has also voids nucleation capabilities as topological derivative based methods. It can be viewed as combining the elements of SIMP and topology derivative approaches.

The obtained numerical results that the optimal domains contain the areas with low values of density function in the central part of the body and near the fixed edges. The normal contact stress is almost constant along the optimal topology domain boundary and has been significantly reduced comparing to the initial one. The proposed algorithm is robust and finds optimal topologies which seems to be in accordance with the physical reasoning and engineering experience. Since the elliptic inequalities constrained topology optimization problems are generally nonconvex it is well known that their numerically obtained solutions are dependent on the initial design. Therefore the obtained optimal topology domains are likely of local character.

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