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ON ROBUST $PI^{\lambda}D^{\mu}$ CONTROL FOR TIME–DELAY NON–INTEGER ORDER PLANTS

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Abstract

This paper deals wityh the robust control of time-delay non-integer order plants by means of standard non-integer order controllers. Precisely, a method for determining the parameter regions where a $PI^{\lambda}D^{\mu}$ controller ensures a given modulus margin (inverse of the H_{∞} norm of the sensitivity function) is presented. This method, which is conceptually simple and intuitive, extends to the aforementioned general classes of controllers and plants the approach recently followed by the present authors in the integer order case. A Matlab program to plot the loci of constant modulus margin and those of constant crossover frequency has been developed and applied to some numerical examples taken from the literature.

I. INTRODUCTION

Non-integer order systems have been considered with increasing interest in the recent control literature because many plants can be described more satisfactorily by models of this kind [1], [2] or because non-integer order controllers provide a better performance than classic integer order ones [3]. In fact, it has been shown that in many instances fractional order PID controllers outperform the best integer order PID controllers [4], [5], [6]. In the following, we consider the most general situation where the order of the controller is non–integer and the plant is described by a non–integer order model plus a time delay . Obviously, the other cases are but special instances of this one.

Robustness plays a fundamental role in non-integer order control, too [7], [8], [9], [10]. In particular, it is very important to determine the set of $PI^{\lambda}D^{\mu}$ controllers that satisfy certain stability margins. Among these margins, the modulus margin (also called H_{∞} margin because it is the inverse of the H_{∞} norm of the sensitivity function) seems to be the most meaningful [11], [12], [13]. Determining the controllers that ensure a given modulus margin, however, is not an easy task even for integer order systems. Such problem has been tackled, e.g., in [14] and [15] for integer order time-delay plants and PID controllers using different approaches. Here, we extend the essentially graphic method in [15] to the aforementioned general case of $PI^{\lambda}D^{\mu}$ controllers and non-integer order time-delay plants. The entire stability region in the controller parameter space has already been determined in [16], [17] and, for particular classes of fractional order controllers and time–delay plants, in [18], [19] where, however, no indication is given regarding the loci of constant modulus margin. In [20], drawing on the procedure described in [21] for integer order systems, a design technique based on shaping the sensitivity function has been proposed for minimum–phase commensurate order plants without time delay.

The remainder of this paper is organized as follows. Section II specifies the adopted plant and controller representations, and formulates the problem to be solved. The equation of the stability boundary in the controller parameter space is derived in Section III along the lines followed in [15] for the integer order case. Section IV gives the equations of the loci of constant gain and phase crossover frequency. The loci of constant modulus margin are determined in Section V. The use of these loci for controller synthesis is illustrated in Section VI by means of numerical examples taken from the literature. A few concluding remarks are made in Section VII. To plot the aforementioned loci, the Matlab program described in the Appendix has been developed.

II. PROBLEM STATEMENT

Consider a unity–feedback control system and assume that the controlled plant is described by the transfer function:

$$G(s) = \frac{n(s)}{d(s)}\, e^{-Ts} = \frac{b_n s^{\beta_n} + b_{n-1} s^{\beta_{n-1}} + \ldots + b_1 s^{\beta_1} + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \ldots + a_1 s^{\alpha_1} + a_0 s^{\alpha_0}}\, e^{-Ts}, \tag{1}$$

where: $\beta_n > \beta_{n-1} > \cdots > \beta_1 > \beta_0 \geq 0$, $\alpha_n > \alpha_{n-1} > \cdots > \alpha_1 > \alpha_0 \geq 0$, and $T \geq 0$ is a time delay.
It is also assumed that plant is controlled by means of a standard $PI^{\lambda}D^{\mu}$ controller described by the transfer function:

$$C(s) = k_p + k_i s^{\lambda} + k_d \frac{1}{s^{\mu}}, \tag{2}$$

where $\lambda > 0$, $\mu > 0$, and k_p, k_i, k_d are the proportional, integral and derivative gain, respectively.

Clearly, this setting encompasses any combination of integer or non–integer time–delay plant and integer or non–integer controller. Moreover, both α_0 and β_0 can be equal to zero, i.e., $\alpha_0 = \beta_0 = 0$, even if this case has no particular interest.

The problem we refer to is that of finding a controller of form (2) in such a way that the overall unity–feedback control system is stable with a modulus margin greater than a prescribed value.

In the usual case of integer order controllers, there are three design parameters, i.e., k_p, k_i and k_d, whereas the aforementioned control problem allows for two more design parameters, i.e., λ and μ, and this greater flexibility can be exploited to achieve a better performance.

III. STABILITY REGIONS

The Nyquist diagram of the loop function:

$$L(s) = C(s)G(s) \tag{3}$$

crosses the unit circle centred at the origin with a phase equal to $m_{\varphi} - \pi$, where m_{φ} is the phase margin, if:

$$L(\jmath\omega_a) = e^{\jmath(m_{\varphi}-\pi)}, \tag{4}$$

where ω_a denotes the gain crossover frequency and $\jmath$ denotes the imaginary unit.
Taking into account (1) and (2), the interpolation condition (4) can be written as

$$\left[k_p(\jmath\omega_a)^{\lambda} + k_i + k_d(\jmath\omega_a)^{\lambda+\mu}\right] n(\jmath\omega_a) = d(\jmath\omega_a)(\jmath\omega_a)^{\lambda}\, e^{\jmath(T\omega_a + m_{\varphi} - \pi)}. \tag{5}$$

Decomposing $n(\jmath\omega_a)$ and $d(\jmath\omega_a)$ into their real and imaginary parts, i.e.,

$$n(\jmath\omega_a) = n_r(\omega_a) + \jmath\, n_i(\omega_a), \quad d(\jmath\omega_a) = d_r(\omega_a) + \jmath\, d_i(\omega_a), \tag{6}$$

and expressing $(\jmath\omega_a)^{\lambda}$ as

$$(\jmath\omega_a)^{\lambda} = \omega_a^{\lambda}\, e^{\jmath\lambda\frac{\pi}{2}} = \omega_a^{\lambda}\Big(\cos\lambda\frac{\pi}{2} + \jmath\sin\lambda\frac{\pi}{2}\Big), \tag{7}$$

eqn. (5) can be rewritten as

$$\left[k_p\omega_a^\lambda\Big(\cos\lambda\frac{\pi}{2}+\jmath\sin\lambda\frac{\pi}{2}\Big)+k_i+k_d\omega_a^{\lambda+\mu}\Big(\cos(\lambda+\mu)\frac{\pi}{2}+\jmath\sin(\lambda+\mu)\frac{\pi}{2}\Big)\right]\Big[n_r(\omega_a)+\jmath n_i(\omega_a)\Big]=$$
$$\Big[d_r(\omega_a)+\jmath d_i(\omega_a)\Big]\omega_a^\lambda\left[\cos\Big(\omega_aT+m_\varphi+\lambda\frac{\pi}{2}-\pi\Big)+\jmath\sin\Big(\omega_aT+m_\varphi+\lambda\frac{\pi}{2}-\pi\Big)\right] \tag{8}$$

which can be split into the following two equations relating the real and imaginary parts on both its sides:

$$\Big[k_p\,\omega_a^\lambda\cos\lambda\frac{\pi}{2}+k_i+k_d\,\omega_a^{\lambda+\mu}\cos(\lambda+\mu)\frac{\pi}{2}\Big]\,n_r(\omega_a)-\Big[k_p\,\omega_a^\lambda\sin\lambda\frac{\pi}{2}+k_d\,\omega_a^{\lambda+\mu}\sin(\lambda+\mu)\frac{\pi}{2}\Big]\,n_i(\omega_a)=$$
$$-\omega_a^\lambda d_r(\omega_a)\cos\Big(\omega_aT+m_\varphi+\lambda\frac{\pi}{2}\Big)+\omega_a^\lambda d_i(\omega_a)\sin\Big(\omega_aT+m_\varphi+\lambda\frac{\pi}{2}\Big), \tag{9}$$

$$\Big[k_p\,\omega_a^\lambda\cos\lambda\frac{\pi}{2}+k_i+k_d\,\omega_a^{\lambda+\mu}\cos(\lambda+\mu)\frac{\pi}{2}\Big]\,n_i[\omega_a)+\Big[k_p\,\omega_a^\lambda\sin\lambda\frac{\pi}{2}+k_d\,\omega_a^{\lambda+\mu}\sin(\lambda+\mu)\frac{\pi}{2}\Big]\,n_r(\omega_a)=$$
$$-\omega_a^\lambda d_r(\omega_a)\sin\Big(\omega_aT+m_\varphi+\lambda\frac{\pi}{2}\Big)-\omega_a^\lambda d_i(\omega_a)\cos\Big(\omega_aT+m_\varphi+\lambda\frac{\pi}{2}\Big), \tag{10}$$

leading to

$$k_p\sin\lambda\frac{\pi}{2}+k_d\,\omega_a^\mu\sin(\lambda+\mu)\frac{\pi}{2}=-A(\omega_a)\,\sin\Big(\omega_aT+m_\varphi+\lambda\frac{\pi}{2}\Big)+B(\omega_a)\,\cos\Big(\omega_aT+m_\varphi+\lambda\frac{\pi}{2}\Big), \tag{11}$$

$$k_i\sin\lambda\frac{\pi}{2}-k_d\,\omega_a^{\lambda+\mu}\sin\mu\frac{\pi}{2}=\omega_a^\lambda\Big[A(\omega_a)\,\sin\big(\omega_aT+m_\varphi\big)-B(\omega_a)\,\cos\big(\omega_aT+m_\varphi\big)\Big], \tag{12}$$

where

$$A(\omega)=\frac{d_r(\omega)n_r(\omega)+d_i(\omega)n_i(\omega)}{n_r^2(\omega)+n_r^2(\omega)}, \tag{13}$$

$$B(\omega)=\frac{d_r(\omega)n_i(\jmath\omega)-d_i(\jmath\omega)n_r(\omega)}{n_r^2(\omega)+n_i^2(\omega)}. \tag{14}$$

For $k_d=$ const, eqns. (11) and (12) supply the parametric equations (with parameter ω_a) of a curve in the cross section k_d =const of the (k_p,k_i,k_d)–space, whereas, for either $k_p=$ const or $k_i=$ const, they describe a curve on the plane (k_i,k_d) or, respectively, (k_p,k_d).

When $m_\varphi=0$, at each point of these curves $1+L(\jmath\omega_a)=0$, i.e., the characteristic equation has purely imaginary roots . Therefore, on the aforementioned parameter planes, these curves separate regions characterized by different numbers of RHP and LHP roots of the system characteristic equation, and some of these regions may correspond to a stable behaviour. This property was proved in [15] for integer order systems and controllers. The set of all stabilizing fractional order PID controllers has also been determined in [22].

For $\lambda=\mu=1$ (integer order PID controller) eqns. (11) and (12) simplify to

$$k_p=-A(\omega_a)\,\cos\Big(\omega_aT+m_\varphi\Big)-B(\omega_a)\,\sin\Big(\omega_aT+m_\varphi\Big), \tag{15}$$

$$k_i-k_d\,\omega_a^2=\omega_a\Big[A(\omega_a)\,\sin\big(\omega_aT+m_\varphi\big)-B(\omega_a)\,\cos\big(\omega_aT+m_\varphi\big)\Big], \tag{16}$$

which of course coincide with eqns. (10) and (11) in [15].

The particular cases of PD^μ and PI^λ controllers can simply be obtained from (11)–(12) by setting $k_i=0$ or, respectively, $k_d=0$ and then eliminating either λ or μ from the resulting equations. For example, setting $k_i=0$ in (11)–(12) (PD^μ controller) leads to

$$k_p\sin\lambda\frac{\pi}{2}+k_d\,\omega_a^\mu\sin(\lambda+\mu)\frac{\pi}{2}=-A(\omega_a)\,\sin\Big(\omega_aT+m_\varphi+\lambda\frac{\pi}{2}\Big)+B(\omega_a)\,\cos\Big(\omega_aT+m_\varphi+\lambda\frac{\pi}{2}\Big), \tag{17}$$

$$-k_d\,\omega_a^{\lambda+\mu}\sin\mu\frac{\pi}{2} = \omega_a^{\lambda}\Big[A(\omega_a)\,\sin\big(\omega_a T + m_\varphi\big) - B(\omega_a)\,\cos\big(\omega_a T + m_\varphi\big)\Big]. \tag{18}$$

To eliminate λ from (17) and (18), it is enough to multiply (17) by $\sin\mu\frac{\pi}{2}$ and (18) by $\sin(\lambda+\mu)\frac{\pi}{2}$, and then combine the resulting two equations, thus arriving at

$$k_p\sin\mu\frac{\pi}{2} = A(\omega_a)\,\sin\Big(\omega_a T + m_\varphi - \mu\frac{\pi}{2}\Big) - B(\omega_a)\,\cos\Big(\omega_a T + m_\varphi - \mu\frac{\pi}{2}\Big), \tag{19}$$

$$k_d\,\omega_a^{\mu}\sin\mu\frac{\pi}{2} = -A(\omega_a)\,\sin\big(\omega_a T + m_\varphi\big) + B(\omega_a)\,\cos\big(\omega_a T + m_\varphi\big). \tag{20}$$

As an example, Fig. ?? shows some curves (19)–(20) for various values of m_φ when the plant is described by (43) (see Section VI) and the exponential of the PD^μ controller is $\mu = 0.5$. Clearly, the stability region is obtained for $m_\varphi = 0$.

It is even simpler is to find the equations for a PI^λ controller from (11)–(12): essentially, it is enough to set $k_d = 0$ in (11) and (12).

A procedure similar to that adopted for finding the loci of constant m_φ can be followed to determine the locus of the (k_p, k_i, k_d)–points ensuring that the diagram of $L(\jmath\omega)$ crosses the real axis at $-1/m_g + \jmath 0$, $m_g > 0$, for some angular frequency ω_b (phase crossover frequency), i.e.:

$$L(\jmath\omega_b) = -\frac{1}{m_g}. \tag{21}$$

It turns out that this locus is defined by the parametric equations (with parameter ω_b):

$$k_p\sin\lambda\frac{\pi}{2} + k_d\,\omega_b^{\mu}\sin(\lambda+\mu)\frac{\pi}{2} = \frac{1}{m_g}\Big[B(\omega_b)\,\cos\Big(\omega_b T + \lambda\frac{\pi}{2}\Big) - A(\omega_b)\,\sin\Big(\omega_b T + \lambda\frac{\pi}{2}\Big)\Big], \tag{22}$$

$$k_i\sin\lambda\frac{\pi}{2} - k_d\,\omega_b^{\lambda+\mu}\sin\mu\frac{\pi}{2} = \frac{\omega_b^{\lambda}}{m_g}\Big[-B(\omega_b)\,\cos\omega_b T + A(\omega_b)\,\sin\omega_b T\Big], \tag{23}$$

which, in the case of an integer order controller ($\lambda = \mu = 1$), simplify to

$$k_p = -\frac{1}{m_g}\Big[A(\omega_b)\,\cos\omega_b T + B(\omega_b)\,\sin\omega_b T\Big], \tag{24}$$

$$k_i - k_d\,\omega_b^2 = \frac{\omega_b}{m_g}\Big[A(\omega_b)\,\sin\omega_b T - B(\omega_b)\,\cos\omega_b T\Big]. \tag{25}$$

Again, the equations for either a PD^μ or a PI^λ controller can immediately be obtained from (22)–(23). Precisely, the equations for the former are:

$$k_p\sin\mu\frac{\pi}{2} = \frac{1}{m_g}\Big[A(\omega_b)\,\sin\Big(\omega_b T - \mu\frac{\pi}{2}\Big) - B(\omega_b)\,\cos\Big(\omega_b T - \mu\frac{\pi}{2}\Big)\Big], \tag{26}$$

$$k_d\,\omega_b^{\mu}\sin\mu\frac{\pi}{2} = \frac{1}{m_g}\Big[-A(\omega_b)\,\sin\omega_b T + B(\omega_b)\,\cos\omega_b T\Big], \tag{27}$$

whereas the equations for the PI^λ controller can be found simply by setting $k_d = 0$ in (22)–(23).

IV. LOCI OF CONSTANT CROSSOVER FREQUENCY

At the gain crossover frequency ω_a, the square magnitude of the loop function is equal to 1, i.e.,

$$|L(\jmath\omega_a)|^2 = |C(\jmath\omega_a)|^2 |G(\jmath\omega_a)|^2 = 1, \tag{28}$$

where

$$|G(\jmath\omega_a)|^2 = \frac{n_r^2(\omega_a) + n_i^2(\omega_a)}{d_r^2(\omega_a) + d_i^2(\omega_a)} \tag{29}$$

and

$$|C(\jmath\omega_a)|^2 = \frac{|k_p(\jmath\omega_a)^\lambda + k_i + k_p(\jmath\omega_a)^{\lambda+\mu}|^2}{\omega_a^{2\lambda}} \tag{30}$$

which, according to (7), can be rewritten as

$$|C(\jmath\omega_a)|^2 = \frac{|k_p\omega_a^\lambda(\cos\lambda\frac{\pi}{2} + \jmath\sin\lambda\frac{\pi}{2}) + k_i + k_p\omega_a^{\lambda+\mu}(\cos\lambda\frac{\pi}{2} + \jmath\sin\lambda\frac{\pi}{2})|^2}{\omega_a^{2\lambda}} =$$

$$\frac{1}{\omega_a^{2\lambda}}\left[\left(k_p\omega_a^\lambda\right)^2 + k_i^2 + \left(k_d\omega_a^{\lambda+\mu}\right)^2 + 2k_ik_p\omega_a^\lambda\cos\lambda\frac{\pi}{2} + 2k_ik_d\omega_a^{\lambda+\mu}\cos(\lambda+\mu)\frac{\pi}{2} + 2k_p\omega_a^\lambda k_d\omega_a^{\lambda+\mu}\cos\mu\frac{\pi}{2}\right]. \tag{31}$$

Using (29) and (31), eqn. (28) leads to:

$$\left(k_p\omega_a^\lambda\right)^2 + k_i^2 + \left(k_d\omega_a^{\lambda+\mu}\right)^2 + 2k_ik_p\omega_a^\lambda\cos\lambda\frac{\pi}{2} + 2k_ik_d\omega_a^{\lambda+\mu}\cos(\lambda+\mu)\frac{\pi}{2} + 2k_p\omega_a^\lambda k_d\omega_a^{\lambda+\mu}\cos\mu\frac{\pi}{2}$$

$$= \omega_a^{2\lambda}\,\frac{d_r^2(\omega_a) + d_i^2(\omega_a)}{n_r^2(\omega_a) + n_i^2(\omega_a)}\,. \tag{32}$$

which is independent of the time delay T.

It is easily verified that the last equation represents an ellipsoid in the parameter space (excluding degenerate cases). It follows that its intersections with the planes $k_d = \text{const}$, $k_i = \text{const}$ and $k_p = \text{const}$ are ellipses.

In particular, to find the intersections with the planes $k_d = \text{const}$, eqn. (32) can conveniently be written as

$$\left[k_p\omega_a^\lambda + \frac{\sin(\lambda+\mu)\frac{\pi}{2}}{\sin\lambda\frac{\pi}{2}}k_d\omega_a^{\lambda+\mu}\right]^2 + \left[k_i - \frac{\sin\mu\frac{\pi}{2}}{\sin\lambda\frac{\pi}{2}}k_d\omega_a^{\lambda+\mu}\right]^2 +$$

$$2\left[k_p\omega_a^\lambda + \frac{\sin(\lambda+\mu)\frac{\pi}{2}}{\sin\lambda\frac{\pi}{2}}k_d\omega_a^{\lambda+\mu}\right]\left[k_i - \frac{\sin\mu\frac{\pi}{2}}{\sin\lambda\frac{\pi}{2}}k_d\omega_a^{\lambda+\mu}\right]\cos\lambda\frac{\pi}{2} = \omega_a^{2\lambda}\,\frac{d_r^2(\omega_a) + d_i^2(\omega_a)}{n_r^2(\omega_a) + n_i^2(\omega_a)} \tag{33}$$

which is the equation of an ellipse on the plane (k_p, k_i) centred at

$$\hat{k}_p = -\frac{\sin(\lambda+\mu)\frac{\pi}{2}}{\sin\lambda\frac{\pi}{2}}k_d\omega_a^\lambda\,, \quad \hat{k}_i = \frac{\sin\mu\frac{\pi}{2}}{\sin\lambda\frac{\pi}{2}}k_d\omega_a^{\lambda+\mu}. \tag{34}$$

For $\lambda = \mu = 1$ the coordinates of the centre are simply $\hat{k}_p = 0$, $\hat{k}_i = \omega_a^2 k_d$.

By plotting the ellipses for a number of gain crossover frequencies, it is possible to assign the value of ω_a to the points of the curve (11)–(12) on the considered plane and, thus, evaluate the feedback system passband.

On the plane $k_d = \text{const}$, the loci of constant ω_b are just straight lines through the origin. Precisely, from (23) and (22) it follows that:

$$k_i = m(\omega_b)\cdot k_p, \tag{35}$$

whose slope is

$$m(\omega_b) = \frac{\omega_b^\lambda\left[-B(\omega_b)\cos\omega_bT + A(\omega_b)\sin\omega_bT\right] + m_gk_d\,\omega_b^{\lambda+\mu}\sin\mu\frac{\pi}{2}}{\left[B(\omega_b)\cos\left(\omega_bT + \lambda\frac{\pi}{2}\right) - A(\omega_b)\sin\left(\omega_bT + \lambda\frac{\pi}{2}\right)\right] - m_gk_d\,\omega_b^\mu\sin(\lambda+\mu)\frac{\pi}{2}}. \tag{36}$$

The loci on the planes $k_i = \text{const}$ and $k_p = \text{const}$ can be determined in the same way.

V. LOCI OF CONSTANT MODULUS MARGIN

An indicator of system robustness that is more adequate than m_φ and m_g is the modulus margin defined as:

$$\delta := \min_\omega |1 + L(\jmath\omega)|. \tag{37}$$

As is well known, it represents the minimal distance of the Nyquist diagram of the loop function from the critical point $-1 + \jmath 0$ and corresponds to the reciprocal of the infinity norm of the sensitivity function. Clearly, the locus of the parameter points where $\delta = \text{const}$ is the envelope of the loci:

$$|1 + L(\jmath\omega)| = \delta, \ \forall\omega, \tag{38}$$

which is equivalent to

$$L(\jmath\omega) + \overline{L(\jmath\omega)} + |L(\jmath\omega)|^2 = \delta^2 - 1\,, \tag{39}$$

where the overbar denotes complex conjugate, and thus to

$$\frac{k_p(\jmath\omega)^\lambda + k_i + k_d(\jmath\omega)^{\lambda+\mu}}{(\jmath\omega)^\lambda}\,\frac{n_r(\omega) + \jmath\, n_i(\omega)}{d_r(\omega) + \jmath\, d_i(\omega)}\, e^{-\jmath\omega T} +$$
$$\frac{k_p\overline{(\jmath\omega)^\lambda} + k_i + k_d\overline{(\jmath\omega)^{\lambda+\mu}}}{\overline{(\jmath\omega)^\lambda}}\,\frac{n_r(\omega) - \jmath\, n_i(\omega)}{d_r(\omega) - \jmath\, d_i(\omega)}\, e^{\jmath\omega T} + |L(\jmath\omega)|^2 = \delta^2 - 1. \tag{40}$$

Recalling (13) and (14), eqn. (40) can be written as

$$\overline{(\jmath\omega)^\lambda}\Big[k_p(\jmath\omega)^\lambda + k_i + k_d(\jmath\omega)^{\lambda+\mu}\Big]\Big[A(\omega) + \jmath\, B(\omega)\Big]\, e^{-\jmath\omega T} +$$
$$(\jmath\omega)^\lambda\Big[k_p\overline{(\jmath\omega)^\lambda} + k_i + k_d\overline{(\jmath\omega)^{\lambda+\mu}}\Big]\Big[A(\omega) - \jmath\, B(\omega)\Big]\, e^{\jmath\omega T} +$$
$$\omega^{2\lambda}\,\frac{d_r^2(\omega) + d_i^2(\omega)}{n_r^2(\omega) + n_i^2(\omega)}|L(\jmath\omega)|^2 = \omega^{2\lambda}\,\frac{d_r^2(\omega) + d_i^2(\omega)}{n_r^2(\omega) + n_i^2(\omega)}(\delta^2 - 1)\,, \tag{41}$$

which, after some trivial manipulations, leads to

$$\left(k_p\omega^\lambda\right)^2 + k_i^2 + \left(k_d\omega^{\lambda+\mu}\right)^2 + 2k_i\left(k_p\omega^\lambda\right)\cos\lambda\frac{\pi}{2} + 2k_i\left(k_d\omega^{\lambda+\mu}\right)\cos(\lambda+\mu)\frac{\pi}{2} + 2\left(k_p\omega^\lambda\right)\left(k_d\omega^{\lambda+\mu}\right)\cos\mu\frac{\pi}{2}$$
$$+2k_p\omega^{2\lambda}\Big[A(\omega)\cos\omega T + B(\omega)\sin\omega T\Big] + 2k_i\omega^\lambda\Big[A(\omega)\cos(\omega T + \lambda\frac{\pi}{2}) + B(\omega)\sin(\omega T + \lambda\frac{\pi}{2})\Big]$$
$$+2k_d\omega^{2\lambda+\mu}\Big[A(\omega)\cos(\omega T - \mu\frac{\pi}{2}) + B(\omega)\sin(\omega T - \mu\frac{\pi}{2})\Big] = \omega^{2\lambda}\,\frac{d_r^2(\omega) + d_i^2(\omega)}{n_r^2(\omega) + n_i^2(\omega)}(\delta^2 - 1)\,. \tag{42}$$

Again, this is the equation of an ellipsoid. To find the center of the ellipses on the cross sections of this quadric, a procedure similar to that leading from (32) to (33) can be adopted.

VI. EXAMPLES

[18], [23] [24], [25], [26], [27]

A. *Example 1*

Assume that the plant transfer function is:

$$G(s) = \frac{1}{5s}e^{-s}. \tag{43}$$

Since (43) has already a pole in the origin, it is reasonable to adopt a PD^μ controller. This simple, yet meaningful, example has been considered in [18] and then in [23], where the parameters k_p and k_d of a PD^μ controller have been chosen so as to minimize the Integral of the Absolute Error (IAE).

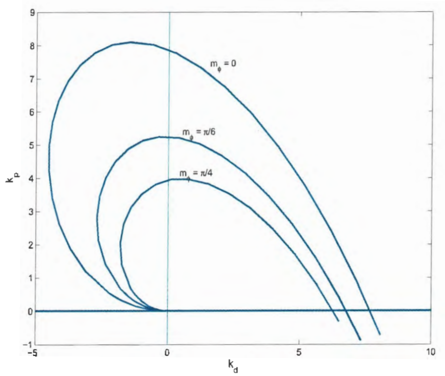

Fig. 1. Locus described by (19) and (20) for $\mu = 0.5$ and $m_\varphi = 0$, $m_\varphi = \pi/6$, $m_\varphi = \pi/4$ when the plant transfer function is given by (43). The stability region lies between the curve for $m_\varphi = 0$ and the k_d–axis ($k_p = 0$).

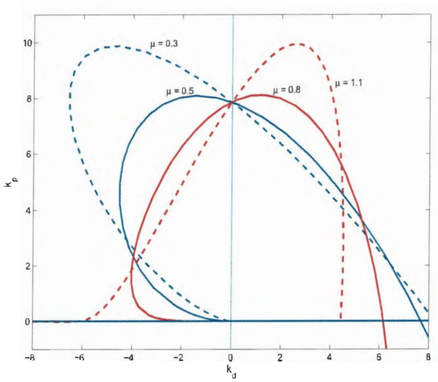

Fig. 2. Stability regions for different values of the exponent μ of the derivative term (regions below the curves corresponding to each value of μ and above the horizontal axis).

VII. CONCLUSIONS

The loci of constant crossover frequency and stability margins in the controller parameter space have been determined as an aid in the design of non–integer standard controllers for non–integer time–delay plants. Particular attention has been given to the modulus margin that accounts well for system robustness. The suggested procedures are simpler and more intuitive than alternative techniques and can easily be implemented using the Matlab® program illustrated in the Appendix. A couple of examples taken from the literature have been worked out to show how the aforementioned loci can be exploited to find the most satisfactory values of the controller parameters.

REFERENCES

[1] Y.K. Chen, I. Petráš, and D. Xue, "Fractional order control - A tutorial", in Proc. American Control Conf., St. Louis, MI, USA, pp. 1397–1411, June 10–12, 2009 (Paper WeC02.1).

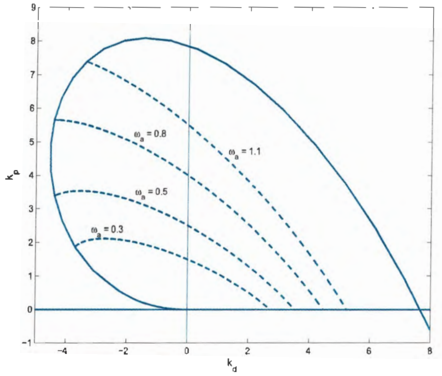

Fig. 3. Loci of constant gain crossover frequency ω_a inside the stability region for $\mu = 0.5$.

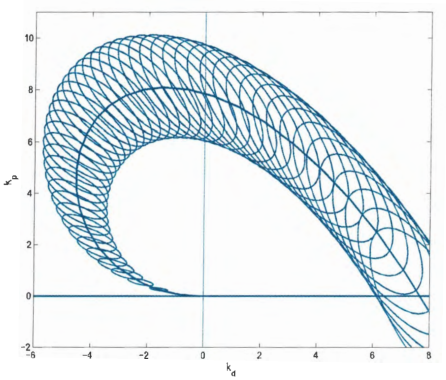

Fig. 4. Ellipses corresponding to $\delta = 0.2$ with $\mu = 0.5$ and $\omega \in (0.001, 2.4)$ rad/s.

[2] I. Petráš, "Stability of fractional–order systems with rational orders: A survey", Fract. Calc. Appl. Anal., vol. 12, no. 3, pp. 269–298, March 2009.

[3] I. Podlubny, "Fractional–order systems and $PI^{\lambda}D^{\mu}$ controllers", IEEE Trans. Automat. Contr., vol. 44, no. 1, pp. 208–213, Jan. 1999.

[4] Y. Luo and Y.Q. Chen, "Fractional order [proportional derivative] controller for a class of fractional order systems", Automatica, vol. 45, no. 10, pp. 2446–2450, Oct. 2009.

[5] H. Chao, Y. Luo, and Y.Q. Chen, "Fractional order flight control of a small fixed–wing UAV: Controller design and simulation study", in Proc. ASME 2009 Int. Design Engineering Tech. Conf. and Compiters and Information in Engineering Conf., San Diego, CA, USA, pp. , August 30 – September 2, 2009.

[6] R. Singhal, S. Padhee, and G. Kaur, "Design of fractional order PID controllers for speed control of DC motor", Int. J. Scientific Research Publ., vol. 2, no. 6, pp. 1–8, June 2012.

[7] T. Emami and J.M. Watkins, "Complementary sensitivity design of PID controllers for arbitrary–order transfer functions with time delay", in Proc. ASME 2008 Dynamic Systems and Control Conf., Ann Arbor, MI, USA, pp. 747–753, Oct. 20–22, 2008.

[8] T. Emami and J.M. Watkins, "A unified approach for sensitivity design of PID controllers in the frequency domain", WSEAS Trans. Systems and Control, vol. 4, no. 5, pp. 221–231, May 2009.

[9] S.H. HosseinNia, I. Tejado, and B.M. Vinagre, "Robust fractional order PI controller for switching systems", in Proc. 5th Symp. Fractional Differentiation and its Applications, Hohai, University of Nanjing, China, May 14–17, 2012 (arXiv:1205.5910).

[10] De–Jin Wang and Xue–Li Gao, "H_{∞} design with fractional–order PD^{μ} controllers", Automatica, vol. 48, no. 5, pp. 974–977, May

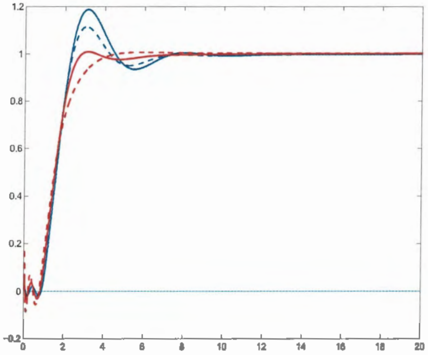

Fig. 5. Step responses of the feedback control system for $k_p = 3.0$ and $k_d = 1$ with $\mu = 0.3$ (solid blue), $\mu = 0.5$ (dashed blue), $\mu = 0.8$ (solid red) and $\mu = 1.1$ (dashed red).

2012.

[11] D. Garcia, A. Karimi, and R. Longchamp, "Robust PID controller tuning with specification on modulus margin", in Proc. American Control Conf., Boston, Ma, vol. 4, pp. 3297–3302, June 30 – July 2, 2004.

[12] W. Krajewski, A. Lepschy, and U. Viaro, "Designing PI controllers for robust stability and performance", IEEE Trans. Control Syst. Technol., vol. 12, no. 6, pp. 973–983, Nov. 2004.

[13] W. Krajewski, A. Lepschy, S. Miani, and U. Viaro, "Frequency–domain approach to robust PI control", J. Franklin Inst., vol. 342, no. 6, pp. 674–687, Sept. 2005.

[14] T. Emami and J.M. Watkins, "Robust performance design of PID controllers with inverse multiplicative uncertainty", in Proc. American Control Conf., San Francisco, CA, USA, pp. 5001–5006, June 29 – July 01, 2011.

[15] W. Krajewski and U. Viaro, "On robust PID control for time–delay plants", in Proc.17th IEEE Int. Conf. Methods and Models in Automation and Robotics, Miedzyzdroje, Poland, Aug. 27–30, 2012 (paper ID 1005 - C2L-A).

[16] S. E. Hamamci, "An algorithm for stabilization of fractional–order time delay systems using fractional–order PID controllers", IEEE Trans. Automat. Contr., vol. 52, no. 10, pp. 1964–1969, Oct. 2007.

[17] A. Ruszewski, "Stability regions of closed loop system with time delay inertial plant of fractional order and fractional order PI controller", Bull. Polish Acad. Sciences, Tech. Sciences, vol 56, no. 4, pp. 329–332, Dec. 2008.

[18] S. E. Hamamci and M. Koksal, "Calculation of all stabilizing fractional–order PD controllers for integrating time delay systems", Computers and Mathematics with Applications, vol. 59, no. 5, pp. 1621–1629, Mar. 2010.

[19] M. A. Rahimian and M. S. Tavazoei, "Stabilizing fractional–order PI and PD controllers: An integer–order implemented system approach", Proc. of the Institution of Mechanical Engineers, Part I: Journal of Systems and Control Engineering, vol. 224, no. 8, pp. 893–903, Dec. 2010.

[20] F. Merrikh–Bayat and M. Karimi–Ghartemani, "Controller design for minimum–phase fractional systems of commensurate order based on shaping the sensitivity function", in Proc. 17th IFAC World Congress, Seoul, Korea, pp. 15279–15284, July 6–11, 2008.

[21] J. Doyle, B. Francis, and A. Tannenbaum, *Feedback Control Theory*, Macmillan Publishing Co., New York, 1992.

[22] Y.K. Lee and J.M. Watkins, "Determination of all stabilizing fractional–order PID controllers", in Proc. American Control Conference, San Francisco, CA, USA, pp. 5007–5012, June 29–July 01, 2011.

[23] T.V. Moghaddam, N. Bigdeli, and K. Afshar, "A new hybrid method for tuning fractional order proportional–integral–derivative (PID) controllers in frequency domain", Int. J. Physical Sciences, vol. 6, no. 34, pp. 7699–7708, 16 Dec. 2011.

[24] R. Caponetto, L. Fortuna, and D. Porto, "A new tuning strategy for non-integer order PID controller", in Proc. 1st IFAC Workshop Fractional Differentiation and its Applications, Bordeaux, France, pp. 168–173, July 19–21, 2004.

[25] D. Valério and J. Sà da Costa, "Tuning of fractional PID controllers with Ziegler..Nichols–type rules", Signal Processing, vol. 86, no. 10, pp. 2771–2784, Oct. 2006.

[26] C.A. Monje, B.M. Vinagre, V. Feliu, and Y.Q. Chen, "Tuning and auto–tuning of fractional order controllers for industry applications", Control Engineering Practice, vol. 16, no. 7, pp. 798–812, July 2008.

[27] S. Das, S. Saha, S. Das, and A. Gupta, "On the selection of tuning methodology of FOPID controllers for the control of higher order processes", ISA Transactions, vol. 50, no. 3, pp. 376–388, July 2011.