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M. Libura

Instytut Badań Systemowych
Polska Akademia Nauk

Systems Research Institute
Polish Academy of Sciences

POLSKA AKADEMIA NAUK

Instytut Badań Systemowych

ul. Newelska 6

01-447 Warszawa

tel.: (+48) (22) 3810100

fax: (+48) (22) 3810105

Kierownik Pracowni zgłaszający pracę:
Prof. dr hab. inż. Krzysztof C. Kiwiel

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Marek Libura

Systems Research Institute
Polish Academy of Sciences
Newelska 6, 01-447 Warszawa, Poland
E-mail: Marek.Libura@ibspan.waw.pl

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Abstract

We consider so-called *generic combinatorial optimization problem*, where the set of feasible solutions is some family of nonempty subsets of a finite ground set with specified positive initial weights of elements, and the objective function represents the total weight of elements of the feasible solution. We assume that the set of feasible solutions is fixed, but the weights of elements may be perturbed or are given with errors. All possible realizations of weights form the set of *scenarios*. A feasible solution, which for a given set of scenarios guarantees the minimum value of the worst-case relative regret among all the feasible solutions, is called a *robust solution*.

In this paper we deal with so-called *robustness analysis* for the generic combinatorial optimization problem. Its main goal consists in finding subsets of scenarios for which an initially optimal solution of the problem remains robust. Thus, the robustness analysis may be considered as a natural extension of the standard *sensitivity analysis* in combinatorial optimization. Main results of the paper concern the *robustness region*, *robustness radius* and *robustness tolerances*, which are introduced as direct analogues of the stability region, stability radius and stability tolerances considered in the sensitivity analysis.

Keywords: combinatorial optimization, sensitivity analysis, robustness analysis, robustness region, robustness radius, robustness tolerances.

1 Introduction

We consider a combinatorial optimization problem in the following generic form:

$$v(\mathcal{F}, c) = \min\{w(F, c) : \ F \in \mathcal{F}\}, \tag{1}$$

where the set of feasible solutions $\mathcal{F}$ is a family of nonempty subsets of a given ground set $E = \{e_1, \ldots, e_n\}$ and $c = (c(e_1), \ldots, c(e_n))^{\mathsf{T}} \in \mathbb{R}^n$ denotes the vector of weights of the elements of E. For $c \in \mathbb{R}^n$ and $F \in \mathcal{F}$, the objective function in (1) represents the total weight of this solution, i.e.,

$$w(F, c) = \sum_{e \in F} c(e).$$

Numerous discrete optimization problems, like e.g. the traveling salesman problem, the minimum spanning tree problem, the shortest path problem, the linear 0–1 programming problem, can be stated in this general form.

We will assume that the set of feasible solutions $\mathcal{F}$ in problem (1) is fixed but the vector of weights can change or it is given with errors. Let $\mathcal{C} \subseteq \mathbb{R}^n$ be the set of all possible realizations of the vector c, called *scenarios*. Consider an initial scenario $c^o \in \mathcal{C}$ and let $\Omega(c^o) = \arg\min\{w(F, c^o) : \ F \in \mathcal{F}\}$ denote the set of optimal solutions in (1) for $c = c^o$.

Given an optimal solution $F^o \in \Omega(c^o)$ an important question concerns the stability of this solution on the set of possible scenarios $\mathcal{C}$. This question belongs to so-called *sensitivity (stability) analysis*, which is regarded an essential step of any optimization procedure (see e.g. Greenberg [5], Libura [9], Sotskov et al. [18]). The main goal of the sensitivity analysis consists in finding a subset of scenarios, for which the solution F^o remains *optimal*.

In this paper we consider a natural extension of the standard sensitivity analysis, which we will call the *robustness analysis* of initially optimal solutions. Namely, as main goal of this analysis, we will consider a problem of determining a subset of scenarios for which the solution F^o remains *robust*.

There are various concepts of the robustness of solutions in optimization and there are many possible robustness measures as well (see e.g. Averbakh [1], Ben-Tal, Niemirowski [2], Bertsimas, Sim [3], Kouvelis, Yu [7], Mulvey et al. [15], Roy [16]). In this paper we will use as a robustness measure the maximum relative error (worst case relative regret) of the solution considered, over the set of all scenarios.

In standard sensitivity analysis one seeks for the maximal under inclusion subset $S(F^o) \subseteq \mathbb{R}^n$ of the weights vectors in problem (1) for which the solution F^o remains optimal. Such a set is called the *optimality* (or – *stability*) *region* of the solution F^o. It is obvious that an optimal solution $F^o \in \Omega(c^o)$

is robust for arbitrary scenario $c \in S(F^o)$. But this solution may remain robust for significantly larger set of scenarios. This motivates studying an analogue of the stability region which we will call a *robustness region* of the feasible solution F and denote $R(F)$. Formally, $R(F)$ denotes the maximal subset of scenarios in $\mathbb{R}^n$ for which F is a robust solution.

Moreover, in case of multiple optimal solutions it may happen, that the solutions belonging to the set $\Omega(c^o)$ are quite different from the robustness point of view. The following simple example illustrates this situation.

Consider an undirected graph $G = (V, E)$, where $V = \{1, 2, 3, 4, 5\}$ and $E = \{e_1, \ldots, e_7\} = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,4\}, \{3,4\}, \{3,5\}, \{4,5\}\}$. Let $\mathcal{F}$ be a family of subsets of E corresponding to all spanning trees in G, and let $c^o = (2, 2, 2, 2, 1, 2, 2)^\intercal$ be a vector of the initial weights of edges in G. Then the combinatorial optimization problem (1) for $c = c^o$ is the minimum spanning tree problem in the weighted graph G.

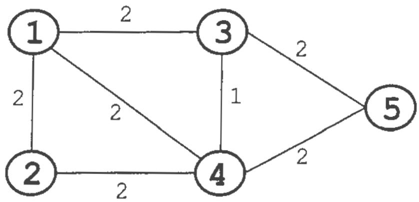

Figure 1: Graph G with indicated weights of edges.

The graph G with indicated weights of its edges is shown in Fig. 1. In Fig. 2 all of the spanning trees in G with corresponding weights for $c = c^o$ are presented. It is easy to check that the set of optimal solutions contains ten spanning trees: $\Omega(c^o) = \{T_3, T_4, T_8, T_9, T_{11}, T_{12}, T_{16}, T_{17}, T_{19}, T_{20}\}$. All of them are, obviously, robust for a set of scenarios $\mathcal{C} = \{c^o\}$. But only two optimal solutions, namely T_{11} and T_{12}, appear robust for an neighborhood of the initial vector c^o. For all other optimal solutions we can construct arbitrarily small nonzero perturbations of weights which destroy their robustness. Such a solutions may be regarded unacceptable from the robustness point of view. It is therefore important to ask for a method of selecting the optimal solutions which preserve its robustness in a neighborhood of an initial vector of weights. In the following will call such solutions *robust optimal* solutions. Next section describes a characterization of optimal robust solutions obtained in Libura [12].

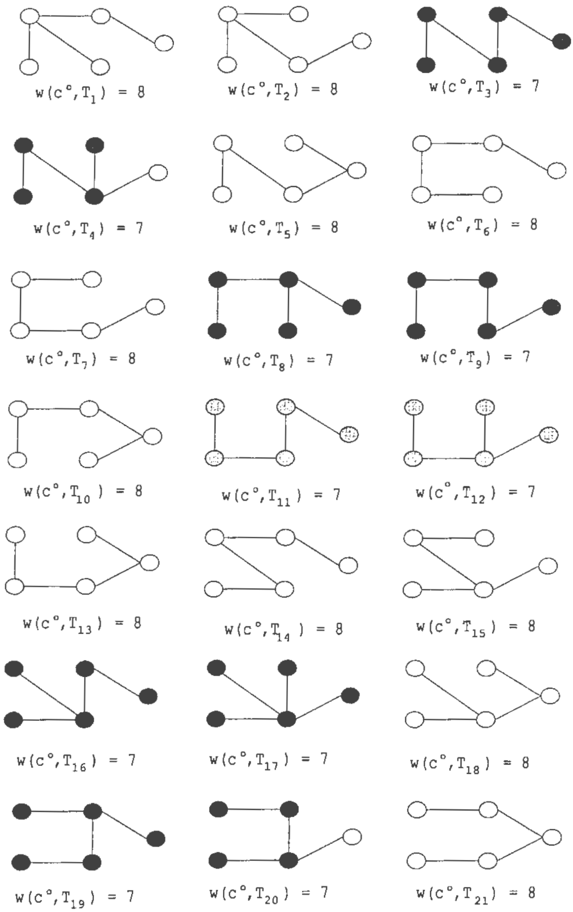

Figure 2: All the feasible solutions of the problem from Example 1.

2 Robust optimal solutions

We recall that an optimal solution $F^o \in \Omega(c^o)$ is called a *robust optimal solution* in $c = c^o$ if and only if it remains a robust solution in some neighborhood of c^o. Let us denote a subset of robust optimal solutions by $\Omega_r(c^o)$. The following theorem (Libura [12]) characterizes a subset $\Omega_r(c^o)$.

Theorem 1 *Let*

$$b = \max_{X\in\Omega(c^o)} \min_{Y\in\Omega(c^o)} w(X \cap Y, c^o). \tag{2}$$

Then

$$\Omega_r(c^o) = \left\{ F \in \Omega(c^o) : \min_{F'\in\Omega} w(F \cap F', c^o) = b \right\}. \tag{3}$$

The main drawback of the above characterization is that it requires using the whole set of optimal solutions of the considered optimization problem. It is an open question, whether this can be avoided.

3 Optimality and robustness regions

In the following will assume that for any $F \in \mathcal{F}$ and $c \in \mathcal{C}$ the inequality $w(F, c) > 0$ holds.

Consider a feasible solution $F \in \mathcal{F}$ and an initial scenario $c^o \in \mathcal{C}$. The quality of the solution F for the scenario c^o can be measured by its relative error $r(F, c^o)$, where

$$r(F, c^o) = \max_{F'\in\mathcal{F}} \frac{w(F, c^o) - w(F', c^o)}{w(F', c^o)} = \frac{w(F, c^o) - v(\mathcal{F}, c^o)}{v(\mathcal{F}, c^o)}. \tag{4}$$

A feasible solution $F^o \in \mathcal{F}$ is called an optimal solution for the scenario c^o if and only if $r(F^o, c^o) \leq r(F, c^o)$ for any $F \in \mathcal{F}$. Let $\Omega(c^o)$ denote the set of optimal solutions in problem (1) for the scenario c^o. It is obvious that for arbitrary $F \in \Omega(c^o)$ we have $r(F, c^o) = 0$.

Consider now a particular optimal solution $F^o \in \Omega(c^o)$. The main object studied in the sensitivity analysis for combinatorial optimization problems is so-called optimality region $S(F^o, \mathcal{C})$ of the solution F^o, defined as the maximal under inclusion subset of scenarios, for which this solution remains optimal, i.e.,

$$S(F^o, \mathcal{C}) = \{c \in \mathcal{C} : r(F^o, c) = 0\}.$$

Denote $S(F^o) = S(F^o, \mathbb{R}^n)$. We have therefore $S(F^o, \mathcal{C}) = S(F^o) \cap \mathcal{C}$. It is well known that the optimality region $S(F^o)$ is a convex polyhedral cone in $\mathbb{R}^n$ (see e.g. Greenberg [5], Libura [9]). This follows directly from the theory of linear programming. Namely, let $\xi(F) \in \mathbb{B}^n$, where $\mathbb{B} = \{0, 1\}$, denote the characteristic vector of the subset $F \subseteq E$. The generic combinatorial problem (1) is equivalent (see e.g. Schrijver [17]) to the following linear program:

$$\min\{c^{\mathsf{T}}x : \ x \in \text{conv.hull } \Phi(\mathcal{F})\},$$

where $\Phi(\mathcal{F})$ is a polyhedral convex set. This means that $\Phi(\mathcal{F})$ can be, at least in principle, described by a system of linear inequalities

$$\Phi(\mathcal{F}) = \left\{x \in \mathbb{B}^n : \ h_i^{\mathsf{T}}(x) \leq \overline{h}_i, \quad i \in I\right\}. \tag{5}$$

Let $I^o \subseteq I$ be a subset of inequalities binding in $x^o = \xi(F^o)$, i.e., $h_i^{\mathsf{T}}(x^o) = \overline{h}_i$ for $i \in I^o$. Then

$$S(F^o) = -\text{cone}\,\{h_i, \ i \in I^o\}. \tag{6}$$

Although polyhedral description (5) of the set $\Phi(\mathcal{F})$ may contain very large number of faces, it can be exploited in various approximations of the optimality region $S(F^o)$ (see e.g. Libura et al. [14]) and appears useful in sensitivity analysis.

In the following we will define an analogue of the optimality region $S(F^o)$ in the robustness analysis framework.

Let for $F \in \mathcal{F}$ and for a given set of scenarios $\mathcal{C} \subseteq \mathbb{R}^n$,

$$Z(F, \mathcal{C}) = \max_{c \in \mathcal{C}} r(F, c).$$

We will call $Z(F, \mathcal{C})$ the *worst-case relative regret* of the solution F over the set of scenarios $\mathcal{C}$.

A feasible solution $F^* \in \mathcal{F}$ will be called a *robust solution* for the set of scenarios $\mathcal{C} \subseteq \mathbb{R}^n$ if and only if the following inequalities hold:

$$Z(F^*, \mathcal{C}) \leq Z(F, \mathcal{C}) \qquad \text{for any} \quad F \in \mathcal{F}. \tag{7}$$

Thus, a feasible solution is robust if it guarantees the minimum value of the worst-case relative regret on the set $\mathcal{C}$ among all the feasible solutions.

Consider an optimal solution $F^o \in \Omega(c^o)$. It is obvious that F^o is a robust solution for scenario c^o, and that it is robust for arbitrary scenario $c \in S(F^o, \mathcal{C})$ as well. But it may happen that F^o remains robust also for other scenarios. Actually, we will be interested in the maximal under inclusion subset of scenarios, for which the solution F^o is a robust solution: such a subset will be denoted $R(F, \mathcal{C})$ and called the *robustness region* of the initially optimal solution F^o. Formally,

$$R(F^o, \mathcal{C}) = \{c \in \mathcal{C} : \ Z(F^o, R(F^o, \mathcal{C})) \leq Z(F, R(F^o, \mathcal{C})) \ \text{ for any } \ F \in \mathcal{F}\}.$$

In case $\mathcal{C} = \mathbb{R}^n$ we will use simplified notation $R(F^o) = R(F^o, \mathbb{R}^n)$. Observe anyway that this time - in contrast to stability region - we can not express $R(F^o, \mathcal{C})$ as an intersection of the sets $R(F^o)$ and $\mathcal{C}$.

The above definition of the robustness region leads to significant difficulties with calculating this set for particular combinatorial optimization problems. It appears that there is no direct relation between $R(F^o)$ and polyhedral description of the convex hull of characteristic vectors of feasible solutions as in case of the set $S(F^o)$. Therefore, it is reasonable to consider various subsets of the robustness region, which may appear easier to analyze and - simultaneously - give some insight into robustness properties of the solutions considered. The main role in such analysis is played by appropriate choice of particular sets of scenarios.

4 Scenarios

The set of scenarios $\mathcal{C}$ plays a crucial role in describing an uncertainty concerning the data of the optimization problem. In this paper we will use the same sets of scenarios in sensitivity analysis and in robustness analysis contexts, although the interpretations in both cases will be actually different.

In sensitivity analysis the set $\mathcal{C}$ represents all of the possible data changes we are interested in. In robustness analysis this set describes all possible perturbations of the data, which we want to be hedged against. Frequently, a choice of the set $\mathcal{C}$ will be determined by various simplifying assumptions wee will make in case of approximate analysis. Moreover, appropriate choice of the set of scenarios will lead to definitions of such objects as tolerances of weights, optimality radius, accuracy radius, robustness radius etc.

In the following we discuss several particular choices of the set of scenarios, and we introduce corresponding definitions of main objects studied in sensitivity and robustness analysis. We start with the sensitivity analysis and then we describe analogous definitions in the robustness analysis context.

4.1 Basic scenarios

In sensitivity analysis the set $\mathcal{C} = \mathbb{R}^n$ may be regarded as a basic set of scenarios and it is actually a starting point for any further analysis. The main object studied for this particular set of scenarios is the optimality region $S(F)$ of a feasible solution $F \in \mathcal{F}$. Nevertheless, sometimes it is necessary to avoid negative weights of elements which may have no reasonable interpretation. In such a case we consider a restricted set of scenarios $\mathcal{C}_+ = \{c \in \mathbb{R}^n : \ c \geq 0\}$.

In fact, in the sensitivity analysis context a choice of the set of scenarios corresponds mainly to various simplifications. A standard approach here consists in an assumption that only weights of elements belonging to some given subset $Q \subseteq E$ may be perturbed while all remaining weights are equal to their initial values given by the vector $c^o \in \mathbb{R}^n$. This leads to the following set of scenarios, whic we will consider a basic set of scenarios:

$$\mathcal{C}(Q, c^o) = \{c \in \mathbb{R}^n : \ c(e) = c^o(e) \ \text{ for } \ e \notin Q\}.$$

The most frequently studied special case corresponds to an assumption that only a single weight of particular element $e \in E$ may be perturbed, i.e., $Q = \{e\}$. This leads to so-called *tolerances* of weight, which are considered in numerous papers (see e.g. Chakravarti, Wagelmans [4], Libura [8], van Hoesel, Wagelmans [21], Sotskov et al. [18], Tarjan [19], Turkensteen et al. [20], Wendell [22]).

Let $F^o \in \Omega(c^o)$. From the convexity of the set $S(F^o)$ it follows directly that

$$\begin{aligned} S(F^o, \mathcal{C}(\{e\}, c^o) = \{c \in \mathbb{R}^n \ \ : \ \ & c(e_i) = c^o(e_i) \text{ for } \ e_i \neq e, \\ & c^o - t^-(e) \leq c(e) \leq c^o + t^+(e)\}, \end{aligned}$$

where $t^+(e), t^-(e) \in \mathbb{R} \cup \{\infty\}$ denote, respectively, so-called *upper* and *lower tolerance* of the weight $c(e)$.

Let $\mathcal{F}^e = \{F \in \mathcal{F} : \ e \in F\}$ and $\mathcal{F}_e = \{F \in \mathcal{F} : \ e \notin F\}$. It is well known (see e.g. Libura [8, 9], Sotskov et al. [18]), that the following facts hold:

Proposition 1 *If* $e \in X^o$, *then* $t^-(e) = \infty, \ \ t^+(e) = v(\mathcal{F}_e, c^o) - v(\mathcal{F}, c^o)$. *If* $e \notin X^o$, *then* $t^+(e) = \infty, \ \ t^-(e) = v(\mathcal{F}^e, c^o) - v(\mathcal{F}, c^o)$.

According to standard conventions, we take $v(\mathcal{F}_e, c^o) = \infty$ or $v(\mathcal{F}^e, c^o) = \infty$ if $\mathcal{F}_e = \emptyset$ or $\mathcal{F}^e = \emptyset$, respectively. Observe that given an algorithm for solving problem (1) for arbitrary $c \in \mathbb{R}^n$ and $\mathcal{F} \subseteq 2^E$, we may use them also to calculate values $v(\mathcal{F}_e, c^o)$ and $v(\mathcal{F}^e, c^o)$. From Proposition 1 it follows therefore that if the optimization problem (1) is polynomially solvable, then also the tolerances $t^+(e)$, $t^-(e)$ for $e \in E$, can be computed in polynomial time. Moreover, the opposite implication also holds under some mild assumptions (see Chakravarti, Wagelmans [4], van Hoesel, Wagelmans [21]).

In Libura [13] similar results are obtained in the robustness context. We will present them after describing two important families of scenarios, which form subsets of the basic set of scenarios $\mathcal{C}(Q, c^o)$.

4.2 Families of scenarios based on $\mathcal{C}(Q, c^o)$

In the basic set of scenarios $\mathcal{C}(Q, c^o)$ we allow arbitrary perturbations of the weights for all elements belonging to the subset $Q \subseteq E$. It appears interesting to consider some restrictions of these changes and – simultaneously – to allow additional simple parametrizations of the perturbations. This will lead to two main families of scenarios, which we will denote denote $T_\delta(Q, c^o)$, $K_\varrho(Q, c^o)$, and define for scalar parameters $\delta \in [0, 1)$, and $\varrho \in [0, \varrho(Q, c^o))$, where $\varrho(Q, c^o) = \min\{c^o(e) : e \in Q\}$, respectively:

$$
\begin{aligned}
T_\delta(Q, c^o) &= \{c \in \mathcal{C}(Q, c^o) : |c(e) - c^o(e)| \leq \delta \cdot c^o(e) \quad \text{for } e \in Q\} &(8)\\
K_\varrho(Q, c^o) &= \{c \in \mathcal{C}(Q, c^o) : |c(e) - c^o(e)| \leq \varrho \quad \text{for } e \in Q\}. &(9)
\end{aligned}
$$

Thus, in the set of scenarios $K_\varrho(Q, c^o)$ for a given scalar parameter ϱ we allow additive perturbations of any weight of element from the subset Q, which do not exceed ϱ. In case of the set of scenarios $T_\delta(Q, c^o)$ we are interested in percentage perturbations of these weights, controlled by the parameter δ. Both introduced families of scenarios can be used in the sensitivity analysis to define so-called *stability function* and *accuracy function* (see Libura [12]).

For an optimal solution F^o, an arbitrary subset of elements $Q \subseteq E$, and $\delta \in [0, 1)$, the value of the *accuracy function* $a(F^o, X, \delta)$ is defined as the maximum of the relative error of the solution F^o over the set $T_\delta(Q, c^o)$, i.e.,

$$a(F^o, Q, \delta) = \max_{c \in T_\delta(Q, c^o)} r(F^o, c). \tag{10}$$

In a similar way the *stability function* $s(F^o, \varrho)$ is defined. Namely, for $\varrho \in [0, \varrho(Q, c^o))$ and $Q \subseteq E$

$$s(F^o, Q, \delta) = \max_{c \in K_\varrho(Q, c^o)} r(F^o, c). \tag{11}$$

Denote for $S', S'' \subseteq E$, $S' \otimes S'' = (S' \setminus S'') \cup (S'' \setminus S')$. In Libura [11] general formulae for calculating accuracy function and stability function are given:

Theorem 2 *For $F^o \in \Omega(c^o)$, $Q \subseteq E$, and $\delta \in [0, 1)$,*

$$a(F^o, Q, \delta) = \max_{F \in \mathcal{F}} \frac{w(F^o, c^o) - w(F, c^o) + \delta \cdot w((F^o \otimes F) \cap Q), c^o)}{w(F, c^o) - \delta \cdot w(F \cap Q, c^o)}. \quad (12)$$

For $F^o \in \Omega(c^o)$, $Q \subseteq E$, and $\varrho \in [0, \varrho(X, c^o))$,

$$s(F^o, Q, \varrho) = \max_{F \in \mathcal{F}} \frac{w(F^o, c^o) - w(F, c^o) + \varrho \cdot |(F^o \otimes F) \cap Q|}{w(F, c^o) - \varrho \cdot |F \cap Q|}. \quad (13)$$

The accuracy function and the stability function can be now used to define so-called *accuracy radius* and *stability radius* as well as to derive formulae to calculate exact and approximate values of these radii. Analogous radii can be introduced in the framework of the robustness analysis.

Observe that if F^o is an optimal solution of the problem (1) then, obviously, $a(F^o, Q, 0) = 0$. It is of special interest to know the maximum value of δ for which $a(F^o, Q, \delta) = 0$. This value is called the *accuracy radius* of the solution F^o with respect to the set Q and is denoted by $r^a(F^o, Q)$. Formally

$$r^a(F^o, Q) = \sup\{\delta \in [0, 1) : a(F^o, Q, \delta) = 0\}. \quad (14)$$

A practical importance of the accuracy radius consists in the fact, that given the value $r = r^a(F^o, Q)$ we know, that the weight of any element e belonging to the set Q may be perturbed (increased or decreased) arbitrarily by $r \cdot 100\%$ (or less) without destroying the optimality of F^o. Similarly, if we know that the weights of elements in Q are estimated with the accuracy $r \cdot 100\%$, then we can guarantee that the solution F^o, calculated for the estimated vector of weights c^o, is also optimal for the actual vector of weights.

In an analogous way the *stability radius* $r^s(F^o, Q)$ of the solution F^o with respect to the set Q can be defined. Formally,

$$r^s(F^o, Q) \;=\; \sup\{\delta \in [0, \varrho(Q, c^o)) : s(F^o, Q, \delta) = 0\}. \quad (15)$$

Observe that the value of stability radius gives the maximum absolute deviation of any weight of element from the set Q which do not destroy the optimality of the solution F^o.

Let

$$\mathcal{F}_Q = \{F \in \mathcal{F} : w((F^o \otimes F) \cap Q, c^o) \neq 0\}$$

and

$$\mathcal{F}'_Q = \{F \in \mathcal{F} : (F^o \otimes F) \cap Q \neq 0\}.$$

The following theorem (see Libura [11]) gives general formulae for calculating the accuracy radius and the stability radius of the solution F^o with respect to the set Q.

Theorem 3 *For $F^o \in \Omega(c^o)$ and $Q \subseteq E$,*

$$r^a(F^o, Q) = \min\left\{ 1, \min_{F\in\mathcal{F}_Q} \frac{w(F,c^o) - w(F^o,c^o)}{w((F^o \otimes F)\cap Q), c^o)} \right\} \tag{16}$$

and

$$r^s(F^o, Q) = \min\left\{ \varrho(Q,c^o), \min_{F\in\mathcal{F}'_Q} \frac{w(F,c^o) - w(F^o,c^o)}{|(F \otimes F^o)\cap Q|} \right\}. \tag{17}$$

Analogous radii can be introduced in the framework of the robustness analysis. Namely, instead of studying maximum perturbations for which a given initial solution remains optimal, we may seek for the maximum perturbations preserving the robustness of this solution. In particular, in Libura [12] an analogue of the accuracy radius – called the *robustness radius* is considered.

Let $F^o \in \Omega(c^o)$. The robustness radius of F^o is denoted $r^J(F^o, Q)$ and defined as the maximum value of the parameter δ for which the solution F^o is robust under the set of scenarios $T_\delta(Q, c^o)$. Formally,

$$\begin{aligned} r^r(F^o,Q) &= \sup\{\delta \in [0,1) : \\ &\qquad Z(F^o, T_\delta(Q,c^o)) \leq Z(F, T_\delta(Q,c^o)) \quad \text{for any} \quad F \in \mathcal{F}\}. \end{aligned}$$

No formula like (16) is known for calculating the robustness radius. In Libura [12] some evaluations of the robustness radius are given for $Q = E$. The following facts hold:

Theorem 4 *If F^o is a single optimal solution of problem* (1) *for $c = c^o$, then*

$$r^r(F^o, E) \geq \begin{cases} \frac{a}{2-a} & if \quad a < 1, \\ 1 & otherwise, \end{cases} \tag{18}$$

where

$$a = \min_{F\in\mathcal{F}\setminus\Omega(c^o)} \frac{w(F,c^o) - v(\mathcal{F},c^o)}{v(\mathcal{F},c^o)}. \tag{19}$$

Theorem 5 *If $F^o \in \Omega_r(c^o)$ and $a \geq \frac{b}{1-b}$, then*

$$r^r(F^o, E) \geq \begin{cases} \frac{a}{2(1-b)-a} & if \quad a < 1-b, \\ 1 & otherwise. \end{cases} \tag{20}$$

If $F^o \in \Omega_r(c^o)$ and $a < \frac{b}{1-b}$, then

$$r^r(F^o, E) \geq \begin{cases} \min\left\{\frac{a}{2(1-b)-a}, \frac{a}{2b+2ab-a}\right\} & if \quad a < 1-b, \\ \frac{a}{2b+2ab-a} & otherwise. \end{cases} \tag{21}$$

The situation simplifies significantly in case $Q = \{e\}$ for some $e \in E$. Then the robustness radius becomes an analogue of the tolerance of single weight in the sensitivity analysis. Namely, for $e \in E$ and $F^o \in \Omega(c^o)$ we introduce so-called robustness tolerance of the weight $c(e)$, which we denote $t^r(e)$ and formally define in the following way:

$$t^r(e) = \sup\left\{\delta \in [0,1): \ Z\left(F^o, \mathcal{C}(c^o, \{e\}, \delta)\right) \leq Z\left(F, \mathcal{C}(c^o, \{e\}, \delta)\right), \ F \in \mathcal{F}\right\}.$$

Thus, $t^r(e)$ is the maximum value of the parameter δ, such that F^o remains robust for the set of scenarios $\mathcal{C}(c^o, \{e\}, \delta)$. This case we are able to show a result, which is a close analogue of Proposition 1. Namely, in Libura [13] it is proved that the following fact holds:

Theorem 6 *For $F^o \in \Omega(c^o)$,*

$$t^r(e) = \begin{cases} 1 & \text{if} \quad e \in F^o, \\ \min\left\{1, [v(\mathcal{F}^e, c^o)^2 - v(\mathcal{F}, c^o)^2]^{\frac{1}{2}} \cdot c^o(e)^{-1}\right\} & \text{if} \quad e \notin F^o. \end{cases} \tag{22}$$

Observe that this – as in case of the standard sensitivity analysis – leads to polynomial solvability of the robustness tolerance problem provided that the original optimization problem is polynomially solvable itself.

5 Conclusions

This paper deals with the robustness analysis regarded as a natural extension of the standard sensitivity analysis for combinatorial optimization problems. It is shown, that it is reasonably to define analogues of such objects as stability region, stability radius, accuracy radius, tolerances of weights and to study in the framework of the robustness analysis such objects as robustness region, robustness radius and robustness tolerances. All of them have natural interpretations and give some insight in the quality of a given optimal solution from the robustness point of view.

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