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A METHOD FOR THE INTEGER–ORDER APPROXIMATION OF FRACTIONAL–ORDER SYSTEMS

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Abstract

A procedure for approximating fractional–order systems by means of integer–order state–space models is presented. It is based on the rational approximation of fractional–order operators suggested by Oustaloup. First, a matrix differential equation is obtained from the original fractional–order representation. Then, this equation is realized in a state–space form that has a sparse block–companion structure. The dimension of the resulting integer–order model can be reduced using an efficient algorithm for rational L_2 approximation. Two numerical examples are worked out to show the performance of the suggested technique.

I. INTRODUCTION

Many natural and artificial systems can profitably be modelled or controlled by means of fractional–order systems [1], [2], [12]. Indeed, there is already a vast and qualified literature on this subject [7], [3], [15], [14] so that further motivation for their study is superfluous.

One drawback of the use of fractional–order models is the irrational nature of their transfer function and, therefore, the infinite dimensionality of their state space. Therefore, various methods have been developed to obtain from these models other models that are more suitable for simulation and control purposes. Most of them are based on the approximation, possibly over a suitable frequency range, of the fractional order system by means of an integer–order model (see, e.g., [18], [20], [23], [11], [21].

This paper, which expands and updates a previous paper of the same authors [10], adheres to the same philosophy. Precisely, starting from the original state–space form of the original fractional–order system (Section II) and applying the integer–order approximation of a fractional operator suggested in [20], a matrix fraction description of the integer–order approximating model is obtained (Section III). From this model it is easy to derive a block–companion integer–order state–space representation that is particularly suited to simulation (Section IV). However, the dimension of this model increases with its accuracy, which can make the design of a controller difficult. To reduce this dimension without diminishing appreciably the response accuracy, resort can be made to the efficient algorithm for L_2 model reduction suggested in [] (Section V). Two numerical examples taken from the literature confirm the validity of such an approach (Section VI).

II. BASIC NOTIONS

Let us briefly recall the basics of fractional–order system representation. Various definitions of fractional–order operators have been proposed over the years; it suffices to mention those of Grünwald–Letnikov, Riemann–Liouville and Caputo [19]. The last one is probably the most frequently used in engineering applications. Indicating by $D^\lambda z(t) = \frac{d^\lambda z}{dt^\alpha}$, $\lambda \in \mathbb{R}_+$, the Caputo derivative of the time–dependent variable $z(t)$, its Laplace transform is

$$\mathscr{L}\{D^\lambda z(t)\} = s^\lambda \mathscr{L}\{z(t)\} - \sum_{i=0}^{[\lambda]-1} s^{\lambda-i-1} \frac{d^i z}{dt^i}(0)\,, \tag{1}$$

where $[\lambda]$ denotes the integer part of λ.
The standard input–output representation of a time–invariant fractional–order system is:

$$y(t) + \sum_{i=1}^{n} a_i D^{\alpha_i} y(t) = \sum_{i=1}^{p} b_i D^{\beta_i} u(t), \tag{2}$$

where, as usual, $u(t)$ and $y(t)$ denote the input and output functions, and $a_i, b_i \in \mathbb{R}$, $\alpha_i, \beta_i \in \mathbb{R}_+$.
The system transfer function $G(s)$ can immediately be obtained by transforming (2) according to (1) with zero initial conditions:

$$G(s) = \frac{b(s)}{a(s)} = \frac{\sum_{i=1}^{p} b_i s^{\beta_i}}{1 + \sum_{i=1}^{n} a_i s^{\alpha_i}}. \tag{3}$$

Clearly, if all powers in (3) are multiples of the same real number $\rho \in (0,1)$ (which qualifies the system as a commensurate–order one), (3) simplifies to

$$G(s) = \frac{\sum_{i=1}^{p} b_i (s^\rho)^i}{1 + \sum_{i=1}^{n} a_i (s^\rho)^i} \tag{4}$$

which is a rational function of s^ρ.
The standard state–space model of a fractional–order system takes the form:

$$D^{(\gamma)}(x)(t) = Ax(t) + bu(t), \tag{5}$$

$$y(t) = cx(t) + du(t), \tag{6}$$

where vector $x = [x_1, x_2, \cdots, x_\ell]^T$ denotes the state, $\bar{\gamma} = [\gamma_1, \gamma_2, \cdots, \gamma_\ell]^T$ with $0 < \gamma_i < 2$,

$$D^{(\bar{\gamma})}(x) = \left[D^{\gamma_1} x_1, D^{\gamma_2} x_2, \cdots, D^{\gamma_\ell} x_\ell\right]^T$$

and $A \in \mathbb{R}^{\ell\times\ell}$, $b \in \mathbb{R}^{\ell\times 1}$, $c \in \mathbb{R}^{1\times\ell}$, $d \in \mathbb{R}$. A general method for realizing a fractional–order transfer function in a state–space form has been presented in [4].

III. APPROXIMATION OF FRACTIONAL OPERATORS

Among the various approaches to find a rational filter approximating a fractional differentiator [16], [11], the most popular is almost certainly the one due to Oustaloup [17] by which the fractional differentiator operator s^α, $0 \le \alpha \le 1$, is replaced by a rational filter $\mathcal{D}^\alpha(s)$ whose zeros and poles are distributed over a frequency band $[\omega_m, \omega_M]$ centred at

$$\omega_u = \sqrt{\omega_m \omega_M}. \tag{7}$$

Precisely the approximating filter is formed by the cascade of $2N+1$ first–order cells, i.e.,

$$\mathcal{D}^\alpha(s) = K_\alpha \prod_{k=-N}^{N} \frac{1 + \frac{s}{\omega_k'}}{1 + \frac{s}{\omega_k}}, \tag{8}$$

where ω_k' and ω_k are computed recursively according to

$$\omega_0' = \delta^{-\frac{1}{2}}\omega_u\,, \quad \omega_0 = \delta^{\frac{1}{2}}\omega_u\,,$$

$$\frac{\omega_{k+1}'}{\omega_k'} = \frac{\omega_{k+1}}{\omega_k} = \delta\,\eta > 1,$$

$$\frac{\omega_k}{\omega_k'} = \delta > 0\,, \quad \frac{\omega_{k+1}'}{\omega_k} = \eta > 0,$$

$$\omega_{-N}' = \eta^{\frac{1}{2}}\omega_m\,, \quad \omega_N' = \eta^{-\frac{1}{2}}\omega_M\,,$$

with [18]

$$\delta = \left(\frac{\omega_M}{\omega_m}\right)^{\frac{\alpha}{2N+1}}, \quad \eta = \left(\frac{\omega_M}{\omega_m}\right)^{\frac{1-\alpha}{2N+1}}.$$

The gain K_α is chosen so as to ensure that s^α has the same magnitude as (8) at ω_u.The number of filter cells is clearly related to the goodness of the approximation.
Variants of (8) have also been suggested to avoid the so–called border effects and compensate for the null asymptotic behaviour at low and high frequencies [25], [24].

The fractional–order integrator operator $1/s^\alpha$ can be approximated in a way consistent with that adopted for the differentiator operator. Precisely, the approximation of the fractional integrator operator can be taken [20] as

$$\mathcal{I}^\alpha(s) = \frac{K_\alpha}{s}\prod_{k=-N}^{N}\frac{1+\dfrac{s}{\omega_k'}}{1+\dfrac{s}{\omega_k}}, \tag{9}$$

which behaves (almost) like $1/s^\alpha$ in an interval $[\omega_m, \omega_M]$.

Functions $\mathcal{D}^\alpha(s)$ and $\mathcal{I}^\alpha(s)$ allow us to find rational (integer–order) models of practically any fractional system. However, the direct application of these operators often leads to high–dimensional models. Consider, for example, the fractional system put forth in [21] whose transfer function is

$$G(s) = \frac{s^{1.56}+4}{s^{3.46}+10s^{2.69}+20s^{1.56}+4}. \tag{10}$$

By setting $\omega_m = 10^{-3}$, $\omega_M = 10^3$, $N = 10$, and using (8), the order of the integer–order approximating transfer function turns out to be 84.

IV. SYSTEM APPROXIMATION

Equation (5) corresponds to the set of scalar equations:

$$\left.\begin{array}{rcl} D^{\gamma_1}x_1 & = & \sum_{i=1}^{\ell} a_{1i}x_i + b_1 u, \\ D^{\gamma_2}x_2 & = & \sum_{i=1}^{\ell} a_{2i}x_i + b_2 u, \\ & \vdots & \\ D^{\gamma_\ell}x_\ell & = & \sum_{i=1}^{\ell} a_{\ell i}x_i + b_\ell u\,. \end{array}\right\} \tag{11}$$

By transforming (11) with zero initial conditions and approximating the fractional–order integrators $1/s^{\gamma_k}$ according to (9) as

$$\mathcal{I}^{\gamma_k}(s) = \frac{\sum_{j=0}^{m} f_{k,j}s^j}{s\sum_{j=0}^{m} g_{k,j}s^j}, \tag{12}$$

where $m = 2N + 1$, the following set of equations is obtained:

$$\left.\begin{array}{rcl} s\sum_{j=0}^{m} g_{1,j}s^j\bar{X}_1 &=& \sum_{j=0}^{m} f_{1,j}s^j(\sum_{i=1}^{\ell} a_{1i}\bar{X}_i + b_1U), \\ s\sum_{j=0}^{m} g_{2,j}s^j\bar{X}_2 &=& \sum_{j=0}^{m} f_{2,j}s^j(\sum_{i=1}^{\ell} a_{2i}\bar{X}_i + b_2U), \\ &\vdots& \\ s\sum_{j=0}^{m} g_{\ell,j}s^j\bar{X}_\ell &=& \sum_{j=0}^{m} f_{\ell,j}s^j(\sum_{i=1}^{\ell} a_{\ell i}\bar{X}_i + b_\ell U), \end{array}\right\}$$

in which, due to the aforementioned approximation, $\bar{X}_k$ does not coincide with the Laplace transform of x_k.

The time–domain differential equations obtained from this set by inverse Laplace transformation can be expressed in compact form as:

$$\begin{aligned} &A_{m+1}\bar{x}^{(m+1)} + A_m\bar{x}^{(m)} + \cdots + A_1\bar{x}^{(1)} + A_0\bar{x} = \\ &\quad B_m u^{(m)} + \cdots + B_1u^{(1)} + B_0u\,, \end{aligned} \tag{13}$$

where

- $\bar{x} = [\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_\ell] = \mathscr{L}^{-1}\{\bar{X} = [\bar{X}_1, \bar{X}_2, \cdots, \bar{X}_\ell]\}$,
- the integers between round brackets denote the orders of differentiation, and

$$\begin{aligned} A_{m+1} &= \text{diag}\{g_{1,m}, g_{2,m}, \cdots, g_{\ell,m}\}, \\ A_k &= \text{diag}\{g_{1,k-1}, g_{2,k-1}, \cdots, g_{\ell,k-1}\} - \\ &\quad \text{diag}\{f_{1,k}, f_{2,k}, \cdots, f_{\ell,k}\}\,A, \quad k = 1, \ldots, m, \\ A_0 &= -\text{diag}\{f_{1,0}, f_{2,0}, \cdots, f_{\ell,0}\}\,A, \\ B_k &= \text{diag}\{f_{1,k}, f_{2,k}, \cdots, f_{\ell,k}\}\,b, \quad k = 0, \ldots, m. \end{aligned}$$

Since coefficients $g_{1,m}, g_{2,m}, \cdots, g_{\ell,m}$ are different from 0 according to (8) and (9), A_{m+1} can be taken as the identity matrix I. Eqn. (13) is thus equivalent to the left MFD:

$$\begin{aligned} \bar{X} &= (s^{m+1}I + s^mA_m + \cdots + s^1A_1 + A_0)^{-1}(B_ms^m + \\ &\quad B_{m-1}s^{m-1} + \cdots + B_1s^1 + B_0)\,U \end{aligned} \tag{14}$$

from which the following state–space integer–order model approximating (5)–(6) is immediately obtained:

$$\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t), \tag{15}$$

$$\hat{y}(t) = \hat{C}\hat{x}(t) + du(t), \tag{16}$$

where $\hat{x} \in \mathbb{R}^{(2N+2)\ell}$, matrix $\hat{A} \in \mathbb{R}^{(2N+2)\ell\times(2N+2)\ell}$ has the block–companion form

$$\hat{A} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -A_0 \\ I & 0 & \cdots & 0 & -A_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -A_{m-1} \\ 0 & 0 & \cdots & I & -A_m \end{bmatrix}, \tag{17}$$

matrices $\hat{B}$, $\hat{C}$ are given by

$$\hat{B} = \begin{bmatrix} B_0 \\ B_1 \\ \vdots \\ B_{m-1} \\ B_m \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} 0 & 0 & \cdots & 0 & c \end{bmatrix}, \tag{18}$$

with c and d as in (5).

The method proposed in [21] brings about approximating models of the same order. The present method, however, leads in a straightforward way to accurate models (see Section VI) whose sparse structure is particularly suited to simulation and model reduction.

V. MODEL REDUCTION

The simplest way to improve the approximation of a fractional–order system is to make N larger. As a result, the dimension of the integer–order models increases significantly, which leads, e.g., to the design of complex and expensive controllers.

This problem can be avoided as follows. First, a large value of N is selected, thus arriving at a very accurate integer–order model of high dimension. Then, a suitable order–reduction procedure is applied to this model.

An approach of this kind has been suggested in [13], [22] where the high–order model initially obtained has been reduced using either balanced truncation, singular perturbation or Padé techniques. However, as pointed out in Section III, a critical factor in the approximation of fractional–order systems is the frequency range $[\omega_m, \omega_M]$. In fact, the deviation of $\mathcal{D}^\alpha(s)$ from the fractional differentiator s^α becomes smaller as this frequency interval becomes wider [20]. The reduction criterion based on the minimization of the unweighted L_2 norm of the impulse–response error seems to be more appropriate [26] since it involves an infinite frequency band. To this purpose, the efficient iterative–interpolation algorithm for L_2 model reduction [8], [5], [6], [9] is used in Section VI. The implementation of this algorithm also benefit from the particular structure of the approximating model derived according to the procedure of Section IV.

VI. EXAMPLES

Three examples taken from the literature are considered in this section which has essentially three purposes: (i) to test the approximation technique, (ii) to evaluate the effects of the value of N on the approximation accuracy and the dimension of the approximating model , and (iii) to show that the dimension of the integer–order models initially derived can be reduced by means of the aforementioned iterative–interpolation algorithm for L_2 model reduction without deteriorating appreciably the response accuracy.

A. Example 1

Consider first the system put forth in [21] whose state–space equations are:

$$\begin{bmatrix} D^{1.56}x_1(t) \\ D^{1.13}x_2(t) \\ D^{0.77}x_3(t) \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -20 & -10 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t), \tag{19}$$

$$y(t) = \begin{bmatrix} 4 & 1 & 0 \end{bmatrix} x(t). \tag{20}$$

Choosing $N = 5$, $\omega_m = 10^{-3}$ and $\omega_M = 10^3$, the procedure outlined in Section IV leads to a 36–th order model with block–companion structure. This integer–order model has subsequently been reduced to a 4–th order one by means of the aforementioned iterative–interpolation algorithm for L_2 model reduction The step responses of these two models are compared in Fig. 1 with the original step response computed according to the Matlab code described in [3] The responses practically coincide so that the $4th$–order model can be used safely for controller design.

It is interesting to notice (see Fig. 2) that the 12–th order and 18–th order models obtained with the procedure of Section III for $N = 1$ and, respectively, $N = 2$ exhibit a response that is much worse than the response of the L_2-optimal 4–th order model obtained from the intermediate 36–th order approximation.

Fig. 1. Step responses of: (i) the system (19)–(20) (solid line), (ii) the 36–th order model obtained for $N = 5$ and $\omega_m = 10^{-3}$, $\omega_M = 10^3$ (dashed line), and (iii) the 4–th order L_2–optimal reduced model (dotted line).

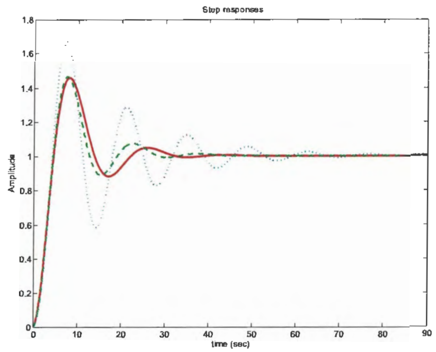

Fig. 2. Step responses of: (i) the system (19)–(20) (solid line), (ii) the 18-th order model obtained for $N = 2$ (dashed line), and (iii) the 12–th order model obtained for $N = 1$ (dotted line).

B. Example 2

Consider now the fractional–order system put forth in [13] whose state–space equations are:

$$\begin{bmatrix} D^{0.4}x_1(t) \\ D^{0.4}x_2(t) \\ D^{0.6}x_3(t) \\ D^{0.6}x_4(t) \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -164 & 20 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 800 & 0 & -800 & -40 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 164 \\ 0 \\ 0 \end{bmatrix} u(t), \tag{21}$$

$$y(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} x(t). \tag{22}$$

Choosing $N = 10$, $\omega_m = 10^{-5}$ and $\omega_M = 10^5$ leads to an 88–th order model whose response reproduces accurately the original response, as shown in Fig. 3. Any attempt to obtain a simpler model with an acceptable response by lowering the value of N leads to models whose responses differ too much from the original one (see Fig. 3 where the effect of the bandwidth $[\omega_m, \omega_M]$ on the response accuracy is also pointed out).

However, the order of the model corresponding to $N = 10$, i.e., 88, can subsequently be reduced using the aforementioned procedure for L_2 model reduction. Fig. 4 shows the excellent response of the 6–th order model obtained in this way.

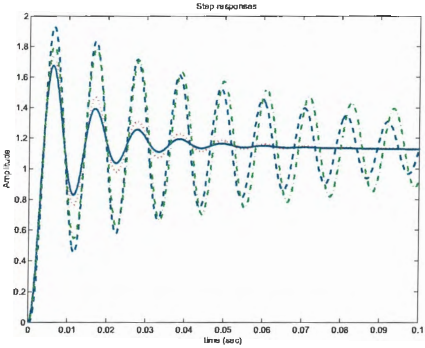

Fig. 3. Step responses of: (i) the system (21)–(22) (solid line), (ii) the 88–th order model obtained for $N = 10$, $\omega_m = 10^{-5}$ and $\omega_M = 10^5$ (dotted line), (iii) the 48–th order model obtained for $N = 5$, $\omega_m = 10^{-5}$ and $\omega_M = 10^5$ (dashdot line), and (iv) the 88–th order model obtained for $N = 10$, $\omega_m = 10^{-3}$ and $\omega_M = 10^3$ (dashed line).

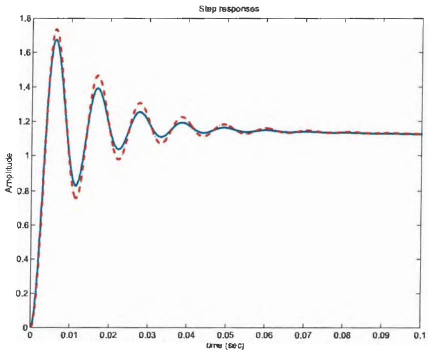

Fig. 4. Step responses of: (i) the system (21)–(22) (solid line), (ii) the 88–th order model obtained for $N = 10$, $\omega_m = 10^{-5}$ and $\omega_M = 10^5$ (dotted line), and (iii) the 6–th order L_2–optimal reduced model (dashed line).

C. Example 3

Let us finally apply the suggested approximation procedure to the state–space model:

$$\begin{bmatrix} D^{0.8}x_1(t) \\ D^{0.8}x_2(t) \\ D^{0.8}x_3(t) \\ D^{0.8}x_4(t) \\ D^{0.8}x_5(t) \\ D^{0.8}x_6(t) \end{bmatrix} =$$

$$\begin{bmatrix} -6 & -6 & -4.4688 & -7.3047 & -6.1719 & -3.4688 \\ 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} x(t)$$

$$+\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T u(t), \tag{23}$$

$$y(t) = \begin{bmatrix} 0.5 & 0.5625 & 0.2422 & 0.2266 & 0.1172 & 0.0313 \end{bmatrix} x(t), \tag{24}$$

corresponding to the transfer function:

$$G(s) = \frac{(s^{0.8}+4)(s^{1.6}+2s^{0.8}+4)(s^{1.6}+3s^{0.8}+1)}{(s^{0.8}+1)(s^{0.8}+3)(s^{1.6}-2s^{0.8}+37)(s^{1.6}+4s^{0.8}+8)}, \tag{25}$$

for which a simplified fractional–order model has been derived in [22]. Choosing $\omega_m = 10^{-5}$, $\omega_M = 10^5$ and $N = 5$ leads to a 72–nd order model whose accuracy is already quite satisfactory (see dotted line in Fig. 5). For $N = 7$ and the same frequency interval $[\omega_m, \omega_M]$, the suggested procedure leads to a 96–th order model whose step response reproduces the original response almost perfectly (see solid line in Fig. 5).

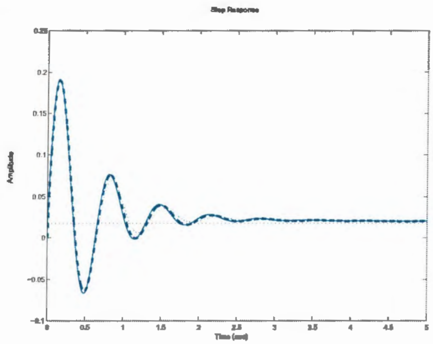

Fig. 5. Step responses of: (i) the system (23)–(24) (dashed line), (ii) the 72–th order model obtained for $N = 5$, $\omega_m = 10^{-5}$ and $\omega_M = 10^5$ (dotted line), and (iii) the 96–th order model for $N = 7$, $\omega_m = 10^{-5}$ and $\omega_M = 10^5$ (solid line).

The iterative–interpolation algorithm for L_2 model reduction has been applied to both these integer–order models. The step responses of the 7–th order models obtained in this way are compared with the original response in Fig. 6. Their accuracy is again remarkable.

VII. CONCLUSIONS

A simple procedure to find integer–order state–space models approximating a given fractional–order system has been presented. The sparse structure of these models lends itself well to simulation and control design. However, their state dimension increases rapidly with their accuracy. Examples have shown that, to reduce the dimension of the integer–order models without deteriorating appreciably the goodness of fit, resort can profitably be made to an efficient algorithm for L_2–optimal model reduction which takes advantage of the privileged structure of the approximating models.

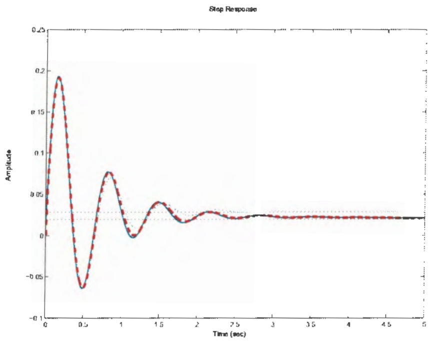

Fig. 6. Step responses of: (i) the original system (23)–(24) (dashed line), (ii) the L_2–optimal 7–th order model obtained from the 96–th order model (solid line), and (iii) the L_2–optimal 7–th order model obtained from the 72–nd order model (dashed line).

REFERENCES

[1] M. Buslowicz, "Stability analysis of linear continuous–time fractional systems of commensurate order", *Journal of Automation, Mobile Robotics & Intelligent Systems*, vol. 3, no. 1, pp. 12–17, 2009.

[2] R. Caponetto, G. Dongola, L. Fortuna, and I. Petras, *Fractional Order Systems: Modeling and Control Applications*, Series oin Nonlinear Science, World Scientific Publishing Co., Singapore, 2010.

[3] Y. Chen, I. Petras, and D. Xue, "Fractional order control – A tutorial", in *Proc. American Control Conf.*, St. Louis, MO, USA, June 10–12, 2009, pp. 1397–1411.

[4] T. Djamah, R. Mansouri, M. Bettayeb, and S. Djennoune, "State space realization of fractional order systems", American Institute of Physics (AIP) Conference Proceedings (2nd Mediterranean Conference on Intelligent Systems and Automation - CISA '09 - Zarzis, Tunisia, March 23-25, 2009), vol. 1107, pp. 37–42, 2009.

[5] A. Ferrante, W. Krajewski, A. Lepschy, and U. Viaro, "Convergent algorithm for L_2 model reduction", *Automatica*, vol. 35, no. 1, pp. 75–79, 1999.

[6] S. Gugercin, A. C. Antoulas, and C. A. Beattie, "A rational Krylov iteration for optimal H_2 model reduction", in *Proc. 17th Int. Symp. Mathematical Theory of Networks and Systems*, Kyoto, Japan, July 24–28, 2006, pp. 1665–1667.

[7] T. Kaczorek, "Fractional positive continuous–time linear systems and their reachibility", *Int. J. Appl. Math. Comput. Sci.*, vol. 18, no 2, pp. 223–228, 2008.

[8] W. Krajewski, A. Lepschy, M. Redivo-Zaglia, and U. Viaro, "A program for solving the L_2 reduced–order model problem with fixed denominator degree", *Numerical Algorithms*, vol. 9, no. 2, pp. 355–377, 1995.

[9] W. Krajewski and U. Viaro "Iterative–interpolation algorithms for L_2 model reduction", *Control and Cybernetics*, vol. 38, no. 2, pp. 543–554, 2009.

[10] W. Krajewski and U. Viaro "On the rational approximation of fractional-order systems", in *Proc. 16th IEEE Int. Conf. Methods and Models in Automation and Robotics*, Miedzyzdroje, Poland, August 22-25, 2011, pp. 132–136.

[11] B. T. Krishna, "Studies on fractional order differentiators and integrators: A survey", *Signal Processing*, vol. 91, no. 3, pp. 386–426, 2011.

[12] R. Magin, M. D. Ortigueira, I. Podlubny, and J. Trujillo, "On the fractional signals and systems", *Signal Processing*, vol. 91, no. 3, pp. 350–371, 2011.

[13] R. Mansouri, M. Bettayeb, and S. Djennoune , "Optimal reduced–order approximation of fractional dynamical systems", in H. Arioui, R. Merzouki, and H. A. Abhassi, Eds., *Intelligent Systems and Automation, 1st Mediterranean Conference*, American Institute of Physics, 2008, pp. 127–132.

[14] R. Matušu, "Application of fractional order calculus to control theory", *Int. J. Math. Models in Applied Sciences*, vol. 5, no. 7, pp. 1162–1169, 2011.

[15] C.A. Monje, Y. Chen, B.M. Vinagre, D. Xue, and V. Feliu, *Fractional–Order Systems and Controls: Fundamentals and Applications*, Series on Advances in Industrial Control, Springer, London, 2010.

[16] P. Ostalczyk, *Zarys rachunku rĂşĂijniczkowo-caĂĆkowego uĂĆamkowych rzĂŹdĂşw. Teoria i zastosowania w automatyce*, Wyd. Politechniki ĂĄĂşdzkiej, 2008 (in Polish).

[17] A. Oustaloup, *La dérivation non entière. Théorie, synthèse et applications*, Hermès Edition, Paris, 1995.

[18] A. Oustaloup, F. Levron, B. Mathieu, and F. M. Nanot, "Frequency–band complex noninteger differentiator: Characterization and synthesis", *IEEE Trans. Circuits and Systems – I: Fundamental Theory and Applications*, vol. 47, no. 1, pp. 25–39, 2000.

[19] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, CA, USA, 1999.

[20] T. Poinot and J.-C. Trigeassou, "A method for modelling and simulation of fractional systems", *Signal Processing*, vol. 83, no. 11, pp. 2319–2333, 2003.

[21] M. Rachid, B. Maamar, and D. Said, "Comparison between two approximation methods of state space fractional systems", *Signal Processing*, vol. 91, no. 3, pp. 461–469, 2011.

[22] M. Tavakoli–Kakhki and M. Haeri, "Model reduction in commensurate fractional–order linear systems", in *Proc. IMechE, Part I: J. of Systems and Control Engineering*, vol. 223, no. 4, pp. 493–505, 2009.

[23] M. S. Tavazoei and M. Haeri, "Rational approximations in the simulation and implementation of fractional–order dynamics: A descriptor system approach", *Automatica*, vol. 46, pp. 94–100, 2010.

[24] M. Thomassin and Rachid Malti, "Multivariable identification of continuous–time fractional systems", in *Proc. ASME IDETC/CIE Conf.*, San Diego, CA, USA, Aug. 30 – Sept. 2, 2009, pp. 1187–1195.

[25] J.-C. Trigeassou, T. Poinot, J. Lin, A. Oustaloup , and F. Levron, "Modeling and identification of a non integer order system", In *Proc. European Control Conf.*, Karlsruhe, Germany, 1999.

[26] D. Xue and Y. Q. Chen, "Suboptimum H_2 pseudo–rational approximations to fractional order linear time invariant systems", in J. Sabatier, O. P. Agrawal, and J. A. Tenreiro Machado (Eds.), *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, Springer, Dordrecht, The Netherlands, 2007, pp. 61–76.