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A note on robustness tolerances for combinatorial optimization problems

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Abstract

We consider so-called *generic combinatorial optimization problem*, where the set of feasible solutions is some family of nonempty subsets of a finite ground set with specified positive initial weights of elements, and the objective function represents the total weight of elements of the feasible solution. We assume that the set of feasible solutions is fixed, but the weights of elements may be perturbed or are given with errors. All possible realizations of weights form the set of *scenarios*.

A feasible solution, which for a given set of scenarios guarantees the minimum value of the worst-case relative regret among all the feasible solutions, is called a *robust solution*. The maximum percentage perturbation of a single weight, which does not destroy the robustness of a given solution, is called the *robustness tolerance* of this weight with respect to the solution considered.

In this paper we give formulae for computing the robustness tolerances with respect to an optimal solution obtained for some initial weights and we show that this can be done in polynomial time whenever the optimization problem is polynomially solvable itself.

Key words: Combinatorial problems, Robust solutions, Robustness tolerances

1. Introduction

We consider a combinatorial optimization problem in the following generic form:

$$v(c) = \min\{w(c, F) : \ F \in \mathcal{F}\}, \tag{1}$$

where the set of feasible solutions $\mathcal{F}$ is a family of nonempty subsets of a given ground set $E = \{e_1, \ldots, e_n\}$ and $c = (c(e_1), \ldots, c(e_n))^{\mathsf{T}} \in \mathbb{R}^n$ denotes the vector of weights of the elements of E. For $c \in \mathbb{R}^n$ and $F \in \mathcal{F}$, the objective function

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in (1) represents the total weight of this solution, i.e.,

$$w(c, F) = \sum_{e \in F} c(e).$$

Numerous discrete optimization problems, like e.g. the traveling salesman problem, the minimum spanning tree problem, the shortest path problem, the linear 0–1 programming problem, can be stated in this general form.

We will assume that the set of feasible solutions $\mathcal{F}$ in problem (1) is fixed but the vector of weights can change or it is given with errors. Let $\mathcal{C} \subseteq \mathbb{R}^n$ be a set of all possible realizations of the vector c, called the *scenarios*. Consider an initial scenario $c^o \in \mathcal{C}$ and let $\Omega(c^o) = \arg\min\{w(c^o, F) : \ F \in \mathcal{F}\}$ denote the set of optimal solutions in (1) for $c = c^o$.

Given an optimal solution $X^o \in \Omega(c^o)$ an important question concerns the stability of this solution on the set of possible scenarios $\mathcal{C}$. This question belongs to so-called sensitivity (stability) analysis, which is regarded an essential step of any optimization procedure (see e.g. Greenberg [5], Libura [9], Sotskov et al. [15]). The main goal of the sensitivity analysis consists in finding a subset of scenarios, for which the solution X^o remains *optimal*. In this paper we consider a natural extension of the sensitivity analysis. Namely, we will consider a problem of determining a subset of scenarios for which the solution X^o remains *robust*.

There are various concepts of the robustness of solutions in optimization (see e.g. Averbakh [1], Ben-Tal, Niemirowski [2], Bertsimas, Sim [3], Kouvelis, Yu [7], Mulvey et al. [13], Roy [14]). In this paper we will use as a robustness measure the maximum relative error (worst case relative regret) of the solution considered, over the set of all scenarios. Namely, assume that for any $F \in \mathcal{F}$ and $c \in \mathcal{C}$, we have $w(c, F) > 0$. Let $Z(F, \mathcal{C})$ denote the *worst-case relative regret* of the solution F on the set $\mathcal{C}$, i.e.,

$$Z(F, \mathcal{C}) = \max_{c \in \mathcal{C}} \max_{Y \in \mathcal{F}} \frac{w(c, F) - w(c, Y)}{w(c, Y)}. \tag{2}$$

A feasible solution $X \in \mathcal{F}$ will be called a *robust* solution for the set of scenarios $\mathcal{C} \subseteq \mathbb{R}^n$ if and only if the following inequalities hold:

$$Z(X, \mathcal{C}) \leq Z(F, \mathcal{C}) \qquad \text{for any} \quad F \in \mathcal{F}. \tag{3}$$

Thus, a feasible solution is robust if it guarantees the minimum value of the worst-case relative regret on the set $\mathcal{C}$ among all the feasible solutions.

In sensitivity analysis one seeks for the maximal under inclusion subset $S(X^o) \subseteq \mathbb{R}^n$ of the weights vectors in problem (1) for which the solution X^o remains optimal. Such a set is called the *optimality* (or – *stability*) *region* of the solution X^o. It is well known (see e.g. Greenberg [5], Libura [9]) that the optimality region is a polyhedral convex cone in $\mathbb{R}^n$. It is also obvious that an optimal solution $X^o \in \Omega(c^o)$ is robust for arbitrary scenario $c \in S(X^o)$. But this solution may remain robust for significantly larger set of scenarios. Moreover, in case of multiple optimal solutions it may happen, that the solutions

belonging to the set $\Omega(c^o)$ are quite different from the robustness point of view. This motivates studying an analogue of the stability region which we will call a *robustness region* of the feasible solution X and denote $R(X)$. Formally, $R(X)$ denotes the maximal subset of scenarios in $\mathbb{R}^n$ for which X is a robust solution.

It is rather difficult to find the robustness region of a given feasible solution of the combinatorial optimization problem (1); some attempts to obtain a subset of $R(X^o)$ for $X^o \in \Omega(c^o)$ are made in Libura [12], where the maximal ball with a center in c^o, belonging to the robustness region of X^o is investigated. Moreover, frequently it is reasonably to assume that only some data of the problem considered are subject to perturbations. In standard sensitivity analysis a particular case, when only a single scalar parameter may vary, received significant attention. This leads to the analysis of so-called *tolerances* of weights in problem (1). There are numerous papers devoted to problem of finding the tolerances as well as exploiting them in optimization algorithms (see e.g. Chakravarti, Wagelmans [4], Libura [8], van Hoesel, Wagelmans [18], Sotskov et al. [15], Tarjan [16], Turkensteen et al. [17], Wendell [19]). In the following we study an analogue of the sensitivity analysis tolerances in the robustness analysis framework. The main result of the paper is a simple formula for calculating so-called robustness tolerance of any weight in problem (1). This – as in case of standard sensitivity analysis – leads to a polynomial algorithm of finding robustness tolerances whereas the problem (1) is polynomially solvable itself.

2. Optimality and robustness tolerances

Let X^o be an optimal solution in problem (1) for $c = c^o$. Assume that only the weight of a single element $e \in E$ can be perturbed, i.e., $c(e_i) = c^o(e_i)$ for $e_i \neq e$. It is known that then X^o remains optimal if and only if the following inequalities holds:

$$c^o(e) - t^-(e) \leq c(e) \leq c^o(e) + t^+(e), \tag{4}$$

where $t^+(e), t^-(e) \in \mathbb{R} \cup \{\infty\}$ denote, respectively, so-called *upper* and *lower tolerance* of the weight $c(e)$.
Let $\mathcal{F}^e = \{F \in \mathcal{F} : e \in F\}$, $\mathcal{F}_e = \{F \in \mathcal{F} : e \notin F\}$, and

$$v^e(c) = \min_{F \in \mathcal{F}^e} w(c, F), \tag{5}$$

$$v_e(c) = \min_{F \in \mathcal{F}_e} w(c, F). \tag{6}$$

According to standard conventions, we take $v^e(c) = \infty$ or $v_e(c) = \infty$ if $\mathcal{F}^e = \emptyset$ or $\mathcal{F}_e = \emptyset$, respectively. Observe that given an algorithm for solving problem (1) for arbitrary $c \in \mathbb{R}^n$ and $\mathcal{F} \subseteq 2^E$, we may use them also for solving the optimization problems (5), (6). It is well known (see e.g. Libura [8, 9], Sotskov et al. [15]), that the following facts hold:

Proposition 1. *If $e \in X^o$, then $t^-(e) = \infty$ and $t^+(e) = v_e(c^o) - v(c^o)$.*
If $e \notin X^o$, then $t^+(e) = \infty$ and $t^-(e) = v^e(c^o) - v(c^o)$.

From Proposition 1 it follows that if the optimization problem (1) is polynomially solvable, then also the tolerances $t^+(e)$, $t^-(e)$, $e \in E$, can be computed in polynomial time. Moreover, the opposite implication also holds under some mild assumptions (see Chakravarti, Wagelmans [4], van Hoesel, Wagelmans [18]).

In the following we will introduce an analogue of the tolerances $t^+(e)$, $t^-(e)$ in the robustness analysis context. Our approach is similar to the Wendell's tolerance approach in linear programming (see Wendell [19]), which is actually more general, since it allows simultaneous changes of all weights in the objective function or right-hand-side vector of linear program.

Consider the following model of perturbations of the weights of elements in problem (1): Assume that some initial vector of weights $c^o > 0$ is given as well as a subset $Q \subseteq E$ is specified. The set Q represents all of the elements, for which the weights may be perturbed simultaneously and independently, Moreover, assume that the maximum percentage perturbation of any weight does not exceed $\delta \cdot 100\%$ of its initial value for some $\delta \in [0,1)$. This means that for a given value of δ we are faced with the set of scenarios $\mathcal{C}(c^o, Q, \delta)$, where

$$\mathcal{C}(c^o, Q, \delta) = \left\{(c(e_1), \ldots, c(e_n))^T \in \mathbb{R}^n : \ |c(e_i) - c^o(e_i)| \leq \delta \cdot c^o(e_i), \text{ if } e_i \in Q; \right.$$
$$\left. c(e_i) = c^o(e_i), \quad \text{if } \ e_i \notin Q\right\}.$$

Consider an optimal solution $X^o \in \Omega(c^o)$ and let $Q = \{e\}$, where $e \in E$. Obviously, X^o is a robust solution for the set of scenarios $\mathcal{C}(c^o, \{e\}, 0) = \{c^o\}$. The maximum value $t^r(e)$ of the parameter δ, such that X^o remains robust for the set of scenarios $\mathcal{C}(c^o, \{e\}, \delta)$, will be called the *robustness tolerance* of the weight $c(e)$. Formally,

$$t^r(e) = \sup\{\delta \in [0,1) : Z(X^o, \mathcal{C}(c^o, \{e\}, \delta)) \leq Z(X, \mathcal{C}(c^o, \{e\}, \delta)) \ \forall\, X \in \mathcal{F}\}.$$

In order to find the exact values of the robustness tolerances for $e \in E$ we will exploit some properties of the so-called *accuracy function* of a feasible solution of problem (1) introduced in Libura [11].

Let X be an arbitrary feasible solution of problem (1). Given $Q \subseteq E$ and $\delta \in [0,1)$, the value $a(X, Q, \delta)$ of the accuracy function $a(X, Q, \cdot) : [0,1) \to \mathbb{R}$ is equal to the maximum relative error (relative regret) of the solution X on the set of scenarios $\mathcal{C}(c^o, Q, \delta)$, i.e.,

$$a(X, Q, \delta) = \max_{c \in \mathcal{C}(c^o, Q, \delta)} \max_{Y \in \mathcal{F}} \frac{w(c, X) - w(c, Y)}{w(c, Y)}. \tag{7}$$

Observe that this means, that $a(X, Q, \delta) = Z(X, \mathcal{C})$ for $\mathcal{C} = \mathcal{C}(c^o, Q, \delta)$. The properties of the accuracy function can be therefore directly used in the robustness analysis for the set of scenarios $\mathcal{C}(c^o, Q, \delta)$. In particular, it is shown in Libura [11] that the following fact holds:

Lemma 1. *For $X \in \mathcal{F}$, $Q \subseteq E$ and $\delta \in [0,1)$,*

$$a(X, Q, \delta) = \max_{Y \in \mathcal{F}} \frac{w(c^o, X) - w(c^o, Y) + \delta \cdot w(c^o, (X \otimes Y) \cap Q)}{w(c^o, Y) - \delta \cdot w(c^o, Y \cap Q)}, \tag{8}$$

where $X \otimes Y = (X \setminus Y) \cup (Y \setminus X)$.

Formula (8) can be easily specified for the case $Q = \{e\}$ and we get the following corrolary:

Corollary 1. *For $X \in \mathcal{F}$, $e \in E$, and $\delta \in [0,1)$,*

$$Z\left(X, \mathcal{C}(c^o, \{e\}, \delta)\right) = a(X, \{e\}, \delta) = \max\{a', a''\}, \tag{9}$$

where

$$\begin{aligned} a' &= \frac{w(c^o, X) - v_e(c^o) + \delta \cdot w(c^o, X \cap \{e\})}{v_e(c^o)}, \\ a'' &= \frac{w(c^o, X) - v^e(c^o) + \delta \cdot [c^o(e) - w(c^o, X \cap \{e\})]}{v^e(c^o) - \delta \cdot c^o(e)}. \end{aligned}$$

It will be convenient to state formulae for calculating $Z\left(X, \mathcal{C}(c^o, \{e\}, \delta)\right)$ separately in both cases: $e \in X$ and $e \notin X$. To simplify the notation let for $X \in \mathcal{F}$, $e \in X$, and $\delta \in [0,1)$, $Z_e(X, \delta) = Z\left(X, \mathcal{C}(c^o, \{e\}, \delta)\right)$. Now from Corollary 1 we have the following facts:

If $X \in \mathcal{F}^e$ and $\delta \in [0,1)$, then

$$Z_e(X, \delta) = \max\left\{\frac{w(c^o, X) - v_e(c^o) + \delta \cdot c^o(e)}{v_e(c^o)}, \frac{w(c^o, X) - v^e(c^o)}{v^e(c^o) - \delta \cdot c^o(e)}\right\}. \tag{10}$$

If $X \in \mathcal{F}_e$ and $\delta \in [0,1)$, then

$$Z_e(X, \delta) = \max\left\{\frac{w(c^o, X) - v_e(c^o)}{v_e(c^o)}, \frac{w(c^o, X) - v^e(c^o) + \delta \cdot c^o(e)}{v^e(c^o) - \delta \cdot c^o(e)}\right\}. \tag{11}$$

The following theorem gives simple formulae for calculating the robustness tolerances $t^r(e)$, $e \in E$, for an initially optimal solution $X^o \in \Omega(c^o)$.

Theorem 1. *For $X^o \in \Omega(c^o)$,*

$$t^r(e) = \begin{cases} 1 & \text{if } e \in X^o, \\ \min\left\{1, \frac{[v^e(c^o)^2 - v(c^o)^2]^{\frac{1}{2}}}{c^o(e)}\right\} & \text{if } e \notin X^o. \end{cases} \tag{12}$$

PROOF. Let $X^o \in \Omega(c^o)$. From the definition of the robustness tolerances we have for $e \in E$, $t^r(e) = \sup\{\delta \in [0,1) : Z_e(X^o, \delta) \leq Z_e(X, \delta) \text{ for any } X \in \mathcal{F}\}$.

(i) Consider first the case when $e \in X^o$, which implies $v(c^o) = v^e(c^o) \leq v_e(c^o)$. It is easy to see that then $Z_e(X^o, \delta) \leq Z_e(X, \delta)$ for arbitrary $X \in \mathcal{F}$ and $\delta \in [0,1)$. Indeed, from (10) we have for $X = X^o$,

$$Z_e(X^o, \delta) = \max\left\{\frac{v(c^o) - v_e(c^o) + \delta \cdot c^o(e)}{v_e(c^o)}, 0\right\},$$

and for any $X \in \mathcal{F}^e$,

$$\begin{aligned} Z_e(X,\delta) &= \max\left\{\frac{w(c^o,X)-v_e(c^o)+\delta\cdot c^o(e)}{v_e(c^o)},\ \frac{w(c^o,X)-v^e(c^o)}{v^e(c^o)-\delta\cdot c^o(e)}\right\} \\ &\geq \max\left\{\frac{v(c^o)-v_e(c^o)+\delta\cdot c^o(e)}{v_e(c^o)},\ 0\right\} = Z_e(X^o,\delta). \end{aligned}$$

According to (11) also for any $X \in \mathcal{F}_e$ and $\delta \in [0,1)$,

$$\begin{aligned} Z_e(X,\delta) &= \max\left\{\frac{w(c^o,X)-v_e(c^o)}{v_e(c^o)},\ \frac{w(c^o,X)-v^e(c^o)+\delta\cdot c^o(e)}{v^e(c^o)-\delta\cdot c^o(e)}\right\} \\ &\geq \max\left\{0,\ \frac{v(c^o)-v^e(c^o)+\delta\cdot c^o(e)}{v^e(c^o)-\delta\cdot c^o(e)}\right\} \\ &\geq \max\left\{0,\ \frac{v(c^o)-v_e(c^o)+\delta\cdot c^o(e)}{v_e(c^o)-\delta\cdot c^o(e)}\right\} = Z_e(X^o,\delta). \end{aligned}$$

This means that X^o remains robust for arbitrary $\delta \in [0,1)$, which implies that $t^r(e) = 1$ when $e \in X^o$.

(ii) Consider now the case $e \notin X^o$, which implies $v(c^o) = v_e(c^o) \leq v^e(c^o)$. From (11) for any $\delta \in [0,1)$,

$$Z_e(X^o,\delta) = \max\left\{0,\ \frac{v(c^o)-v^e(c^o)+\delta\cdot c^o(e)}{v^e(c^o)-\delta\cdot c^o(e)}\right\},$$

and for arbitrary $X \in \mathcal{F}_e$ we have

$$\begin{aligned} Z_e(X,\delta) &= \max\left\{\frac{w(c^o,X)-v_e(c^o)}{v_e(c^o)},\ \frac{w(c^o,X)-v^e(c^o)+\delta\cdot c^o(e)}{v^e(c^o)-\delta\cdot c^o(e)}\right\} \\ &\geq \max\left\{0,\ \frac{v(c^o)-v^e(c^o)+\delta\cdot c^o(e)}{v^e(c^o)-\delta\cdot c^o(e)}\right\}. \end{aligned}$$

This implies $Z_e(X^o,\delta) \leq Z_e(X,\delta)$ for any $X \in \mathcal{F}_e$, $\delta \in [0,1)$, and consequently,

$$t^r(e) = \sup\{\delta \in [0,1) :\ Z_e(X^o,\delta) \leq Z_e(X,\delta) \text{ for all } X \in \mathcal{F}^e\}. \tag{13}$$

If $\mathcal{F}^e = \emptyset$ then $t^r(e) = 1$; assume therefore that $\mathcal{F}^e \neq \emptyset$ and consider $X^e \in \arg\min_{F\in\mathcal{F}^e} w(c^o,F)$. Substituting $v(c^o) = v_e(c^o)$ in (10) we obtain for $\delta \in [0,1)$,

$$\begin{aligned} Z_e(X^e,\delta) &= \max\left\{\frac{v^e(c^o)-v(c^o)+\delta\cdot c^o(e)}{v(c^o)},\ 0\right\} \\ &= \frac{v^e(c^o)-v(c^o)+\delta\cdot c^o(e)}{v(c^o)}, \end{aligned} \tag{14}$$

and for arbitrary $X \in \mathcal{F}^e$,

$$Z_e(X,\delta) = \max\left\{\frac{w(c^o,X)-v(c^o)+\delta\cdot c^o(e)}{v(c^o)},\ \frac{w(c^o,X)-v^e(c^o)}{v^e(c^o)-\delta\cdot c^o(e)}\right\}. \tag{15}$$

Now it is easy to see that for any $X \in \mathcal{F}^e$ and $\delta \in [0,1)$, the inequality $Z_e(X,\delta) \geq Z_e(X^e,\delta)$ holds, and from (13) it follows that for $e \notin X^e$,

$$t^r(e) = \sup\{\delta \in [0,1) : \; Z_e(X^o,\delta) \leq Z_e(X^e,\delta)\}.$$

Finally, using (14), (15), and solving the inequality

$$\max\left\{0, \; \frac{v(c^o) - v^e(c^o) + \delta \cdot c^o(e)}{v^e(c^o) - \delta \cdot c^o(e)}\right\} \leq \frac{v^e(c^o) - v(c^o) + \delta \cdot c^o(e)}{v(c^o)}$$

we obtain for $e \notin X^o$, $t^r(e) = \min\left\{1, \; \frac{\left[v^e(c^o)^2 - v(c^o)^2\right]^{\frac{1}{2}}}{c^o(e)}\right\}$, which proves (12).

Example

Consider an undirected graph $G = (V,E)$, where $V = \{1,2,3,4,5\}$ and $E = \{e_1,\ldots,e_7\} = \{\{1,2\},\{1,3\},\{1,4\},\{2,4\},\{3,4\},\{3,5\},\{4,5\}\}$. Let $\mathcal{F}$ be a family of subsets of E corresponding to all spanning trees in G, and let $c^o = (14,11,14,15,13,18,17)^T$ be a vector of the initial weights of edges in G. Then the combinatorial optimization problem (1) for $c = c^o$ is the minimum spanning tree problem in the weighted graph G. It is easy to verify that there are 21 spanning trees in graph G and that $T^o = \{\{1,2\},\{1,3\},\{3,4\},\{4,5\}\}$ is a single optimal spanning tree for an initial vector of weights c^o.

From Theorem 1 it follows that the robustness tolerances of all the edges belonging to X^o are equal to 1 which means that we can perturb individually the weights of these edges up to 100% of their initial values without destroying the robustness of the solution X^o. Consider therefore some edge from the set $E \setminus X^o$, e.g. the edge $e = \{1,4\}$, and the corresponding set of scenarios $\mathcal{C} = \mathcal{C}(c^o,\{\{1,4\}\},\delta)$. We have $c^o(e) = 14$, $v(c^o) = w(c^o,X^o) = 55$, $v^e(c^o) = 56$. Calculating $t^r(e)$ from (12) we obtain:

$$t^r(e) = \frac{\left(56^2 - 55^2\right)^{\frac{1}{2}}}{14} \approx \frac{10.54}{14} \approx 0.75.$$

Thus, the spanning tree X^o guarantees the minimum value of the worst-case relative regret among all the spanning trees in G if the weight of the edge $e = \{1,4\}$ is perturbed by no more than approximately 75%, and all the remaining weights are unchanged.

In Fig. 1 the worst case regret functions for all the feasible solutions in problem (1) are shown; bold line indicates the worst-case regret function of the spanning tree X^o. Observe that the solution X^o guarantees, indeed the minimum value of the worst-case regret among all the feasible solutions, i.e. it remains a *robust* spanning tree, provided $\delta \leq t^r(e) \approx 0.75$. It is interesting to note that in order to destroy the *optimality* of X^o it is enough to increase the weight of edge e by approximately 7.14%, which corresponds to the first breakpoint of the worst case regret function $Z(X^o,\delta)$ in Fig. 1.

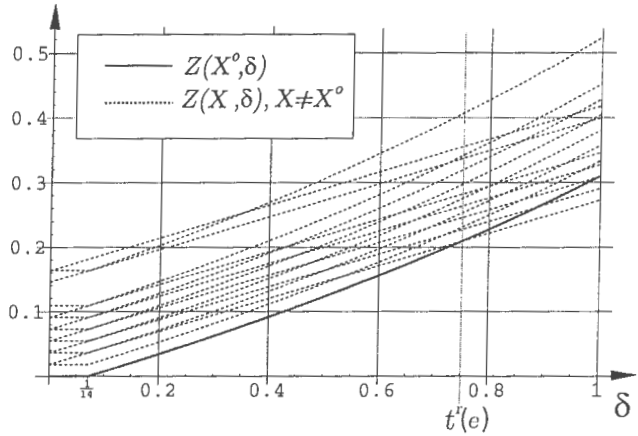

Figure 1: Worst-case regret functions of all spanning trees from Example.

3. Conclusions

In this paper we consider an influence of perturbations of single weights on the robustness of an initially optimal solution for the generic combinatorial optimization problem. Maximum percentage perturbation of the weight, which do not destroy the robustness of the solution considered, is called the robustness tolerance of this weight. It is shown, that the robustness tolerances of the weights for all elements belonging to the optimal solution are equal to 1, which means that these weights may be individually perturbed up to 100% of their initial values without destroying the robustness of this solution. The tolerances of weights for all remaining elements can be computed easily if the optimal value of an auxiliary optimization problem is known. This auxiliary problem consists in forcing an additional requirement, that the element considered does not belong to any feasible solution. Observe that this leads to polynomial solvability of the robustness tolerance problem provided that the original optimization problem is polynomially solvable itself.

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