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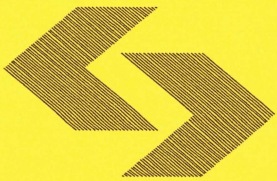

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L_2-optimal Routh–type model reduction

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Abstract

A computationally simple method for generating reduced–order models that minimise the L_2 norm of the approximation error while retaining a number of second–order information indices (energies of the resolvent kernel) is presented. The method exploits the energy–retention property peculiar to the Routh reduction method as well as the interpolation property of L_2–optimal approximations. The method can be adapted to the case in which the steady–state response to step inputs must also be preserved. Two examples taken from the relevant literature show that the suggested techniques may lead to approximations that are not worse than those afforded by alternative techniques.

Keywords: Model reduction, L_2 norm, Routh approximation, Impulse–response energy, Asymptotic response

1. Introduction

The model reduction problem has aroused a continual interest in the engineering community since the dawn of control and system theory [33], [62], its importance being evident not only in system simulation and controller synthesis but also in many problems related to robustness and uncertainty issues. Indeed, despite the dramatic increase of computing capabilities that reduce the need for simplified models, the new challenges facing the control engineer have recently led to a revival of studies on this topic with particular

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emphasis on optimisation and algorithmic efficiency (see, e.g., [1, 2], [4], [7], [10, 11], [12]–[51], [53], [56], [60, 61]).

Besides the reduction methods based on the retention of first–order information indices (e.g., coefficients of suitable series expansions), such as the classic Padé technique and its numerous variants [8, 9] that are characterised by remarkable computational simplicity and ease of implementation, the methods based on second–order information indices, such as Hankel singular values and impulse–response energies [14]–[17], [24], [26, 27], [36], [38], [52], [55], and on suitable quadratic criteria, such as the L_2 norm of the error [7], [12], [19]–[22], [25], [28]–[30], [42], [54], [57]–[59], [61], [63], have enjoyed an increasing popularity since the late Seventies and early Eighties, and dedicated software has been developed for their implementation.

The advantages of the aforementioned methods are related to an intuitively meaningful definition of the reduction criterion, to the possibility of determining bounds on some error norms (e.g., H_∞ norm [16]). However, their computational complexity increases rapidly with the dimensionality of the original system [18], which has stimulated research on efficient numerical algorithms, especially in view of the very high order of certain circuits and structures (see, e.g., [15], [28]). This paper presents a computationally efficient model reduction technique that combines the advantages of the Routh approximation in terms of retention of the resolvent–kernel energies [6] with those of the L_2–optimal rational approximation. Essentially, the reduction algorithm requires: (i) the construction of a Routh table, (ii) the solution of an algebraic equation of degree equal to the order of the approximating model, and (ii) the satisfaction of a set of interpolation conditions (conditions for L_2 optimality [19], [25], [30]). It is also shown how the algorithm can be adapted to obtain the desired asymptotic behaviour in the response to step inputs, a characteristic that is not exhibited by most popular reduction techniques such as the balanced truncation method.

The paper is organized as follows. Section 2 recalls briefly the basic recursion of the Routh algorithm and the energy–retention property of the Routh approximation, while Section 3 reviews briefly the interpolation conditions that are satisfied by the L_2–optimal reduced models. Section 4 presents the suggested reduction algorithms and discusses their computational complexity. Section 5 shows the results of the application of the algorithm to a pair of examples taken from the literature on model reduction.

2. Routh's algorithm and its use

This section recalls the properties of the Routh algorithm that are relevant to the reduction procedure described in Section 4 (for other interesting

properties of this remarkable algorithm see [3], [6], [23] and [43]).

The Routh algorithm generates a sequence of polynomials of descending degree starting from the even and odd parts of a given real polynomial of degree n

$$P_n(s) = Q_n(s) + Q_{n-1}(s) \tag{1}$$

according to the recursion

$$Q_{i-2}(s) = Q_i(s) - q_{i-1}\, s\, Q_{i-1}(s), \tag{2}$$

where

$$Q_i(s) = \sum_{k=0}^{\lfloor i/2 \rfloor} r_{i,i-2k} s^{i-2k}, \quad 0 \le i \le n, \tag{3}$$

and q_{i-1} is the ratio of the leading coefficients of $Q_i(s)$ and $Q_{i-1}(s)$, respectively, i.e.,

$$q_{i-1} = \frac{r_{i,i}}{r_{i-1,i-1}}, \quad 1 \le i \le n. \tag{4}$$

The entries of the row of order i in the standard Routh table for $P_n(s)$ are precisely the coefficients of the decreasing powers of s in (3). As is well known, if and only if $P_n(s)$ is a Hurwitz polynomial, the leading coefficients $r_{j,j}$, like all the other coefficients in the table, are different from zero and have the same sign, so that the entire sequence of $n+1$ polynomials $Q_j(s)$, $j = n, n-1, \ldots, 1, 0$, containing only even or only odd powers of s, can be constructed and all the n quotients (4) are positive (Routh criterion).

As for the first two upper rows, a *complete* polynomial $P_i(s)$ can be associated with every pair of consecutive polynomials in this sequence according to

$$P_i(s) = Q_i(s) + Q_{i-1}(s), \tag{5}$$

thus forming a sequence of real polynomials $\{P_j(s), j = n, \ldots, 1\}$. Clearly, two consecutive polynomials $P_i(s)$ and $P_{i-1}(s)$ share the same even or odd part $Q_{i-1}(s)$, and the Routh table for $P_{i-1}(s)$ coincides with the *tail* of the Routh tables for $P_j(s)$, $j = i, i+1, \ldots, n$. As a consequence, all polynomials in the sequence are Hurwitz if $P_n(s)$ is so. From (2) and (5), the following recursive relation between two consecutive complete polynomials is obtained:

$$P_{i-1}(s) = \left(1 + \frac{q_{i-1}}{2}\, s\right) P_i(s) - (-1)^i \frac{q_{i-1}}{2}\, s\, P_i(-s) \tag{6}$$

which is the two–term form of Routh's algorithm as opposed to the usual three–term (or split) form (2) [31]. It is also called *step–down* form because it generates polynomials of descending degree.

The Routh approximation method (cf., e.g., [6]) uses as the denominator of the reduced–order transfer function a polynomial $P_i(s)$ in the sequence generated from the original denominator $P_n(s)$, which ensures the stability of the reduced model if the original model is stable. Quite interestingly, besides stability, the reduced–order model obtained in this way retains a number of second–order information indices related to the impulse–response energies. To show this, consider the function

$$K_i(s) = \frac{1}{P_i(s)}, \; i < n, \tag{7}$$

and denote by $k_i^{(h)}(t)$ the h–th derivative of its impulse response $k_i(t)$, which is the so–called *resolvent kernel* of the convolution integral that determines the forced response $y_{f,i}(t)$ to an input $u(t)$ of an i–th order LTI system with transfer function

$$G_i(s) = \frac{N_i(s)}{P_i(s)} = N_i(s)\, K_i(s), \tag{8}$$

where $N_i(s) = n_{i,0} + n_{i,1}s + \ldots + n_{i,i-1}s^{i-1}$, that is,

$$y_{f,i}(t) = \int_0^t k_i(t-\tau)\, n_i(\tau)\, d\tau, \tag{9}$$

where

$$n_i(t) = n_{i,0}u(t) + n_{i,1}\frac{du}{dt} + \ldots + n_{i,i-1}\frac{d^{i-1}u}{dt^{i-i}} \tag{10}$$

is the right–hand side of the standard form of the differential equation describing the input–output behaviour of this system.

Denoting by

$$E_{i,h} = \int_0^\infty [k_i^{(h)}(t)]^2 dt \tag{11}$$

the energy of the k–th derivative $k_i^{(h)}(t)$ of $k_i(t)$, it may be proved [6] that

$$E_{i,h} = E_{n,h}, \; h = 0, 1, \ldots, i-1. \tag{12}$$

In other words, the i–th order model whose denominator is formed from the original denominator $P_n(s)$ according to the Routh recursion (6) retains the first i *kernel energies* of the n–th order (original) system with transfer function $G_n(s)$. Also, these energies may be computed recursively using only the entries of the Routh table for $P_n(s)$ [6]. From the kernel energies and the coefficients of $N_i(s)$, the impulse–response energy for the system with transfer function $G_i(s)$ can easily be determined and equated to the impulse–response energy of $G_n(s)$ (see, e.g., [27]). Of course, matching these energies does not

entail minimising the L_2 norm of the *difference* between the original and reduced impulse responses (approximation error).

The next section reviews briefly the conditions under which an i–th order transfer function minimises the L_2 norm of the approximation error subject to the retention of the energies (12).

3. Conditions for L_2 optimality

The *unconstrained* L_2–optimal rational approximation satisfies a set of interpolation conditions (at points that are not available *a priori*) that have been known to the control community for quite some time in the s–domain SISO case [33] and have more recently been extended to MIMO systems represented by transfer function matrices in [25]. By exploiting these interpolation conditions, some efficient reduction algorithms that avoid the direct computation of the gradient of the objective function (L_2 norm of the error) have been developed (see, e.g., [19, 20], [30], [54], [57] in a state–space setting and [28] in an input–output setting). However, these procedures are intrinsically nonlinear, strongly depend on the initial conditions, do not even retain the steady–state value of the step response and, in some cases, might give rise to unstable models of stable systems [61]. These drawbacks justify the search for alternative simpler and more robust techniques, even if they lead to constrained optima or near–optima in the L_2 sense [22], [42], [59]. Such an attempt is made in this paper. Specifically, the transform of the reduced–model kernel is chosen as in (7) with $P_i(s)$ obtained from the original denominator using the Routh recursion (6), thus ensuring the retention of a number of kernel energies; then, the numerator parameters are determined so as to minimise the L_2 error norm. Note, in this regard, that the same kernel energies characterise *all* responses, and not only the impulse response.

Let us denote the difference between the impulse responses $g_n(t)$ and $g_i(t)$ of the systems characterised by the strictly proper transfer functions $G_n(s)$ (original system) and $G_i(s)$ (reduced–order model), respectively, by

$$d_i(t) = g_n(t) - g_i(t) \tag{13}$$

whose Laplace transform, according to (8), is

$$D_i(s) = G_n(s) - G_i(s) = \frac{P_i(s)N_n(s) - P_n(s)N_i(s)}{P_n(s)P_i(s)}. \tag{14}$$

The squared L_2 norm of (13) induced by the usual scalar product is

$$||d_i(t)||^2 = \int_0^\infty d_i(t)\, d_i^*(t)\, dt, \tag{15}$$

where the asterisk denotes complex conjugation. Assuming for simplicity that the i roots $p_{i,h}$, $h = 1, 2, \ldots, i$, of $P_i(s)$ (poles of $G_i(s)$) are distinct, and indicating with

$$\mathcal{F}_i = \text{span}\{e^{p_{i,1}t}, e^{p_{i,2}t}, \ldots, e^{p_{i,i}t}\} \tag{16}$$

the i–th dimensional vector space generated by the modes of the reduced–order system, the index (15) is minimum if, and only if, for *any* function $f_i(t) \in \mathcal{F}_i$, the following orthogonality condition holds [32]:

$$\int_0^\infty d_i(t) f_i^*(t)\, dt = 0. \tag{17}$$

Denoting by $F_i(s)$ the Laplace transform of $f_i(t)$ and recalling that the Laplace transform of $f_i^*(t)$ is $F_i^*(s^*)$, from (17) and Parseval's theorem we obtain

$$\int_0^\infty d_i(t) f_i^*(t)\, dt = \frac{1}{2\pi\jmath} \int_{-\jmath\infty}^{+\jmath\infty} D_i(s) F_i^*(-s^*)\, ds = 0. \tag{18}$$

Therefore, by Cauchy's integral formula, all the poles of $D_i(s)F_i^*(-s^*)$ must lie in the left half–plane like those of $D_i(s)$. Since the poles $-p_{i,h}^*$ of all functions $F_i^*(-s^*)$ are in the right half–plane, it follows that they must be cancelled by the zeros of $D_i(s) = G_n(s) - G_i(s)$, that is, $G_i(s)$ must *interpolate* $G_n(s)$ at the negatives of its own poles $p_{i,h}$ (which are real or in conjugate pairs).

Taking (14) into account, the aforementioned optimality condition can be expressed in the compact form of a polynomial identity as:

$$P_i(s)N_n(s) - P_n(s)N_i(s) = M_{n-1}(s) \prod_{h=1}^{i} (s + p_{i,h}), \tag{19}$$

where $M_{n-1}(s)$ is a (real) polynomial of degree equal, at most, to $n-1$. By equating the coefficients of the equal powers of s on both sides of (19) a system of $n+i$ equations linear in the same number of unknown coefficients of $N_i(s)$ and $M_{n-1}(s)$ can be formed. Polynomial $M_{n-1}(s)$ can then be used, if necessary, to compute the value of the index (15) by resorting again to Cauchy's integral formula and Parseval's theorem. Alternatively, by setting $s = -p_{i,h}$, $h = 1, 2, \ldots, i$, in (19), the following smaller set of i equations *linear* in the i unknown coefficients of $N_i(s)$ is formed:

$$N_i(-p_{i,h}) = P_i(-p_{i,h}) \frac{N_n(-p_{i,h})}{P_n(-p_{i,h})}, \quad h = 1, 2, \ldots, i. \tag{20}$$

In both cases, it is necessary to preliminarily determine the roots $p_{i,h}$ of polynomial $P_i(s)$.

4. Reduction procedure

On the basis of the previous considerations, the following reduction algorithm can be can conceived.

Algorithm A

1. Generate, according to (6), a sequence of polynomials of descending degree from the original denominator polynomial $P_n(s)$ down to the polynomial $P_i(s)$ of the desired reduced degree i.

2. Find the roots $p_{i,h}$, $h = 1, 2, \ldots, i$, of the polynomial equation $P_i(s) = 0$.

3. Determine the i coefficients of the $(i-1)$–th degree numerator polynomial $N_i(s)$ of the transfer function $G_i(s) = N_i(s)/P_i(s)$ approximating $G_n(s) = N_n(s)/P_n(s)$ by solving the system (20) of i linear equations.

The procedure is not computationally demanding. Observe, in particular, that the construction of the entire Routh table for a polynomial of degree n requires $\mathcal{O}(n^2/2)$ elementary operations [6] (but the aforementioned algorithm can be arrested at the i–th row), the computational complexity of the Gauss elimination procedure to solve a system of i linear equation is $\mathcal{O}(i^3)$ [46], while the solution of polynomial equations up to degree 20 does not pose any particular problem in terms both numerical robustness and efficiency [39] (note that, usually, $i \ll n$)). It is also worth mentioning that *fraction–free* Routh tests that increase considerably the numerical accuracy of the classical Routh algorithm have been proposed recently [5]. Clearly, if, after $N_i(s)$ has been determined, it is required to evaluate the integral (15), e.g., by means of (13) and Parseval's theorem, the computational complexity increases, even if not substantially.

As already said, the reduced–order model obtained according to the aforementioned procedure retains i kernel energies and minimises index (15) subject to the Hurwitz denominator $P_i(s)$. However, since index (15) refers to the impulse response (that necessarily tends to zero), the asymptotic behaviour of the response to any other input $u(t)$ is not equal, in general, to that of the original system. A suggestion as to how the method can be adapted to the case in which the steady–state response to step inputs need be preserved, at the expense of the number of parameters left for optimisation, is outlined next. The procedure could be extended to reproduce the asymptotic response to more complicated inputs. However, such an extension would entail a further reduction of the number of optimisation parameters and is not pursued here.

Assume again that the reduced–order model denominator $P_i(s)$ is obtained from the Routh algorithm (6), thus ensuring the retention of stability and a number of kernel energies, and denote by

$$\hat{G}_i(s) = \frac{\hat{N}_i(s)}{P_i(s)} \tag{21}$$

the transfer function of the strictly–proper reduced–order model whose numerator $\hat{N}_i(s)$ must be determined in such a way that the steady–state response to a step input is preserved.

The Laplace transform of the original system's step response can be decomposed as

$$Y_{f,n}(s) = \frac{N_n(s)}{P_n(s)} \frac{1}{s} = \frac{T_n(s)}{P_n(s)} + \frac{K}{s}, \tag{22}$$

where $T_n(s)/P_n(s)$ is the Laplace transform of the transient response and K is the steady–state value. In order for $\hat{G}_i(s)$ to exhibit the same steady state, the transform of its step response should be decomposable as

$$Y_{f,i}(s) = \frac{\hat{N}_i(s)}{P_i(s)} \frac{1}{s} = \frac{T_i(s)}{P_i(s)} + \frac{K}{s}, \tag{23}$$

where the transient component $T_i(s)/P_i(s)$ is strictly proper and the steady–state component K/s matches the one of (22). From (23) it follows that

$$\hat{N}_i(s) = T_i(s)\, s + K\, P_i(s). \tag{24}$$

Since the degree of this polynomial identity is i, by equating the coefficients of the same powers of s on both sides of (24), a system of $i+1$ equations is obtained. Therefore, to admit a unique solution, the number of unknowns must also be $i+1$. Now, if $T_i(s)$ is completely determined by minimising the L_2 norm of the difference between the transient terms:

$$\hat{d}_i(t) = y_{tr,n}(t) - y_{tr,i}(t), \tag{25}$$

where $y_{tr,n}(t) = LT^{-1}[T_n(s)/P_n(s)]$ and $y_{tr,i}(t) = LT^{-1}[T_i(s)/P_i(s)]$, then the number of unknowns in (24) is only i (number of coefficients of $\hat{N}_i(s)$) and no solution exists.

To overcome this problem, a further unknown should be introduced. One way to do this, is to replace $T_i(s)/P_i(s)$ by the sum of the best approximation (in the L_2 sense) of immediately lower order $i-1$, i.e., $T_{i-1}(s)/P_{i-1}(s)$, plus an *auxiliary* stable first–order term $x/(s-q)$ with unknown gain x and pre–specified pole q. Not to influence appreciably the system dynamics, this pole

could be located far to the left of the roots of $P_{i-1}(s)$, but other choices are of course possible (and even advisable). Taking (21) into account, the Laplace transform of the step response of the reduced–order model then becomes:

$$Y_{f,i}(s) = \hat{G}_i(s)\,\frac{1}{s} = \frac{\hat{N}_i(s)}{P_{i-1}(s)(s-q)}\,\frac{1}{s} = \frac{T_{i-1}(s)}{P_{i-1}(s)} + \frac{x}{s-q} + \frac{k}{s} \tag{26}$$

leading to the polynomial identity

$$\hat{N}_i(s) = T_{i-1}(s)\,(s-q)\,s + x\,P_{i-1}(s)\,s + k\,P_{i-1}(s)\,(s-q). \tag{27}$$

In this way the number of unknowns (the i coefficients of $\hat{N}_i(s)$ plus x) matches the number of $i+1$ equations obtained by equating the coefficients of the equal powers of s on both sides of (27).

Note that, using the notation:

$$T_{i-1}(s) = b_{i-1,i-2}s^{i-2} + b_{i-1,i-3}s^{i-3} + \ldots + b_{i-1,0}, \tag{28}$$

$$P_{i-1}(s) = a_{i-1,i-1}s^{i-1} + a_{i-1,i-2}s^{i-2} + \ldots + a_{i-1,0}, \tag{29}$$

the unknown parameter x is obtained from the coefficients of s^i only as:

$$x = -\frac{b_{i-1,i-2}}{a_{i-1,i-1}} - k. \tag{30}$$

Once x has been determined, the computation of the coefficients of $\hat{N}_i(s)$ is straightforward since all terms at the right–hand side of (27) become known. In conclusion, the algorithm for finding an i–th order reduced–order model that retains the steady–state value of the original step response can be presented as follows.

Algorithm B

1. Decompose the original step response transform as in (22).

2. Find the transient component $T_{i-1}(s)/P_{i-1}(s)$ that minimises the L_2 norm of $\hat{d}_{i-1}(t) = y_{tr,n}(t) - y_{tr,i-1}(t)$ with $P_{i-1}(s)$ obtained from the original denominator $P_n(s)$ using the Routh algorithm (6).

3. Choose q.

4. Compute x according to (30).

5. Determine the coefficients of polynomial $\hat{N}_i(s)$ from (27).

6. Form the i–th order approximating transfer function as

$$\hat{G}_i(s) = \frac{\hat{N}_i(s)}{P_{i-1}(s)(s-q)}.$$

The only demanding step of *Algorithm B* is clearly the second. It entails the same operations as *Algorithm A* (referred, however, to the transient component of the step response instead of the impulse response), namely, the construction of a (part of a) Routh table, the solution of a polynomial equation of degree $i-1$, and the solution of a system of $i-1$ linear equations similar to (20), i.e.:

$$T_{i-1}(-p_{i-1,h}) = P_{i-1}(-p_{i-1,h})\frac{T_n(-p_{i-1,h})}{P_n(-p_{i-1,h})}, \quad h = 1, 2, \ldots, i-1, \tag{31}$$

corresponding to a set of $i-1$ interpolation conditions at the negatives of the roots $p_{i-1,h}$, $h = 1, 2, \ldots, i-1$, of $P_{i-1}(s)$. Therefore the computational complexity of *Algorithm B* is not much greater than that of *Algorithm A*, at least if the auxiliary pole q is arbitrarily placed to the left of the other poles of $Y_{f,i}(s)$, as previously suggested. Alternatively, q may be chosen so as to minimise $\|G_n(s) - \hat{G}_i(s)\|$. This result can be obtained by: (i) repeating Steps 3 through 6 for a number of different auxiliary poles q, (ii) computing the related values of the aforementioned norm, and (iii) picking up the pole q that ensures the least value of this norm. Clearly, this alternative choice increases the computational complexity of the procedure but might be worthwhile.

5. Examples

Two examples taken from the literature on model reduction are worked out in this section. The results obtained from the application of *Algorithm A* and *Algorithm B* are compared with those obtained using the popular balanced truncation method as well as the methods employed by authors who recently considered the same examples. As commonly done, the comparison is essentially based on the visual inspection of the responses to impulse and step inputs, on the Bode plots, and on the value of the L_2 norm of the respective impulse–response errors (clearly, the norm of the *step–response* error tends to infinity when the steady–state value of the original system is not matched exactly).

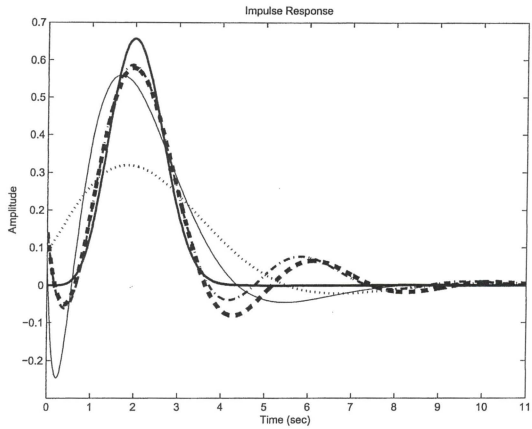

Figure 1: Impulse responses of: (i) the original model (32) (solid line), (ii) the third–order approximation (33) obtained using *Algorithm A* (dashed line), (iii) the third–order approximation (34) obtained using *Algorithm B* (solid thin line), (iv) the third–order model derived in [10] (dotted line), and (v) the third–order model obtained via balanced truncation (dashdotted line).

5.1. Example 1

Consider first the following 9–th order original transfer function [37]

$$G(s) = \frac{s^4 + 35s^3 + 291s^2 + 1093s + 1700}{s^9 + 9s^8 + 66s^7 + 294s^6 + 1029s^5 + 2541s^4 + 4684s^3 + 5856s^2 + 4620s + 1700}, \quad (32)$$

whose poles are $-1, -1 \pm \jmath, -1 \pm \jmath 2, -1 \pm \jmath 3, -1 \pm \jmath 4$. The same original system has been used in [10] to find a third–order approximating model by means of a "biased stability–equation" technique. *Algorithm A* in Section 4 leads to

$$G_3(s) = \frac{0.1399s^2 - 0.8022s + 1.8554}{s^3 + 1.6412s^2 + 3.3077s + 1.8601} \quad (33)$$

whose poles are $-0.7024, -0.4694 \pm \jmath 1.5582$. The squared L_2 norm of the related impulse–response error turns out to be 0.0184, whereas the squared error norm for the model obtained in [10] is 0.1348 and that for the third–order model obtained from balanced truncation is 0.0158. *Algorithm B* with $q = -5.2$ (found using the iterative search outlined at the end of Section 4) leads to

$$\hat{G}_3(s) = \frac{0.0724s^2 - 3.1780s + 5.8933}{s^3 + 6.5248s^2 + 8.0224s + 5.8933} \quad (34)$$

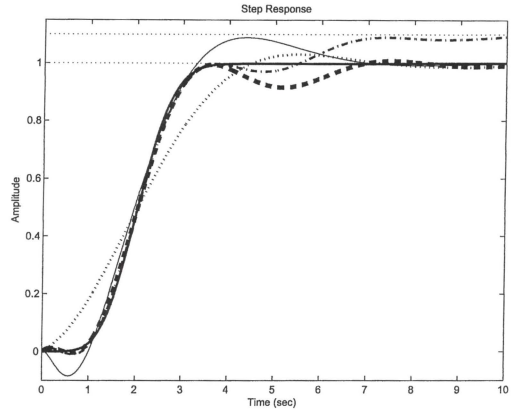

Figure 2: Step responses of: (i) the original model (32) (solid line), (ii) the third–order approximation (33) obtained using *Algorithm A* (dashed line), (iii) the third–order approximation (34) obtained using *Algorithm B* (solid thin line), (iv) the third–order model derived in [10] (dotted line), and (v) the third–order model obtained via balanced truncation (dashdotted line).

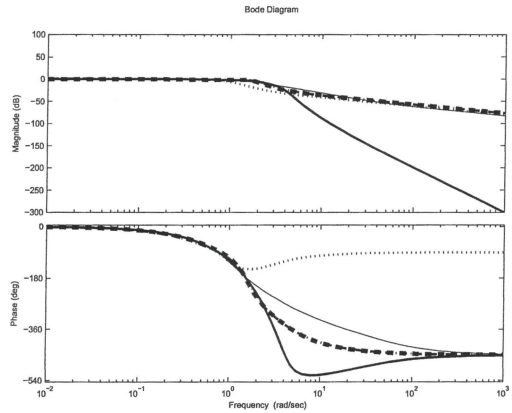

Figure 3: Bode plots of: (i) the original model (32) (solid line), (ii) the third–order approximation (33) obtained using *Algorithm A* (dashed line), (iii) the third–order approximation (34) obtained using *Algorithm B* (solid thin line), (iv) the third–order model derived in [10] (dotted line), and (v) the third–order model obtained via balanced truncation (dashdotted line).

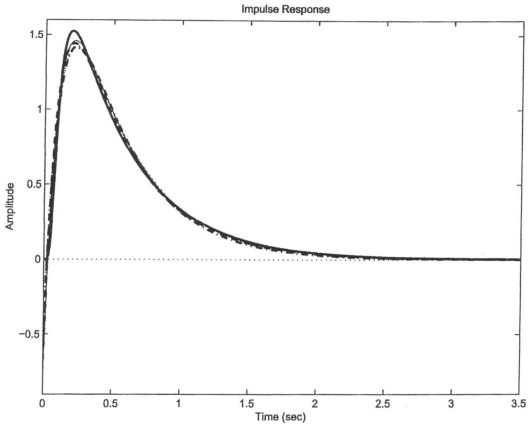

Figure 4: Impulse responses of: (i) the original model (35) (solid line), (ii) the second–order approximation (36) obtained using *Algorithm A* (dashed line), (iii) the second–order approximation (37) obtained using *Algorithm B* (solid thin line), (iv) the second–order model derived in [34] (dotted line), and (v) the second–order model obtained via balanced truncation (dashdotted line).

whose poles are $-5.2, -0.6624 \pm j0.8334$. In this case the squared L_2 error norm turns out to be 0.0662. Figs. 1, 2 and 3 compare, respectively, the impulse responses, the step responses and the Bode plots of (33) and (34) with those obtained using the method suggested in [10] and the balanced truncation method.

5.2. Example 2

Consider the 10-th order system described by (see [41], [34])

$$G(s) = \frac{540.70748 \times 10^{17}}{\prod_{i=1}^{10}(s + b_i)}, \tag{35}$$

where $b_1 = 2.04, b_2 = 18.3, b_3 = 50.13, b_4 = 95.15, b_5 = 148.85, b_6 = 205.16, b_7 = 257.21, b_8 = 298.03, b_9 = 320.97, b_{10} = 404.16$. Assume, as in [34], that a second–order approximating model is needed. *Algorithm A* in Section 4 leads to

$$G_2(s) = \frac{-0.6687s + 23.2918}{s^2 + 13.0793s + 23.6262} \tag{36}$$

whose poles are $-10.9147, -2.1646$. The squared L_2 norm of the related impulse–response error turns out to be 0.0082, whereas the squared error norm for the model obtained in [34] is 0.0074 and that obtained via balanced truncation is 0.0074. *Algorithm B* with $q = -19.1$ (found using the iterative search outlined at the end of Section 4) leads to

$$\hat{G}_2(s) = \frac{-0.3521s + 34.5019}{s^2 + 20.9064s + 34.5019} \tag{37}$$

whose poles are $-19.1, -1.8064$. In this case the squared L_2 error norm turns out to be 0.0398. Figs. 4, 5 and 6 compare, respectively, the impulse responses, the step responses and the Bode plots of (36) and (37) with those obtained using the method suggested in [34] and the balanced truncation method.

6. Conclusions

A model reduction method has been proposed that: (i) preserves stability, (ii) ensures the retention of a number of kernel energies characterising all forced responses, and (iii) minimises the L_2 norm of the approximation error for the desired denominator formed according to the Routh algorithm. The method is characterised by remarkable computational simplicity compared to alternative techniques that refer to second–order information indices and may lead to satisfactory results, as shown by two examples taken from the relevant literature, which makes the method worth trying.

A variant of the method that allows us to reproduce exactly the asymptotic response to step inputs has also been outlined. The computational burden of this variant, based on the decomposition of the step response into the transient and steady–state components, is not appreciably heavier than that of the original method, while the accuracy during the transient does not deteriorate significantly.

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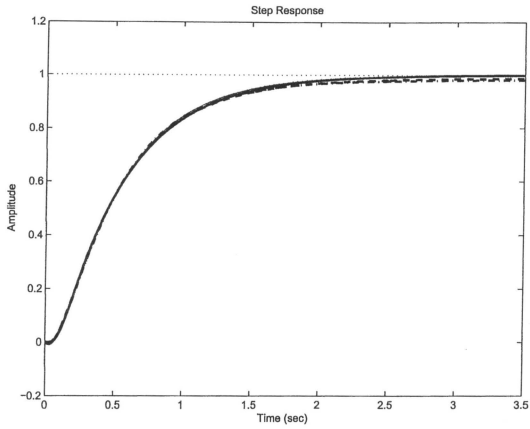

Figure 5: Step responses of: (i) the original model (35) (solid line), (ii) the second–order approximation (36) obtained using *Algorithm A* (dashed line), (iii) the second–order approximation (37) obtained using *Algorithm B* (solid thin line), (iv) the second–order model derived in [34] (dotted line), and (v) the second–order model obtained via balanced truncation (dashdotted line).

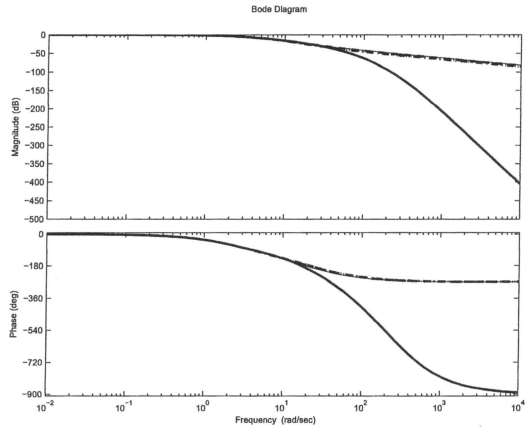

Figure 6: Bode plots of: (i) the original model (35) (solid line), (ii) the second–order approximation (36) obtained using *Algorithm A* (dashed line), (iii) the second–order approximation (37) obtained using *Algorithm B* (solid line), (iv) the second–order model derived in [34] (dotted line), and (v) the second–order model obtained via balanced truncation (dashdotted line).

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