Raport Badawczy

Research Report

RB/29/2015

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Warszawa 2015

L_2–Optimal FOPDT Models of High–Order Rational Plants

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Abstract

Necessary conditions for the L_2 optimality of a first–order–plus–dead–time (FOPDT) model of a high–order plant are derived using classic analytic function theory. They are expressed as a set of three nonlinear equations that partly resemble the interpolation conditions valid for rational approximation. From these conditions a simple procedure to find the optimal FOPDT model is obtained. An example taken from the relevant literature is finally worked out. It turns out that the impulse and step responses of the L_2–optimal FOPDT model fit well those of the original high–order rational system.

Keywords: Approximation, L_2 norm, Time delay, Optimality conditions, Algorithms.

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1. Introduction

The values of the parameters of PID controllers are usually set on the basis of an approximate model of the plant to be controlled (see, e.g., [1], [14, Ch. 3]). Often, industrial plants are characterised by a large pole–zero excess and all of their poles are negative real. As a consequence, their step response is monotonically increasing and exhibits a flat initial part of non–negligible length because a number of successive derivatives are zero at $t = 0$. Therefore, it is reasonable to approximate the transfer function of these plants by means of a first–order–plus–dead–time (FOPDT) model with

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transfer function

$$G_a(s) = \frac{\mu}{s+\lambda} e^{-Ls}. \tag{1}$$

Since the pioneering paper by Ziegler and Nichols [19] a variety of methods have been proposed to derive a transfer function like (1) from the step or harmonic response (usually obtained experimentally) of an original complex system (see, e.g., [17, Ch. 6], [14, Ch. 3] and bibliographies therein). On the other hand, even when an analytical description of a high–order process is already available, it is often useful or even mandatory to resort to a more compact representation for both computational and controller design reasons. Observe, in this regard, that model (1) allows us to deal satisfactorily with robustness issues. For example, the set of all PID controllers ensuring given stability margins can easily be found when the plant is described by (1) [8], [7], [14], [15]. However, despite these advantages, the irrational nature of (1) hinders the application of analytically–based techniques to the approximation of high–order systems by means of FOPDT models. Attempts in this direction have recently been made in [10] and [5] where L_2–optimal FOPDT approximations of either all–pole original systems or, respectively, original systems with only one zero have been sought using a time–domain analytical approach.

This note, too, is concerned with the approximation of a high–order rational model by means of a FOPDT model. The adopted approximation criterion is the minimisation of the L_2 norm of the difference between the original and FOPDT *impulse responses* without restrictions on the structure of the original rational system. The problem is formally stated in Section 2. In Section 3 necessary conditions for the optimality of the approximation are derived using a frequency–domain approach. They correspond to a set of three nonlinear equations in the three unknown parameters of (1) that are somewhat reminiscent of the interpolation conditions valid in the case of rational approximation [6]. In Section 4 these conditions are expressed in a form that is particularly suited to numerical solution. Section 5 applies the suggested approximation procedure to a benchmark example. Some concluding remarks are made in Section 6.

2. Problem statement

Let us write the strictly–proper transfer function of the original high–order plant as

$$G_p(s) = \frac{N_p(s)}{D_p(s)} = \sum_{i=1}^{n} \Big[\sum_{l=1}^{m_i} \frac{k_{i,l}}{(s+a_i)^l} \Big], \tag{2}$$

where the a_i, $i = 1, \ldots, n$, are the negatives of the n distinct roots of the denominator polynomial $D_p(s)$ and the m_i, $i = 1, \ldots, n$, are their respective multiplicities. The following standing assumptions are made.

Assumption 1. *The original plant (2) is BIBO stable, i.e., its poles $-a_i$ are in the open left half-plane (OLHP).*

Assumption 2. *The n distinct poles of (2) are real.*

Assumption 3. *The static gain $G_p(0)$ of (2) is positive.*

Assumption 1 implies that the impulse response $g_p(t) = LT^{-1}[G_p(s)]$ of system (2) tends to zero and its L_2 norm $\|g_p(t)\|$ is finite. Assumption 2 implies that system (2) does not exhibit pseudo-periodic modes, and that all the $k_{i,l}$ are real. The following treatment could be extended to the case in which (2) exhibits poles with nonzero imaginary part, but this extension would entail a substantial increase in notational complexity; on the other hand, the presence of oscillatory modes would almost certainly ask for an approximating model different from (1), for example, a second-order-plus-dead time or SOPDT model (see, e.g., [9]). Assumption 3, which means that the step response of (2) tends to the *positive* value $G_p(0)$, does not entail any loss of generality since one might as well refer to the approximation of $-G_p(s)$.

From (2), the analytic expression of the original impulse response $g_p(t)$ turns out to be:

$$g_p(t) = \sum_{i=1}^{n} \Big[\sum_{l=1}^{m_i} \frac{k_{i,l}}{(l-1)!} t^{l-1} \Big] e^{-a_i t}, \quad t \geq 0, \tag{3}$$

whereas the impulse response of the FOPDT model (1) is

$$g_a(t) = LT^{-1}[G_a(s)] = \mu e^{-\lambda(t-L)}\, \mathbb{1}(t-L), \tag{4}$$

where $\mathbb{1}(t)$ denotes the Heaviside step function which is zero for negative values of the argument and 1 for nonnegative values of the argument.
The difference between the impulse responses (4) and (3), called *approximation error* in the sequel, will be denoted by

$$e(t) = g_a(t) - g_p(t) \tag{5}$$

whose Laplace transform is

$$E(s) = G_a(s) - G_p(s) \tag{6}$$

The approximation problem considered in this paper can now be stated as follows.

Problem 1 (Approximation problem). *Find the three parameters of the FOPDT model (1), with $\mu > 0$, $\lambda > 0$ and $L \geq 0$ in such a way that $\|e(t)\|$ is minimal, where $\|\cdot\|$ denotes the L_2 norm.* $\square$

The constraints $\lambda > 0$ and $L > 0$ are related, respectively, to the BIBO stability and causality of the approximating model. The constraint $\mu > 0$ implies that the *step* response of (1) tends to a positive value (given by μ/λ which coincides with the area under the impulse response curve $g_a(t)$ of the FOPDT model), as is the case for the original system (see Assumption 3).

Observe that a solution certainly exists on every *compact* parameter set $\mathcal{S} \subset \{(\mu, \lambda, L) : \mu > 0, \lambda > 0, L \geq 0\}$ since, by Weierstrass extreme value theorem, every continuous function, such as $\|e(t)\|$, attains its extreme values on a (non–empty) compact set, even if the solution might not correspond to an interior point of $\mathcal{S}$ [13]. Also, it has been proved [10] that the minimum of $\|e(t)\|$ is generically *unique* if $G_p(s)$ is all–pole .

Necessary conditions for an *interior point* of the positive octant to be a point of minimum for $\|e(t)\|$ (and, thus, for $\|e(t)\|^2$ as well) will be derived next in the frequency domain by exploiting Parseval's theorem and simple properties of analytic functions, but exactly the same results could obviously be obtained in the time domain.

Concerning the boundary of this octant, no *minimum* of $\|e(t)\|$ may occur at the points of the plane $\lambda = 0$, because there $g_a(t)$ tends to μ (so that $\|e(t)\|$ diverges, except for $\mu = 0$ where $g_a(t)$ is identically zero).

Moreover, no *minimum* of $\|e(t)\|$ may occur at the points of the plane $\mu = 0$, as shown next.

Lemma 1. *If g_p is* ultimately positive, *i.e., if there exists t_1 such that $g_p(t) > 0$ for all $t > t_1$, then for all $C > 0$ there exist $\gamma > 0$ and $t_0 > 0$ such that, for all $t > t_0$,*

$$g_p(t) > Ce^{-\gamma t}.$$

Proof. Let the a_i's be ordered according to increasing value, i.e., $0 < a_1 < a_2 < \ldots < a_n$. The dominant term of $g_p(t)$, for $t \to \infty$, is then

$$\frac{k_{1,m_1}}{(m_1 - 1)!} t^{m_1 - 1} e^{-a_1 t},$$

which, by the hypothesis on the sign of $g_p(t)$ for $t > t_1$, implies that $k_{1,m_1} > 0$. Moreover, for all $\gamma > a_1$,

$$\lim_{t \to \infty} \frac{g_p(t) - Ce^{-\gamma t}}{t^{m_1 - 1} e^{-a_1 t}} = \frac{k_{1,m_1}}{(m_1 - 1)!} > 0.$$

Consequently, there exists $t_0 > 0$ such that, for all $t > t_0$,

$$\frac{g_p(t) - Ce^{-\gamma t}}{t^{m_1-1}e^{-a_1 t}} > 0 .$$

The claim is proven by noting that $t^{m_1-1}e^{-a_1 t} > 0$ for all $t > t_0$. ■

Corollary 1. *If $g_p(t)$ is ultimately negative, i.e., there exists t_1 such that $g_p(t) < 0$ for all $t > t_1$, then for all $C < 0$ there exist $\gamma > 0$ and $t_0 > 0$ such that, for all $t > t_0$,*

$$g_p(t) < Ce^{-\gamma t} .$$

Proposition 1. *If $G_p(s)$ is not identically zero, then the solution of Problem 1 cannot occur for $\mu = 0$.*

Proof. The approximation error associated with an approximating transfer function that is identically equal to zero is, obviously, $\|g_p(t)\|$. Suppose first that $g_p(t)$ is ultimately positive and consider the transfer function

$$G_d(s) = \frac{Ce^{-\gamma t_0}}{s+\gamma} e^{-s t_0} ,$$

where C, γ and t_0 are as in Lemma 1. The impulse response for this system is

$$g_d(t) = Ce^{-\gamma t_0} e^{-\gamma(t-t_0)} \mathbb{1}(t-t_0) = \begin{cases} 0, & \text{for } t < t_0 , \\ Ce^{-\gamma t}, & \text{for } t \geq t_0 . \end{cases}$$

The L_2 norm of the related approximation error $e_d(t) = g_p(t) - g_d(t)$ is

$$\|e_d(t)\| = \int_0^\infty e_d(t)^2 dt = \int_0^{t_0} e_d(t)^2 dt + \int_{t_0}^\infty e_d(t)^2 dt =$$

$$\int_0^{t_0} g_p(t)^2 dt + \int_{t_0}^\infty (g_p(t) - g_d(t))^2 dt \leq \int_0^{t_0} g_p(t)^2 dt + \int_{t_0}^\infty g_p(t)^2 dt ,$$

where the inequality holds because $g_p(t) > g_d(t) > 0$ for all $t > t_0$. The same result holds when $g_p(t)$ is ultimately negative. Since, by assumption, the impulse response does not exhibit pseudo–periodic modes, one of the two above–mentioned cases must occur and the claim is proven. ■

Remark 1. *To find the candidates to the global optimum on the boundary plane $L = 0$, resort can be made to standard methods of rational approximation (see, e.g., [2]). As already said, this paper focuses on the interior points of the first octant.* □

3. Optimality conditions

Denoting the squared L_2 norm of the approximation error (5) by

$$J := \|e(t)\|^2, \tag{7}$$

according to Parseval's theorem we have

$$J = \frac{1}{2\pi \jmath} \int_{-\jmath\infty}^{+\jmath\infty} E(s)\, E(-s) ds \tag{8}$$

and, taking (6) into account and substituting (1) for $G_a(s)$,

$$J = \frac{1}{2\pi \jmath} \int_{-\jmath\infty}^{+\jmath\infty} \left[\frac{\mu}{s+\lambda} e^{-Ls} - G_p(s)\right] \left[\frac{\mu}{-s+\lambda} e^{Ls} - G_p(-s)\right] ds. \tag{9}$$

The following two lemmas, whose proof is omitted for brevity, will be helpful in the search for the minima of index J.

Lemma 2. *The k-th derivative with respect to s of the function*

$$f_a(s) = \frac{e^{-Ls}}{s+\lambda}$$

is

$$\frac{d^k}{ds^k} f_a(s) = (-1)^k \frac{e^{-Ls}}{(s+\lambda)^{k+1}} \sum_{i=0}^{k} \frac{k!}{i!} L^i (s+\lambda)^i.$$

■

Lemma 3. *The k-th derivative with respect to s of the function*

$$f_b(s) = \frac{e^{-Ls}}{(s+\lambda)^2}$$

is

$$\frac{d^k}{ds^k} f_b(s) = (-1)^k \frac{e^{-Ls}}{(s+\lambda)^{k+2}} \sum_{i=0}^{k} \frac{k!}{i!} (k+1-i)\, L^i\, (s+\lambda)^i.$$

■

The candidate points of minimum of (7) in the open positive parameter octant can be found by setting to zero the partial derivatives of (7) with respect to μ, λ and L.

Now, the partial derivative of (9) with respect to μ is

$$\frac{\partial J}{\partial \mu} = \frac{1}{\pi \jmath} \int_{-\jmath\infty}^{+\jmath\infty} \left[\frac{1}{s+\lambda} e^{-Ls}\right] \left[\frac{\mu}{-s+\lambda} e^{Ls} - G_p(-s)\right] ds =$$

$$\frac{1}{\pi \jmath} \int_{-\jmath\infty}^{+\jmath\infty} \frac{\mu}{(s+\lambda)(-s+\lambda)} ds - \frac{1}{\pi \jmath} \int_{-\jmath\infty}^{+\jmath\infty} \frac{1}{s+\lambda} e^{-Ls}\, G_p(-s)\, ds =$$

$$\frac{\mu}{\lambda} - \frac{1}{\pi \jmath} \int_{-\jmath\infty}^{+\jmath\infty} \frac{1}{s+\lambda} e^{-Ls}\, G_p(-s)\, ds. \qquad (10)$$

Setting this derivative to zero leads to

$$\frac{\mu}{2\lambda} = \frac{1}{2\pi \jmath} \int_{-\jmath\infty}^{+\jmath\infty} \frac{1}{s+\lambda} e^{-Ls}\, G_p(-s)\, ds \qquad (11)$$

from which, replacing $G_p(s)$ by (2), we find

$$\frac{\mu}{2\lambda} = \sum_{i=1}^{n} \left\{ \sum_{l=1}^{m_i} k_{i,l} \left[\frac{1}{2\pi \jmath} \int_{-\jmath\infty}^{+\jmath\infty} \frac{e^{-Ls}}{s+\lambda} \frac{1}{(-s+a_i)^l} ds\right] \right\}. \qquad (12)$$

Since, according to the residue theorem and taking into account Lemma 2

$$\frac{1}{2\pi \jmath} \int_{-\jmath\infty}^{+\jmath\infty} \frac{e^{-Ls}}{s+\lambda} \frac{1}{(-s+a_i)^l} ds = -\mathrm{Res}_{s=a_i} \left[\frac{e^{-Ls}}{s+\lambda} \frac{1}{(-s+a_i)^l}\right] =$$

$$\frac{(-1)^{l+1}}{(l-1)!} \lim_{s\to a_i} \frac{d^{l-1}}{ds^{l-1}} \left(\frac{e^{-Ls}}{s+\lambda}\right) = \frac{e^{-La_i}}{(a_i+\lambda)^l} \sum_{k=0}^{l-1} \frac{1}{k!} L^k (a_i+\lambda)^k, \qquad (13)$$

condition (12) becomes

$$\frac{\mu}{2\lambda} = \sum_{i=1}^{n} \left\{ \sum_{l=1}^{m_i} k_{i,l} \left[\frac{e^{-La_i}}{(a_i+\lambda)^l} \sum_{k=0}^{l-1} \frac{1}{k!} L^k (a_i+\lambda)^k\right] \right\}. \qquad (14)$$

Remark 2. *For $m_i = 1$ and $k_i = k_{i,1}$, $i = 1, \dots, n$ ($G_p(s)$ with simple poles only), condition (21) reduces to*

$$\frac{\mu}{2\lambda} = \sum_{i=1}^{n} \frac{k_i}{a_i+\lambda} e^{-La_i}$$

which, for $L = 0$ (no delay), implies that the L_2-optimal first-order (rational) transfer function interpolates the original n-th order transfer function at the negative λ of its pole $-\lambda$. Such an interpolation property is well-known in L_2-optimal rational approximation [12], [6]. □

A procedure similar to the one followed to find the expression (10) of the partial derivative with respect to μ can be adopted to express the partial derivative of J with respect to λ leading to:

$$\frac{\partial J}{\partial \lambda} = \frac{\mu}{2\lambda^2} - \frac{1}{\pi \jmath} \int_{-\jmath\infty}^{+\jmath\infty} \frac{1}{(s+\lambda)^2} e^{-Ls} G_p(-s) ds. \tag{15}$$

Substituting (2) for $G_p(s)$ in (15), setting the derivative to zero and taking account of the residue theorem and Lemma 3, we eventually find

$$\frac{\mu}{4\lambda^2} = \sum_{i=1}^{n} \left[\sum_{l=1}^{m_i} k_{i,l} \Big[\frac{e^{-La_i}}{(a_i+\lambda)^{l+1}} \sum_{k=0}^{l-1} \frac{l-k}{k!} L^k (a_i+\lambda)^k \Big] \right]. \tag{16}$$

Remark 3. *For $m_i = 1$ and $k_i = k_{i,1}$, $i = 1, \ldots, n$ ($G_p(s)$ with simple poles only), condition (21) reduces to*

$$\frac{\mu}{4\lambda^2} = \sum_{i=1}^{n} \frac{k_i}{(a_i+\lambda)^2} e^{-La_i}$$

which states that, for $L = 0$ (no delay), also the derivative with respect to s of $G_a(s)$ must equal the derivative of $G_p(s)$ at the negative of the pole of $G_a(s)$. This further interpolation condition is also well-known in the L_2-optimal rational approximation of a scalar system [12] (the corresponding condition for multivariable systems can be found in [6]). □

Finally, taking the partial derivative of (9) with respect to L, one successively finds

$$\frac{\partial J}{\partial L} = \frac{1}{\pi\jmath} \int_{-\jmath\infty}^{+\jmath\infty} \Big[-\frac{\mu s}{s+\lambda} e^{-Ls} \Big] \Big[\frac{\mu}{-s+\lambda} e^{-Ls} - G_p(-s) \Big] ds =$$

$$-\frac{\mu}{\pi\jmath} \int_{-\jmath\infty}^{+\jmath\infty} \Big[\frac{\mu s}{(s+\lambda)(-s+\lambda)} - \frac{s+\lambda-\lambda}{s+\lambda} e^{-Ls} G_p(-s) \Big] ds =$$

$$-\mu \Bigg[\frac{1}{\pi\jmath} \int_{-\jmath\infty}^{+\jmath\infty} \frac{\mu s}{(s+\lambda)(-s+\lambda)} ds - \frac{1}{\pi\jmath} \int_{-\jmath\infty}^{+\jmath\infty} e^{-Ls} G_p(-s)\, ds +$$

$$\frac{1}{\pi\jmath} \int_{-\jmath\infty}^{+\jmath\infty} \frac{\lambda}{s+\lambda} e^{-Ls} G_p(-s)\, ds \Bigg]. \tag{17}$$

Setting (17) to zero with $G_p(s)$ as in (2), we finally obtain

$$g_p(L) - \lambda \sum_{i=1}^{n} \left[\sum_{l=1}^{m_i} k_{i,l} \Big[\frac{e^{-La_i}}{(a_i+\lambda)^l} \sum_{k=0}^{l-1} \frac{1}{k!} L^k (a_i+\lambda)^k \Big] \right] = 0, \tag{18}$$

where, according to the definition of inverse Laplace transform,

$$g_p(L) = \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} e^{-Ls}\, G_p(-s)\, ds. \tag{19}$$

Remark 4. *Obviously, condition (23), which for a $G_p(s)$ with only simple poles reduces to*

$$g_p(L) - \lambda \sum_{i=1}^{n} \left[\frac{k_i}{a_i + \lambda} e^{-La_i} \right] = 0, \tag{20}$$

has no counterpart in L_2–optimal rational approximation. □

The previous results prove the following theorem.

Theorem 1 (Optimality conditions). *Necessary conditions for (1) to be the L_2–optimal approximation of (2) in the interior of the positive octant of the parameter space of (1) are (21), (22) and (23).* ■

4. Solution procedure

The optimality conditions obtained in Section 3 consist of three nonlinear equations in the three unknown parameters of the FOPDT model. For convenience, these conditions are rewritten next:

$$\frac{\mu}{2\lambda} = \sum_{i=1}^{n} \left\{ \sum_{l=1}^{m_i} k_{i,l} \Big[\frac{e^{-La_i}}{(a_i+\lambda)^l} \sum_{k=0}^{l-1} \frac{1}{k!} L^k (a_i+\lambda)^k \Big] \right\}, \tag{21}$$

$$\frac{\mu}{4\lambda^2} = \sum_{i=1}^{n} \left[\sum_{l=1}^{m_i} k_{i,l} \Big[\frac{e^{-La_i}}{(a_i+\lambda)^{l+1}} \sum_{k=0}^{l-1} \frac{l-k}{k!} L^k (a_i+\lambda)^k \Big] \right], \tag{22}$$

$$g_p(L) = \lambda \sum_{i=1}^{n} \left[\sum_{l=1}^{m_i} k_{i,l} \Big[\frac{e^{-La_i}}{(a_i+\lambda)^l} \sum_{k=0}^{l-1} \frac{1}{k!} L^k (a_i+\lambda)^k \Big] \right]. \tag{23}$$

Comparing (21) with (22), we obtain

$$\sum_{i=1}^{n} \left[\sum_{l=1}^{m_i} k_{i,l} \frac{e^{-La_i}}{(a_i+\lambda)^l} \Big[\sum_{k=0}^{l-1} \frac{1}{k!} \frac{a_i + \lambda(1-2(l-k))}{a_i+\lambda} L^k (a_i+\lambda)^k \Big] \right] = 0 \tag{24}$$

which contains only parameters λ and L, and, substituting expression (3) with $t = L$ for $g_p(L)$ in (23), we get

$$\sum_{i=1}^{n} (\sum_{l=1}^{m_i} k_{i,l} e^{-La_i} \left[\frac{L^{l-1}}{(l-1)!} - \lambda \frac{1}{(a_i+\lambda)^l} \sum_{k=0}^{l-1} \frac{1}{k!} L^k (a_i+\lambda)^k \right] = 0 \tag{25}$$

which also does not contain μ.

Therefore, to find the triplet of parameter values where the partial derivatives of the index (9) are equal to zero, this procedure can be followed:

- solve for λ and L the system of two equations (24)–(25),
- using these values of λ and L, compute directly the value of μ from

$$\mu = 2\lambda \sum_{i=1}^{n} \left[\sum_{l=1}^{m_i} k_{i,l} \left[\frac{e^{-La_i}}{(a_i+\lambda)^l} \sum_{k=0}^{l-1} \frac{1}{k!} L^k (a_i+\lambda)^k \right] \right] \qquad (26)$$

which corresponds to (21). □

When all of the poles of $G_p(s)$ are simple, equations (24), (25) and (26) simplify, respectively, to

$$\sum_{i=1}^{n} \frac{a_i-\lambda}{a_i+\lambda} \frac{k_i}{a_i+\lambda} e^{-a_i L} = 0, \qquad (27)$$

$$\sum_{i=1}^{n} a_i \frac{k_i}{a_i+\lambda} e^{-a_i L} = 0, \qquad (28)$$

$$\mu = 2\lambda \sum_{i=1}^{n} \frac{k_i}{a_i+\lambda} e^{-a_i L}. \qquad (29)$$

Remark 5. *To solve the nonlinear system (24)–(25) or (27)–(28), resort can be made to standard Matlab® tools, such as the* `fsolve` *function based on the quadratically convergent Newton–Raphson algorithm which requires the specification of a starting point. This point could be chosen according to the second method of Ziegler–Nichols [19].* □

5. Example

To show how well the responses of the L_2–optimal model fit those of the original system, the procedure outlined in Section 4 is applied next to the system described by the transfer function

$$G_p(s) = \frac{(-0.3s+1)(0.08s+1)}{(2s+1)(s+1)(0.4s+1)(0.2s+1)(0.05s+1)^3} \qquad (30)$$

which has also been considered in [16] and [18]. The resulting FOPDT model turns out to be

$$G_a(s) = \frac{0.281}{s + 0.2682}\, e^{-1.31\, s}. \tag{31}$$

The corresponding (minimal) norm of the impulse–response error is $\|e(t)\| = 0.0137$.

The FOPDT model obtained by Yang and Seested [18] using a genetic algorithm is

$$G_g(s) = \frac{1.05}{3.34s + 1}\, e^{-1.4\, s} \tag{32}$$

for which $\|e(t)\| = 0.01497$, and the FOPDT model obtained by Skogestad [16] is

$$G_s(s) = \frac{1}{2.5s + 1}\, e^{-1.47\, s} \tag{33}$$

which matches exactly the original static gain and, consequently, the steady–state value of the step response, but leads to a significantly larger impulse–response error ($\|e(t)\| = 0.0255$).

The impulse responses of the original model (30) and of the three FOPDT models are shown in Fig. 1, whereas their step responses are shown in Fig. 2.

6. Conclusions

The problem of finding a FOLPDT model of a high–order rational plant in such a way that the L_2 norm of the impulse–response error is minimised, has been considered. Necessary conditions of optimality have been derived using standard tools of analytic function theory for the case of plants of general structure, thus extending previous results. On the basis of these conditions, an easily implementable procedure has been suggested to find the parameters of the optimal model. Its numerical complexity compares favourably with that afforded by alternative metaheuristic procedures. A benchmark example has shown that the approximation based on the L_2 criterion leads to models whose impulse and step responses fit well the original responses.

[1] K.J. Åström and T. Hägglund, *PID Controllers: Theory, Design, and Tuning.* 2nd edition. Instrument Society of America (ISA), Research Triangle Park, NC, USA, 1995.

[2] A. Ferrante, W. Krajewski, A. Lepschy, and U. Viaro, "Convergent algorithm for L2 model reduction", Automatica, vol. 35, no. 2, pp. 75-79, 1999.

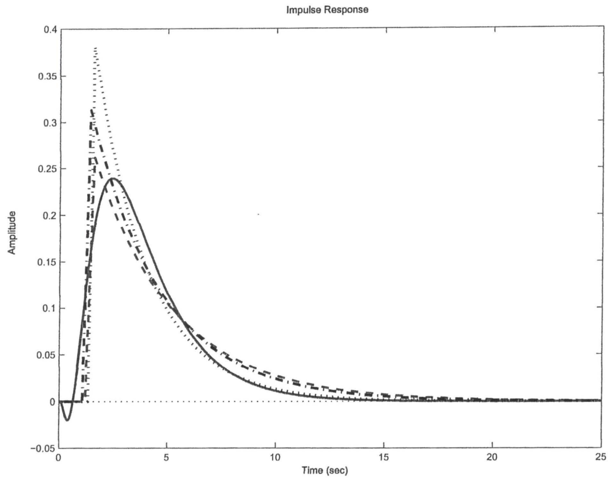

Figure 1: Impulse responses for $G_p(s)$ (solid line), $G_a(s)$ (dashed line), $G_g(s)$ (dashdot line) and $G_s(s)$ (dotted line).

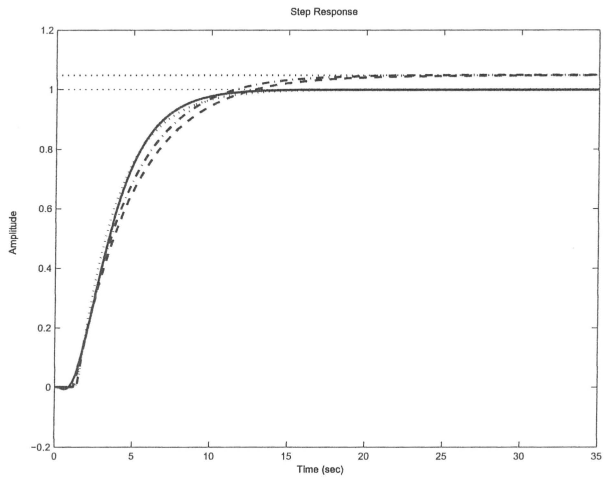

Figure 2: Step responses for $G_p(s)$ (solid line), $G_a(s)$ (dashed line), $G_g(s)$ (dashdot line) and $G_s(s)$ (dotted line).

[3] A. Ferrante, A. Lepschy, and U. Viaro, "Convergence analysis of a fixed–point algorithm", It. J. Pure Appl. Math., no. 9, pp. 179–186, 2001.

[4] A. Ferrante, A. Lepschy, and U. Viaro, "A variant of a convergent fixed–point algorithm that avoids computing Jacobians", It. J. Pure Appl. Math., no. 10, pp. 47–54, 2001.

[5] E. Ghorbani and A. Sedaghati, "FOPDT model fitting to an nth–order all pole NMP process by an optimal method", J. Electrical Eng., vol. 14, edition 3, pp. 87–94, 2014.

[6] W. Krajewski, A. Lepschy, G.A. Mian, and U. Viaro, "Optimality conditions in multivariable L_2 model reduction", J. Franklin Inst., vol. 330, no. 3, pp. 431-439, 1993.

[7] W. Krajewski, A. Lepschy, S. Miani, and U. Viaro, "Frequency-domain approach to robust PI control", J. Franklin Inst., vol. 342, no. 6, pp. 674–687, 2005.

[8] W. Krajewski, A. Lepschy, and U. Viaro, "Designing PI controllers for robust stability and performance", IEEE Trans. Control Syst. Technol., vol. 12, no. 6, pp. 973–983, 2004.

[9] T. Liu, Q.G. Wang, and H.P. Huang, "A tutorial review on process identification from step or relay feedback test", J. Process Control, vol. 23, no. 10, pp. 1597–1623, 2013.

[10] A. Madadi and H.-R. Reza–Alikhani, "Optimal FOPDT model fitting to an nth–order all pole process: Impulse response approach", Proc. IEEE Int. Conf. Control Applications (CCA), Hyderabad, India, 28–30 Aug. 2013, pp. 1129–1134.

[11] J.H. Mathews and K.K. Fink, *Numerical Methods Using Matlab.* 4th edition. Pearson, Upper Saddle River, NJ, USA, 2004.

[12] L. Meier, III, and D.G. Luenberger, "Approximation of linear constant systems", IEEE Trans. Automat. Contr., vol. 12, no. 5, pp. 585–588,1967.

[13] M. Moskowitz and F. Paliogiannis, *Functions of Several Real Variables.* World Scientific, Singapore, 2011.

[14] A. O'Dwyer, *Handbook of PI and PID Controller Tuning Rules.* 3rd edition. Imperial College Press, London, UK, 2009.

[15] G.J. Silva, A. Datta, and S.P. Bhattacharyya, *PID Controllers for Time–Delay Systems.* Birkhäuser, Boston, MA, USA, 2005.

[16] S. Skogestad. "Simple analytic rules for model reduction and PID controller tuning", J. Process Control, vol.13, no.4, pp.291-309, 2003.

[17] D. Xue, Y.Q. Chen, and D.P. Atherton, *Linear Feedback Control: Analysis and Design with MATLAB.* Society for Industrial and Applied Mathematics (SIAM), Series Advances in Design and Control, vol. 14, Philadelphia, PA, USA, 2007.

[18] Z. Yang and G.T. Seested,"Time-delay system identification using genetic algorithm - Part two: FOPDT/SOPDT model approximation", Proc. 3rd IFAC Int. Conf. Intelligent Control and Automation Science, vol. 3, part 1, Chengdu, China, pp. 568–573, 2013.

[19] J.G. Ziegler and N.B. Nichols, "Optimum settings for automatic controllers", Trans. of the American Society of Mechanical Engineers (ASME), vol. 64, no. 11, pp. 759–768, 1942.