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Chapter 1

Fractional–order system forced–response decomposition and its application

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ABSTRACT

This chapter deals with the additive decomposition of the forced response of a fractional–order system. Precisely, it is shown how, by solving a simple polynomial Diophantine equation, this response can almost always be decomposed into the sum of a system–dependent component and an input–dependent component. The system–dependent component is formed from the same modes as the system and, assuming stability, characterises the transient behaviour of the system in the response to sustained inputs. The input–dependent component is formed from the same modes as the input, and accounts for the steady–state or long–term response of a stable system to a persistent input. Simple conditions based on the classical Routh and Mikhailov criteria are provided to check the system input–output stability. Several examples show that the aforementioned decomposition can profitably be exploited to find simplified models in such a way that the asymptotic response is kept unchanged and, at the same time, the transient behaviour is well approximated. The decomposition proves useful also for solving the so–called model–matching problem that is of particular interest in controller synthesis.

KEYWORDS

Rational–order system, Continuous–time system, LTI system, Polynomial Diophantine equation, Stability, Stability criteria, Transient response, Steady–state response, Model reduction

1.1 INTRODUCTION

The theory of fractional–order systems has already attained sufficient maturity to allow its systematic presentation in several books (Azar *et al.*, 2017; Tepljakov, 2017; Kaczorek, 2011; Caponetto *et al.*, 2010; Monje *et al.*, 2010) and to be the subject of many special journal issues (Caponetto *et al.*, 2016; Ionescu *et al.*, 2016; Psychalinos *et al.*, 2016). Despite its increasing popularity, however, some important aspects need further investigation, among them the detailed analysis of the system forced response to inputs with rational–order transform, of which the harmonic and singularity inputs (integrals of the impulse) are distinctive cases. In particular, relatively little attention has been paid to the separate study of the transient and asymptotic responses with some notable exceptions limited to canonical inputs (Monje *et al.*, 2010; Trigeassou *et al.*, 2012; Jakubowska & Walczac, 2016; Semary *et al.*, 2016; Kesarkar & Selvaganesan Narayanasamy, 2016). A more thorough characterisation of the system dynamic behaviour requires the consideration of both the short– and long–term behaviour of the responses to more general inputs. Such an analysis is particularly meaningful in the derivation of simplified models that retain essential properties of the original system, such as stability and performance, and in the synthesis of controllers that satisfy both transient and asymptotic specifications, e.g. on overshoot and steady–state error.

The present contribution aims at a more systematic study of the constituent parts of the forced–response that characterise different aspects of the system behaviour and can conveniently be considered separately. To this purpose, reference is made to the fairly numerous class of inputs with rational–order transform. Following a path similar to that taken in (Casagrande *et al.*, 2017) for integer–order systems and based on (Dorato *et al.*, 1994), the forced response of a stable fractional–order system to a persistent input of this kind is decomposed into the sum of two component: (i) a component with the same pseudo–polynomial denominator as the system transfer function, and (ii) a component with the same pseudo–polynomial denominator as the input transform. The first is called the *system component* of the forced response and the second is called the *input component* because the first is characterised by the same evolution modes of the impulse response, which depends on the system only, and the second by elementary functions that exhibit similar structure (Mittag–Leffler functions) but depend only on the input and will therefore be called "input modes". If there are common modes between the input and the system, a third *resonant component* is also present. However, for the sake of simplicity, this possibility is ruled out (which is necessarily true when the system is stable, so that all of its modes tend to zero as time tends to infinity, and the input is anti–stable, so that all of its modes are persistent).

To ascertain the stability of the rational–order system, resort can be made either to efficient numerical algorithms or to Routh–Hurwitz–like criteria for polynomials with real and complex coefficients. A section of this chapter is dedicated to this problem. As is know, it entails determining the distribution of the roots of a characteristic polynomial with respect to two radii delimiting a sector of the right half–plane (instead of the entire right half–plane, as is the case for integer–order systems).

Robust stability issues are outside the scope of the present contribution and, therefore, are not treated in the sequel. Let us only observe, in this regard, that many results concerning the so–called $\mathcal{D}$–stability can easily be extended to fractional–order systems (Tempo, 1989).

The rest of this chapter is organised as follows. Section 1.2 introduces some essential notation and specifies the families of fractional–order systems and inputs to which the aforementioned decomposition of the forced response applies. Section 1.3 shows how such a decomposition can uniquely be obtained from the Laplace transform of the forced response and, for stable systems, defines its transient and steady–state components. Section 1.4 presents some simple stability conditions. Section 1.5 shows how the decomposition can be used to find simplified models that reproduce the asymptotic response of original complex systems while still approximating well the transient behaviour. Some illustrative examples are worked out in Section 1.6. The results are discussed in Section 1.7 where the relationship between the suggested response decomposition and the model–matching problem, strictly related to controller synthesis, is also pointed out. Possible directions of future research are indicated in Section 1.8.

1.2 NOTATION AND PRELIMINARIES

The transfer function of a continuous–time LTI strictly–proper rational–order system can be written as

$$\widehat{G}(s) = \frac{b_m s^{\frac{m}{q}} + b_{m-1} s^{\frac{m-1}{q}} + \ldots + b_1 s^{\frac{1}{q}} + b_0}{a_n s^{\frac{n}{q}} + a_{n-1} s^{\frac{n-1}{q}} + \ldots + a_1 s^{\frac{1}{q}} + a_0}, \tag{1.1}$$

where q, m, n are positive integers, $m < n$, $q \geq 1$ is the least common denominator (lcd) of the (commensurate) fractional exponents of the Laplace variable s. The numerator and denominator coefficients of (1.1) are assumed to be real.

Consider now the class of inputs whose rational–order Laplace transform can be written as

$$\widehat{U}(s) = \frac{d_k s^{\frac{k}{q}} + d_{k-1} s^{\frac{k-1}{q}} + \ldots + d_1 s^{\frac{1}{q}} + d_0}{c_\ell s^{\frac{\ell}{q}} + c_{\ell-1} s^{\frac{\ell-1}{q}} + \ldots + c_1 s^{\frac{1}{q}} + c_0}, \tag{1.2}$$

where k and ℓ are positive integers, $k < \ell$, and the numerator and denominator coefficients are real. It follows that the lcd of the fractional exponents of both (1.1) and (1.2) is q. This assumption is not much restrictive because it is always possible to express the fractional powers of s in (1.1) and (1.2) in terms of a common lcd, even if this lcd may be larger than the lcd of either function. The class of inputs (1.2) is fairly numerous and includes all inputs whose Laplace transform has only integer powers of s, such as the singularity and harmonic inputs.

By the change of variable

$$w = s^{\frac{1}{q}}, \tag{1.3}$$

functions (1.1) and (1.2) are transformed, respectively, into the following strictly–proper *rational* functions of w:

$$G(w) = \frac{B(w)}{A(w)}, \quad U(w) = \frac{D(w)}{C(w)}, \tag{1.4}$$

where

$$B(w) = b_m w^m + b_{m-1} w^{m-1} + \ldots + b_1 w + b_0, \tag{1.5}$$

$$A(w) = a_n w^n + a_{n-1} w^{n-1} + \ldots + a_1 w + a_0, \tag{1.6}$$

$$D(w) = d_k w^k + d_{k-1} w^{k-1} + \ldots + d_1 w + d_0, \tag{1.7}$$

$$C(w) = c_\ell w^\ell + c_{\ell-1} w^{\ell-1} + \ldots + c_1 w + c_0. \tag{1.8}$$

Correspondingly, the Laplace transform $\widehat{Y}(s)$ of the forced response of the system with transfer function (1.1) to the input with transform (1.2) is converted into the following rational function of w:

$$Y(w) = G(w)\,U(w) = \frac{B(w)\,D(w)}{A(w)\,C(w)}. \tag{1.9}$$

With some abuse of terminology, $G(w)$, $U(w)$ and $Y(w)$ will simply be referred to as system, input and output functions, respectively, because they are directly related via (1.3) to $\widehat{G}(s)$, $\widehat{U}(s)$ and $\widehat{Y}(s) = \widehat{G}(s)\widehat{U}(s)$.

It has been long recognised that the denominator of the rational–order function (1.1) is a multivalued function of s which becomes a single–valued function on a Riemann surface consisting of q sheets with branch cuts along the negative real semi–axis. The first, or principal, sheet contains the *physical* poles of (1.1) (Radwan *et al.*, 2009) corresponding to the so–called structural, or relevant, roots of its denominator (Petráš, 2009). The stability of the rational–order system depends on their location with respect to the imaginary axis. The right half of the first sheet, corresponding to the unstable region, maps into the (minor) sector of the w plane defined by

$$\mathcal{S} \triangleq \left\{ w = \rho e^{\jmath\phi} \; : \; \rho \in \mathbb{R}_+ \, , \, \phi \in \left[-\frac{\pi}{2q}, \frac{\pi}{2q} \right] \right\} . \tag{1.10}$$

As is known, the time–domain expressions of the fractional–order system responses are easily obtained from the partial fraction expansions of the w–domain expressions (Semary *et al.*, 2016; Valerio *et al.*, 2013). It has rightfully been observed in this regard that the Mittag–Leffler functions play for fractional–order systems a role analogous to that played by the exponential modes characterizing the time–domain response of integer–order systems (Rivero *et al.*, 2013; Trzaska, 2008).

Next Section, shows how the expression (1.9) of the forced–response of a fractional–order system can be separated into a component consisting of the same modes as $G(w)$ and a component consisting of the same modes as $U(w)$.

1.3 DECOMPOSITION OF THE FORCED RESPONSE

The rational output (1.9) can be expanded into elementary partial fractions corresponding to its poles which are poles of either $G(w)$ or $U(w)$. If no pole–zero cancellation occurs, *all* (and only) the poles of $G(w)$ and $U(w)$ are poles of $Y(w)$. However, if p is a pole of *both* $G(w)$ and $U(w)$, the multiplicity of pole p in $Y(w)$ is the sum of the multiplicities of the same pole in $G(w)$ and $U(w)$, so that at least one elementary fraction appears in the expansion of $Y(w)$ that is not present in $G(w)$ or $U(w)$. This typically happens in the presence of resonance phenomena where an input frequency coincides with a natural frequency of the system. For the sake of simplicity, in this chapter the following assumption is made. (Indications on the extension of the following procedure to the general case are given in Remark 1 at the end of this section.)

Assumption 1. *Polynomials $A(w)$, $B(w)$, $C(w)$ and $D(w)$ have no common factors.*

Therefore the (strictly–proper) representations (1.4) and (1.9) are irreducible and no cancellation occurs in (1.9); in particular, resonance phenomena are not possible. Under Assumption 1, (1.9) can *uniquely* be decomposed as

$$Y(w) = \frac{X_A(w)}{A(w)} + \frac{X_C(w)}{C(w)} \tag{1.11}$$

where $X_A(w)$ and $X_C(w)$ are the solutions of the polynomial Diophantine equation (Ferrante *et al.*, 2000; Kucera, 1993)

$$X_A(w)C(w) + X_C(w)A(w) = B(w)D(w) \tag{1.12}$$

with $\deg[X_A(w)] < \deg[A(w)] = n$, $\deg[X_C(w)] < \deg[C(w)] = k$ and, by the strict properness of $G(w)$ and $U(w)$, $\deg[B(w)D(w)] < \deg[A(w)] + \deg[C(w)]$. In this case, in fact, equation (1.12) is equivalent to a set of $n + k$ linear equations in the $n + k$ unknown coefficients x_i and y_i of polynomials:

$$X_A(w) = \quad x_{A,n-1}w^{n-1} + x_{A,n-2}w^{n-2} + \cdots + x_{A,1}w + x_{A,0}\,, \tag{1.13}$$

$$X_C(w) = \quad x_{C,n-1}w^{k-1} + x_{C,n-2}w^{k-2} + \cdots + x_{C,1}w + x_{C,0}\,, \tag{1.14}$$

obtained by equating the coefficients of the equal powers of w on both sides of (1.12). By properly ordering the unknowns, this set can be written in a matrix form where the coefficient matrix is nonsingular (Antsaklis & Michel, 2006; Henrion, 1998) (being the Sylvester matrix associated with the polynomials $A(w)$ and $C(w)$ that are co-prime by Assumption 1). It follows from Cramer's rule (Brunetti, 2014) that the aforementioned set of equations admits one, and only one, solution. For clarity of exposition, this result is restated next in the form of a proposition.

Proposition 1. *If $A(w)$ and $C(w)$ are co-prime and if*

$$\deg[B(w)D(w)] < \deg[A(w)] + \deg[C(w)]\,,$$

there is a unique pair of polynomials $X_A(w)$ and $X_C(w)$ with $\deg[X_A(w)] < \deg[A(w)]$ *and* $\deg[X_C(w)] < \deg[C(w)]$ *that solves the polynomial Diophantine equation (1.12).*

The two addenda in (1.11) will be denoted by

$$Y_\Sigma(w) = \frac{X_A(w)}{A(w)}\ ,\ Y_U(w) = \frac{X_C(w)}{C(w)} \tag{1.15}$$

and, borrowing the terminology adopted for integer–order systems (Dorato *et al.*, 1994), will be called the *system component* and *input component* of the output, respectively, because $Y_\Sigma(w)$ is characterised by exactly the same modes as the system (1.1) and $Y_U(w)$ by exactly the same modes as the input (1.2).

If the fractional–order system is asymptotically stable (Petráš, 2009), so is also the system component, and its time–domain counterpart, obtainable by inverse Laplace transformation of the s–domain expression corresponding to $Y_\Sigma(w)$ via (1.3), tends asymptotically to zero. In this case, $Y_\Sigma(w)$ can rightfully be referred to as the *transient response* to input $U(w)$. Instead, if the input is persistent, then the time–domain counterpart of the input component $Y_U(w)$ also persists and can rightfully be referred to as the *steady–state response* or, more in general, the *asymptotic response*.

Since very efficient and fast algorithms exist today to find the roots of a polynomial (Akritas *et al.*, 2008; Jenkins & Traub, 1970) and the related computer programs are readily available, the easiest way to check the stability of a fractional–order system is probably to determine numerically the precise location of the roots of $A(w)$ and see whether some of them lie in the instability sector (1.10). Nevertheless, the problem of finding the root distribution with respect to suitable contours (in particular, the perimeter of circular sectors with *bounded* radius, because upper bounds on the "size" of polynomial roots can be determined easily (Hirst & Macey, 1997)) is certainly of interest for other purposes, such as root clustering or $\mathcal{D}$–stability analysis (see Yedavalli, 2014; Gutman & Jury, 1981 and bibliographies therein), transient characterisation, and stability margin evaluation. This problem is discussed in the following section.

Remark 1. *If, contrary to Assumption 1, $A(w)$ and $C(w)$ are not co–prime, they may be factored as:*

$$A(w) = \overline{A}(w)\, I_A(w),\ \ C(w) = \overline{C}(w)\, I_C(w), \tag{1.16}$$

where $I_A(w)$ is the factor of $A(w)$ containing all and only the roots of $A(w)$ that are roots of $C(w)$ too (with their multiplicities), and $I_C(w)$ is is the factor of $C(w)$ containing all and only the roots of $C(w)$ that are roots of $A(w)$ too (with their multiplicities). Clearly, if all of the common roots are simple $I_A(w) = I_C(w)$. Let $I(w) \triangleq I_A(w)\, I_C(w)$. Since the three pairs $[\overline{A}(w), \overline{C}(w)]$, $[\overline{A}(w), I(w)]$ and $[I(w), \overline{C}(w)]$ are co-prime, $Y(w)$ can uniquely be expressed (Ferrante, 2000) as

$$Y(w) = G(w)U(w) = \frac{X_{\overline{A}}(w)}{\overline{A}(w)} + \frac{X_{\overline{C}}(w)}{\overline{C}(w)} + \frac{X_I(w)}{I(w)}, \tag{1.17}$$

where the first addendum is a combination of modes proper to $G(w)$, the second is a combination of modes proper to $U(w)$, and the third is a combination of modes proper to the "interaction" or "resonant" component $Y_R(w) \triangleq X_I(w)/I(w)$. Some of the modes of $Y_R(w)$ are not contained in both $G(w)$ and $U(w)$ because the multiplicities of the roots of $I(w)$ are greater than the multiplicities of the same roots in $A(w)$ and $C(w)$.

As already said, the possibility of decomposing the forced response into a transient and a steady–state component depends on the system stability. The next section deals with the problem of checking this fundamental property.

1.4 STABILITY CONDITIONS

Although Routh–Hurwitz–like conditions have been derived to determine how the roots of the characteristic pseudo–polynomial of a fractional–order system are distributed among the LHP and RHP half–planes of its principal Riemann sheet or its sectors (Liang *et al.*, 2017), no simple rules are as yet available to establish *directly from polynomial* $A(w)$ whether some of its roots belong to given sectors of the w–plane (for arc angles different from π), except for those given in (Ahmed *et al.*, 2006) that deal with very special cases. Indeed, conditions for *all* of the roots of a polynomial, or the eigenvalues of a matrix, to lie inside a minor LHP sector symmetric with respect to the real axis have been obtained in the Seventies (Gutman, 1979; Anderson *et al.*, 1974) from properties of Kronecker products of matrices (Graham, 1981) or rational maps (Gutman & Jury, 1981). However, the same result, i.e., the confinement of all the roots in the aforementioned minor LHP sector (no roots in the corresponding major sector), had already been obtained well before by means of Routh–Hurwitz arguments (Usher, 1957; Luthi, 1942-43) or could easily have been achieved based on generalisations of the Routh–Hurwitz criteria (Hurwitz, 1895; Routh, 1877) to polynomials with complex coefficients (Frank, 1946; Billarz,1944). New formulations, extensions and improvements of similar algebraic conditions, *including the analysis of the critical cases* and different tabular–form presentations, can be found in (Sivanandam & Sreekala, 2012; Chen & Tsai, 1993; Benidir & Picinbono, 1991; Agashe, 1985; Hwang & Tripathi, 1970) and, more recently, in (Bistritz, 2013) where numerically very efficient variants are presented. A different approach has been followed in (Kaminski *et al.*, 2015) where, for $q > 1$, a test based on regular chains for semi–algebraic sets (Chen *et al.*, 2013) has been suggested. Here, some simple conditions based on the direct application of the Routh test to $A(w)$ are suggested to check whether some (not necessarily all) roots of $A(w)$ lie in an RHP sector symmetric with respect to the real axis.

To this purpose, consider the "forbidden" RHP sector defined by (1.10) (similar considerations hold, of course, for the opposite sector). If no roots of $A(w)$ are in the RHP, which may be checked by means of the standard Routh test, then this sector, as well as the two sectors containing the points of the RHP which do not belong to $\mathcal{S}$, do not contain any root, too. Therefore, the method in (Usher,

1957) can be adopted to determine the number of roots inside any LHP sector symmetric with respect to the negative real semi–axis.

Also, since the number of the real roots in any interval of the real axis can be found easily on the basis of the classic Sturm algorithm (see, e.g., the lecture notes in (Jia, 2016)), whose computational complexity is not greater than that of the Routh algorithm (i.e., $O(n^2)$), only the root distribution of the roots with a nonzero imaginary part need actually be determined. Therefore, for notational simplicity and without loss of generality, the following assumption is made.

Assumption 2. *The real polynomial $A(w)$ has no real roots.*

It follows that the degree n of $A(w)$ is even because its complex roots with nonzero imaginary part are in conjugate pairs.[a]

By combining the information on the root distribution with respect to the imaginary axis with the information on the root distributions with respect to each of the two slanted straight lines with slope $\pm\pi/2q$ (see again (1.10)), some interesting results can be established straightaway. To state them in a compact form, the following notation, illustrated in Fig. 1.1, is introduced:
(i) n_{1u}, n_{1l} denote the number of roots above and, respectively, below the slanted line through the origin with *positive* slope $\pi/2q$,
(ii) n_{2u}, n_{2l} denote the number of roots above and, respectively, below the slanted line through the origin with *negative* slope $-\pi/2q$, and
(iii) n_+, n_- denote the number of roots in the closed RHP and in the open LHP, respectively,
(iv) the *difference* between the number of roots in $\mathcal{S}$ (unstable roots) and the LHP sector (symmetric of $\mathcal{S}$) is denoted by δ.

Clearly,

$$\delta = n_{2u} - n_{1u} = n_{1l} - n_{2l} = n_{2u} - n_{2l} = n_{1l} - n_{1u} \,. \tag{1.18}$$

The following result is obvious.

Proposition 2. *If $\delta > 0$ the fractional–order system is unstable.*

When $\delta \leq 0$, the system may be stable, but the stability conditions also depend on n, n_+ and $n_- = n - n_+$. By considering that, in the absence of real roots, *all of the above numbers are even*, the following two stability conditions can easily be proved. The first only requires the knowledge of n.

Proposition 3. *If $\delta = -i$, i nonnegative, and $n < i + 4$, the fractional–order system is stable.*

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[a] In other words, $A(w)$ can be though of as the even–degree factor containing all of the complex conjugate roots of an original polynomial $\tilde{A}(w) = A(w)A_r(w)$, where $A_r(w)$ is the factor containing all of the real roots of $\tilde{A}(w)$. For simplicity, the following analysis refers to $A(w)$ only. The results, however, can easily be extended to include the real roots.

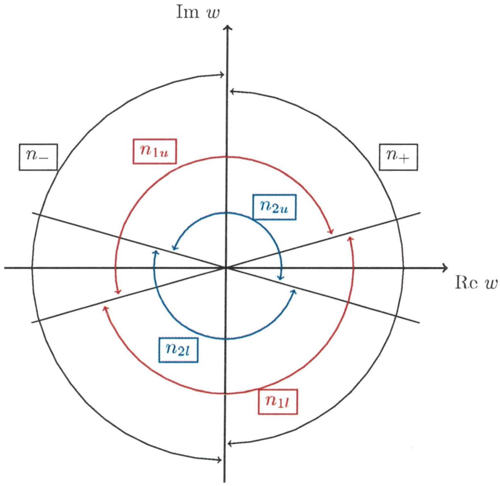

FIGURE 1.1 Notation for: (i) the number of roots in each of the two half–planes separated by each of the two slanted straight lines through the origin with opposite slope (n_{1u}, n_{1l} and n_{2u}, n_{2l}), and (ii) the number of roots in each of the two half–planes separated by the vertical axis (n_+ and n_-).

Proof. Assume that the fractional–order system is unstable. Since all roots appear in conjugate pairs, the number of roots in the instability sector is 2 or more. Therefore, in order for $\delta = -i$, the sector opposite to the instability sector must contain at least $i + 2$ roots, and the polynomial degree n, which is greater than, or equal to, the sum of the roots in both sectors, must be equal, at least, to $i + 4$, contrary to the assumption that $n < i + 4$. □

Proposition 4. *If* $\delta = -i$, *i nonnegative,* $n_- = i + j$, *j nonnegative, and* $n < i + j + 2$, *the fractional–order system is stable.*

Proof. It is enough to consider that, under the adopted assumptions, $n - n_- < 2$ so that no root may lie in the instability sector. □

More general stability conditions require the acquisition of additional information, which in some cases may be worthwhile. For instance, to determine whether some roots lie inside the instability sector $\mathcal{S}$, the slope of the two slanted lines (i.e., the angle $\frac{\pi}{2q}$) can gradually be taken to zero. If the difference between the numbers of roots in the LHP and RHP sectors decreases *monotonically* to zero as the sectors angle tends to zero, then the system is stable. A similar procedure can be applied to detect roots with damping factor in a given range.

By simple adaptation to fractional order systems of the classic Mikhailov stability criterion for integer–order systems (Busłowicz, 2008; Mikhailov, 1938), , the following graphically–based criterion also holds.

Proposition 5. *The fractional–order system is stable if and only if the phase variation of the nth degree polynomial*

$$\widetilde{A}(\rho) \triangleq A(\rho\, e^{J\frac{\pi}{2q}}) \tag{1.19}$$

as ρ *varies from* 0 *to* $+\infty$ *is equal to* $n\,\frac{\pi}{2q}$*:*

$$\Delta \arg[\widetilde{A}(\rho)]_{[0,\infty)} = n\,\frac{\pi}{2q}. \tag{1.20}$$

Proof. Express $A(\rho\, e^{J\frac{\pi}{2q}})$ in factored form as

$$A(\rho\, e^{J\frac{\pi}{2q}}) = a_n \prod_{i=1}^{n} (\rho\, e^{J\frac{\pi}{2q}} - p_i), \tag{1.21}$$

where the p_i, $i = 1, 2, \ldots, n$, are the (possibly repeated) roots of $A(w)$. The phase variation of (1.21) as point $\rho\, e^{J\frac{\pi}{2q}}$ moves along the slanted half–line leaving the origin and making an angle $\pi/(2q)$ with the positive real axis is the sum of the phase variations of its factors which can be regarded as vectors applied at the p_i's and pointing to $\rho\, e^{J\frac{\pi}{2q}}$. Now, if a *real* root, say p_k, is outside the instability sector and thus negative, the initial phase of the corresponding factor (when ρ is equal to zero and the point $\rho\, e^{J\frac{\pi}{2q}}$ coincides with the origin) is zero and its final phase (when $\rho\, e^{J\frac{\pi}{2q}}$ tends to infinity along the aforementioned slanted half–line) is $\pi/(2q)$. If a *complex* root, say p_h, is outside the instability sector, consider the two factors associated with the pair of conjugate poles p_h and $p_h^\star$. The sum of the initial phases of the two vectors $\rho\, e^{J\frac{\pi}{2q}} - p_h$ and $\rho\, e^{J\frac{\pi}{2q}} - p_h^\star$ is also zero, while the final sum of their phases is $2 \cdot \pi/(2q)$, as shown in Fig. 1.2. Therefore, if all the n roots are outside the instability sector, the overall phase variation is (1.20), which proves necessity. The sufficiency of (1.20) can be proved by contradiction. To this purpose, assume that (1.20) holds true but that a real root, say p_j, lies inside the instability sector. The phase variation of the factor associated with p_j, i.e., $\rho\, e^{J\frac{\pi}{2q}} - p_j$, is $-\,[\pi - \pi/(2q)]$ (see Fig. 1.2) so that the overall phase variation is less than (1.20), which contradicts the assumption that (1.20) holds true. A similar reasoning can be used for a pair of conjugate roots inside the instability sector.

Once system stability has been ascertained, efficient approximation methods can be applied to simplify an original complex model. The next section is devoted to such a problem.

1.5 MODEL REDUCTION

The separate consideration of the two components (1.15) of (1.9) can be used for analysis, synthesis and approximation purposes. In this section, it is shown

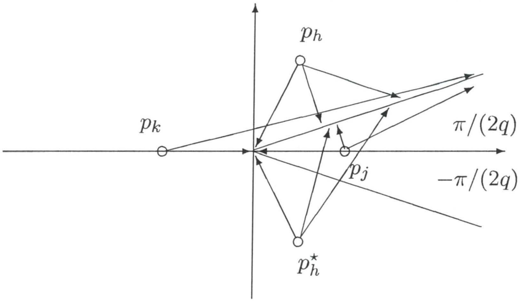

FIGURE 1.2 Vector representation of the phase variation of the factors in (1.20) associated with either real or complex roots outside or inside the RHP minor sector with central angle π/q straddling the positive real axis.

how to obtain reduced–order models that retain the original asymptotic response along the lines of (Casagrande *et al.*, 2017). Essentially, the suggested procedure operates as follows.

1.5.1 Approximation procedure

(i) Find the fractional–order system transfer function (1.1) and determine the Laplace transform (1.2) of the input whose response is of interest.
(ii) Via the change of variable (1.3), convert the Laplace transform of the related response into the rational function $Y(w)$ (see (1.9)).
(iii) Decompose $Y(w)$ according to (1.11) into a system component $Y_\Sigma(w)$ and an input component $Y_U(w)$ (see (1.15)).
(iv) Find a rational function $Y_\Sigma^\nu(w)$ of order $\nu < n$ (usually, $\nu \ll n$) approximating the original system component $Y_\Sigma(w)$ according to any criterion for rational approximation.
(v) Form the reduced–order w–domain transfer function $G_r(w)$ of the reduced–order model in such a way that the reduced–order model response $Y_r(w)$ to $U(w)$ admits $Y_U(w)$ as its input component and $Y_\Sigma^\nu(w)$ as its system component up to an auxiliary additive term of negligible importance $Y_\Sigma^a(w)$, namely:

$$Y_r(w) = G_r(w)\,U(w) = Y_U(w) + Y_\Sigma^\nu(w) + Y_\Sigma^a(w). \tag{1.22}$$

(vi) Construct the simplified transfer function $\widehat{G}_r(s)$ from $G_r(w)$ using again (1.3).

To clarify step (v), some remarks are opportune.

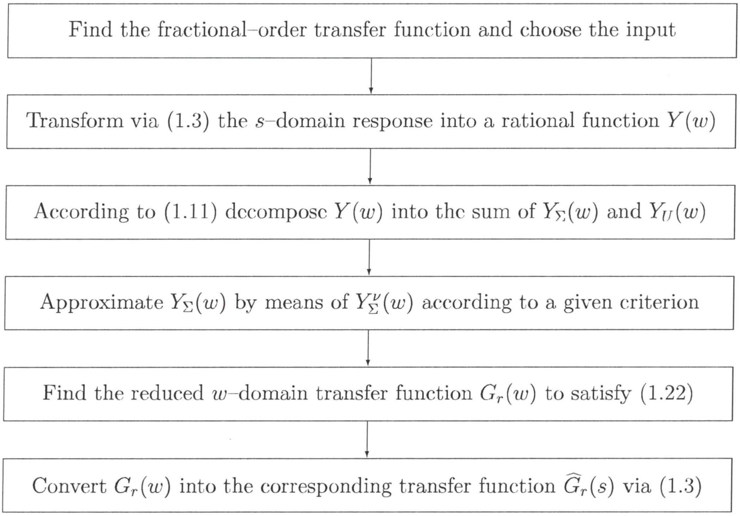

FIGURE 1.3 Basic flow chart of Procedure 1.5.1.

Remark 2. *As explained in detail in (Casagrande* et al., *2017), the introduction of the auxiliary term is necessary to make the number of unknowns in (1.22) equal to the number of equations. If the poles of $Y_\Sigma^a(w)$ are fixed, this result is obtained when the orders of $Y_\Sigma^a(w)$ and $U(s)$ are equal, i.e., the degree of the denominator of $Y_\Sigma^a(w)$ coincides with the degree n_u of the denominator of $U(w)$. The solution is then obtained by equating the coefficients of the equal powers of w at the numerators of the product $W^r(w)\,U(w)$ and of $Y_U(w)+Y_\Sigma^\nu(w)+Y_\Sigma^a(w)$, respectively. The problem turns out to be linear.* □

Remark 3. *If the poles of $Y_\Sigma^a(w)$ are located far to the left of the imaginary axis, this additional term does not alter appreciably the transient dynamics of the system while it leaves unchanged the input component.* □

Remark 4. *Due to the introduction of the auxiliary term, the order r of $W^r(w)$ is greater than the order ν of the function $Y_\Sigma^\nu(w)$ approximating the original system component $Y_\Sigma(w)$. However, since usually $\nu \ll n$, the order $r = \nu + n_u$ is still much smaller than n for the canonical inputs ($n_u \leq 2$ for steps, ramps and sinusoids).* □

Procedure 1.5.1 is schematically represented in Fig. 1.3. It has been applied to several benchmark examples with considerable success; three of them are illustrated in the next section.

1.6 EXAMPLES

The following examples show that the response of the simplified model to the desired input matches closely the original response even during the transient.

1.6.1 Example 1

Consider the fractional–order system described by the transfer function

$$\widehat{G}(s) = \frac{s^4 + 9s^{3.2} + 31s^{2.4} + 58.01s^{1.6} + 60.01s^{0.8} + 16.03}{s^{4.8} + 6s^4 + 48s^{3.2} + 286s^{2.4} + 935s^{1.6} + 1580s^{0.8} + 888}, \tag{1.23}$$

which has also been adopted in (Tavakoli–Kakhki & Haeri, 2009) and (Jiang & Xiao, 2015), and assume that the system is driven by the input

$$\widehat{U}(s) = \frac{1}{s^{0.8}}. \tag{1.24}$$

Remark 5. *Input (1.24) can also be written as*

$$\widehat{U}(s) = s^{0.2} \cdot \frac{1}{s} \tag{1.25}$$

whose time–domain counterpart is

$$\hat{u}(t) = D^{0.2}\mathcal{H}(t) \tag{1.26}$$

which is the fractional derivative of order 0.2 *of the usual step function* $\mathcal{H}(t)$*. It seems reasonable to consider the inputs with Laplace transform:*

$$\widehat{U}(s) = \frac{1}{s^{\alpha}} \tag{1.27}$$

and time–domain expression (Caponetto et al.*, 2010)*

$$u(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \tag{1.28}$$

as the fractional–order equivalents of the canonical inputs for integer–order systems (also called singularity inputs (Tewari, 2011)). Fig. 1.4 shows the time course of (1.28) for three values of α*.* □

By the change of variable

$$w = s^{\frac{1}{5}} \tag{1.29}$$

the response of system (1.23) to input (1.24) is given in the w–domain by the 24th–order rational function

$$Y(w) = G(w)\,U(w) =$$

$$= \frac{w^{20} + 9w^{16} + 31w^{12} + 58.01w^8 + 60.01w^4 + 16.03}{w^{24} + 6w^{20} + 48w^{16} + 286w^{12} + 935w^8 + 1580w^4 + 888} \cdot \frac{1}{w^4} \tag{1.30}$$

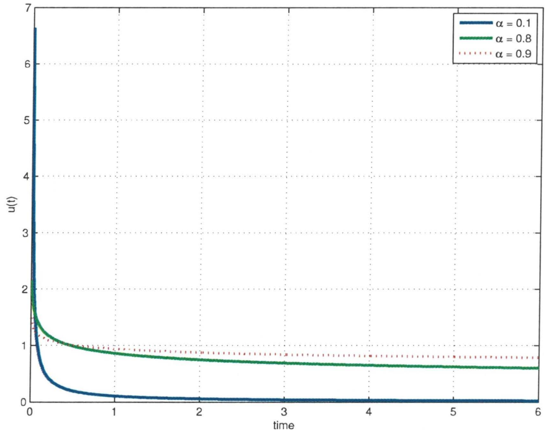

FIGURE 1.4 Input signals of form (1.28) with $\alpha = 0.1$ (lower solid line), $\alpha = 0.8$ (upper solid line), and $\alpha = 0.9$ (dotted line).

whose poles are outside the minor sector delimited by the radii $\rho e^{\pm \jmath \frac{\pi}{10}}$ (see equation (1.10)) so that the fractional–order system is stable.
Since, in this special case, all powers of w are multiples of the same integer 4, the decomposition and reduction procedures can more conveniently be applied to the 6th–order rational function

$$\widetilde{Y}(z) = \frac{z^5 + 9z^4 + 31z^3 + 58.01z^2 + 60.01z + 16.03}{z^6 + 6z^5 + 48z^4 + 286z^3 + 935z^2 + 1580z + 888} \cdot \frac{1}{z}, \qquad (1.31)$$

obtained from (1.30) by setting $z = w^4$. Clearly, the poles of (1.31) are the fourth powers of the poles of (1.30), which means, in particular, that the instability sector in the z–plane is delimited by the radii $\rho e^{\pm \jmath \frac{4\pi}{10}}$ instead of the radii $\rho e^{\pm \jmath \frac{\pi}{10}}$ that enclose the instability sector in the w–plane. Of course, (1.31) could directly be obtained from (1.23) by setting $z = s^{4/5}$.
Function (1.31) can be decomposed into a system–dependent and an input–dependent component as

$$\widetilde{Y}(z) =$$

$$= \frac{-0.0181z^5 + 0.8917z^4 + 8.1335z^3 + 25.8372z^2 + 41.1316z + 31.4882}{z^6 + 6z^5 + 48z^4 + 286z^3 + 935z^2 + 1580z + 888} + \frac{0.0181}{z}. \qquad (1.32)$$

Applying the shifted Padé approximation method suggested in (Tavakoli-Kakhki & Haeri, 2009) with $s_0 = 133$, we find the following 2nd–order approximation of the system–dependent component (first addendum at the right–hand side of

(1.32)):

$$\widetilde{Y}_{\Sigma}^{2}(z)=\frac{-0.0181z+1.0407}{z^2-2.2538z+40.6760}. \tag{1.33}$$

Adding the input–dependent component (second addendum at the right–hand side of (1.32)) to (1.33) as well as an auxiliary 1st–order term with the far–off pole at $z=-100$ (step (v) of Procedure 1.5.1), we find the following 3rd–order approximation of the z–domain system transfer function:

$$\widetilde{G}_r(z)=\frac{z^2+100.7402z+73.4276}{z^3+97.7z^2-184.7z+4067.6} \tag{1.34}$$

which, via the change of variable $z=s^{4/5}$, corresponds to the stable fractional–order transfer function:

$$\widehat{G}_r(s)=\frac{s^{1.6}+100.7402s^{0.8}+73.4276}{s^{2.4}+97.7s^{1.6}-184.7s^{0.8}+4067.6}. \tag{1.35}$$

Instead, by applying the shifted Padé method with $s_0=133$ directly to (1.23), as in (Tavakoli-Kakhki & Haeri, 2009), without consideration of the input component of the forced response, the following simplified stable fractional–order model is obtained:

$$\widehat{G}_{r,TH}(s)=\frac{s^{1.6}+5.0349s^{0.8}+0.3743}{s^{2.4}+2.0349s^{1.6}+29.2696s^{0.8}+145.3930}. \tag{1.36}$$

The responses to input (1.24) of the reduced system (1.35) obtained according to the method suggested in this section and of the reduced system (1.36) are compared in Fig. 1.5 with the response to the same input of the original system (1.23). It is apparent that the retention of the steady–state component leads to a better approximation in the medium to long run.

Since fractional–order systems are infinite–dimensional, or long–memory, systems (Sabatier *et al.*, 2014), it might be argued that the transfer function of a fractional–order system can be considered "simple" if it contains a small number of parameters. From this point of view, functions (1.35) and (1.36) in the previous subsection are indeed simpler than the original transfer function (1.23). Instead, from the point of view of system dimensionality, any integer–order input–output or state–space model simulating the behaviour of a given fractional–order system can be regarded as a simplified model of the fractional–order system (see, e.g., (Krajewski & Viaro, 2014)). In a sense, the w–domain rational function $G(w)$ itself, having a finite integer order, represents the original fractional system in a more compact form.

In this section, simplification is considered to be achieved if the maximum fractional degree of the s-domain approximation is smaller than the maximum fractional degree of the original transfer function, which corresponds to the fact that the (integer) degree of the denominator of $G_r(w)$ is smaller than that of the denominator of $G(w)$, even if the number of parameters in $G_r(w)$ might

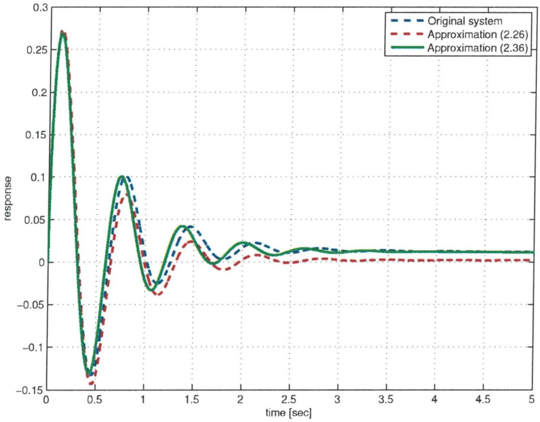

FIGURE 1.5 Responses to $\widehat{U}(s) = 1/s^{0.8}$ of: (i) the original system (1.23) (blue upper dashed line), (ii) the approximation (1.36) (red lower dashed line), and (iii) the approximation (1.35) retaining the steady–state component (green solid line).

sometimes be greater than that in $G(w)$. The last situation typically occurs when a number of intermediate coefficients (between the leading term and the term of lowest degree, usually a constant) in the numerator and denominator of $G(w)$ are missing. Of course, also the choice of the input whose asymptotic term should be preserved influences the model complexity because the fractional powers of s in $\widehat{U}(s)$ contribute to the determination of the minimum common denominator of all fractional exponents of $\widehat{Y}(s) = \widehat{G}(s)\widehat{U}(s)$.

1.6.2 Example 2

As a second example, consider the system analysed in (Tavazoei, 2016; 2011) whose transfer function turns out to be

$$\widehat{G}(s) =$$

$$= \frac{s^{2.7} + 2s^{1.8} + s^{0.9} + 2}{s^{6.3}+4.9s^{5.4}+11.05s^{4.5}+14.07s^{3.6}+10.53s^{2.7}+4.55s^{1.8}+1.05s^{0.9}+0.1}. \tag{1.37}$$

By the change of variable $w = s^{1/10}$, we obtain

$$G(w) =$$

$$= \frac{w^{27} + 2w^{18} + w^{9} + 2}{w^{63}+4.9w^{54}+11.05w^{45}+14.07w^{36}+10.53w^{27}+4.55w^{18}+1.05w^{9}+0.1} \tag{1.38}$$

whose poles are outside the instability sector. Since all powers are multiples of 9, in this case too, the reduction procedure can profitably be applied to an integer–order rational function of smaller degree in the variable $z = w^9$, namely:

$$\widetilde{G}(z) = \frac{z^3 + 2z^2 + z + 2}{z^7 + 4.9z^6 + 11.05z^5 + 14.07z^4 + 10.53z^3 + 4.55z^2 + 1.05z + 0.1} \tag{1.39}$$

whose 3rd–order optimal Hankel–norm approximation (Glover, 1984) turns out to be

$$\widetilde{G}_{r,HN}(z) = \frac{0.5592z^2 - 0.4066z + 0.4178}{z^3 + 0.5841z^2 + 0.1885z + 0.0204} \tag{1.40}$$

and in the s–domain with $z = s^{9/10}$

$$\widehat{G}_{r,HN}(s) = \frac{0.5592s^{1.8} - 0.4066s^{0.9} + 0.4178}{s^{2.7} + 0.5841s^{1.8} + 0.1885s^{0.9} + 0.0204}. \tag{1.41}$$

Let us apply now the procedure based on the retention to the asymptotic response to the input

$$\widehat{U}(s) = \frac{1}{s^{0.9}}. \tag{1.42}$$

To this purpose, the original forced response to input (1.42) is decomposed into the sum of a system–dependent component and an input–dependent component, which in the domain of $z = w^9 = (s^{1/10})^9$ turn out to be, respectively,

$$\widetilde{Y}_\Sigma(z) = \frac{-20z^6 - 98z^5 - 221z^4 - 281.4z^3 - 209.6z^2 - 89z - 20}{z^7 + 4.9z^6 + 11.05z^5 + 14.07z^4 + 10.53z^3 + 4.55z^2 + 1.05z + 0.1}, \tag{1.43}$$

$$\widetilde{Y}_U(z) = \frac{20}{z}. \tag{1.44}$$

The 2nd–order optimal Hankel–norm approximation of (1.43) is

$$\widetilde{Y}^2_\Sigma(z) = \frac{-15.9384z - 9.9977}{z^2 + 0.3796z + 0.0511}. \tag{1.45}$$

By adding to (1.45) an auxiliary term with a far–off pole at -100 (step (v) of Procedure 1.5.1), and combining the resulting sum with the original input–dependent component (1.44), the reduced system transfer function in the z–domain turns out to be

$$\widetilde{G}_r(z) = \frac{402.2097z^2 - 239.7748z + 102.1383}{z^3 + 100.3796z^2 + 38.0102z + 5.1069} \tag{1.46}$$

and in the s–domain with $z = s^{9/10}$

$$\widehat{G}_r(s) = \frac{402.2097s^{1.8} - 239.7748s^{0.9} + 102.1383}{s^{2.7} + 100.3796s^{1.8} + 38.0102s^{0.9} + 5.1069}. \tag{1.47}$$

The responses to (1.42) of the original system (1.37) and of the approximating models (1.41) and (1.47) are shown in Fig. 1.6.

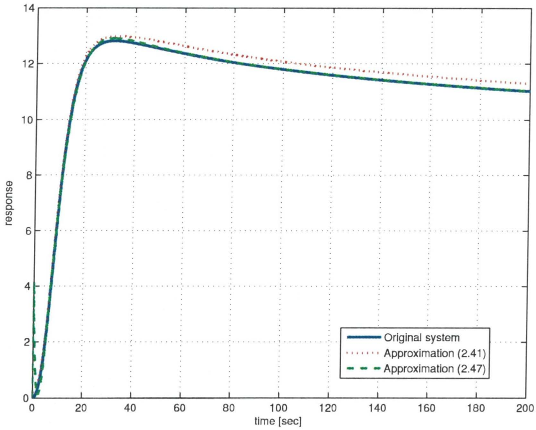

FIGURE 1.6 Responses to $\widehat{U}(s) = 1/s^{0.9}$ of: (i) the original system (1.37) (blue solid line), (ii) the approximating model (1.41) (red dotted line), and (iii) the approximating model (1.47) retaining the asymptotic component of the response (green dashed line).

1.6.3 Example 3

Consider finally the transfer function

$$\widehat{G}(s) = \frac{5s^{0.6} + 2}{s^{3.3} + 3.1s^{2.6} + 2.89s^{1.9} + 2.5s^{1.4} + 1.2} \tag{1.48}$$

taken from (Xue & Chen, 2007), and assume that the input whose asymptotic response must be retained is

$$\widehat{U}(s) = \frac{10}{s^{0.2} - 0.7\,s^{0.1}} \tag{1.49}$$

which has been chosen, rather arbitrarily, to test the system long–term response to non–decaying inputs. The corresponding time–domain signal $\widehat{u}(t)$ is shown in Fig. 1.7. It can be obtained as the step response of a filter with transfer function

$$\widehat{F}(s) = \frac{10\,s^{0.9}}{s^{0.1} - 0.7}\,. \tag{1.50}$$

For $w = s^{0.1}$ function (1.48) becomes

$$G(w) = \frac{5w^6 + 2}{w^{33} + 3.1w^{26} + 2.89w^{19} + 2.5w^{14} + 1.2}\,, \tag{1.51}$$

A 4th–order approximation of (1.51) has been obtained by interpolating (1.51) at $s = 1$ and $s = 2$ with intersection number 2 (retention of 2 time moments)

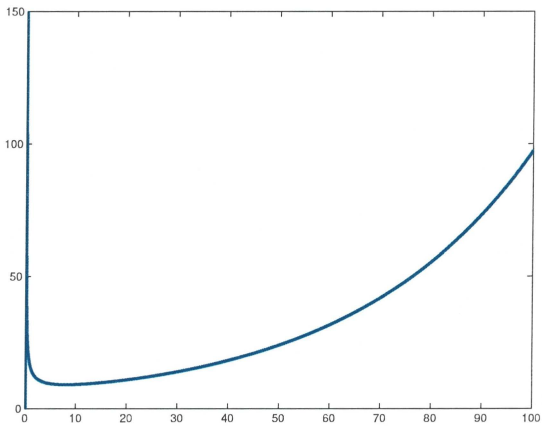

FIGURE 1.7 Time–domain input signal $\widehat{u}(t)$ corresponding to (1.49).

according to the multipoint Padé technique via Lanczos' iteration method (Gallivan *et al.*, 1996). The corresponding simplified fractional–order transfer function turns out to be

$$\widehat{G}_{r,PL}(s) = \frac{9.6597s^{0.3} + 50.106s^{0.2} + 56.107s^{0.1} + 4.753}{s^{0.4} + 35.5s^{0.3} + 161.08s^{0.2} + 173.02s^{0.1} + 14.037}. \tag{1.52}$$

By applying instead the suggested reduction method based on:

(i) the decomposition of $Y(w) = G(w)U(w)$ into a system–dependent component

$$Y_{\Sigma}(w) = \frac{X_A(w)}{w^{33} + 3.1w^{26} + 2.89w^{19} + 2.5w^{14} + 1.2} \tag{1.53}$$

and an input–dependent component

$$Y_U(w) = \frac{X_C(w)}{w^2 + w + 100}, \tag{1.54}$$

(ii) the approximation of (1.53) by means of the same method used to find (1.52) from (1.48), and

(iii) the retention of (1.54),

the following approximating fractional–order transfer function is obtained (after substituting $s^{0.1}$ for w)

$$\widehat{G}_r(s) = \frac{68.447s^{0.3} + 357.21s^{0.2} + 528.5s^{0.1} + 157.17}{s^{0.4} + 3.7838s^{0.3} + 103.73s^{0.2} + 279.32s^{0.1} + 94.3}. \tag{1.55}$$

Fig. 1.8 shows the responses to the input with Laplace transform (1.49) of: (i) the original system (1.48), (ii) the approximating model (1.52), and (iii) the approximating model (1.55) retaining the asymptotic response.

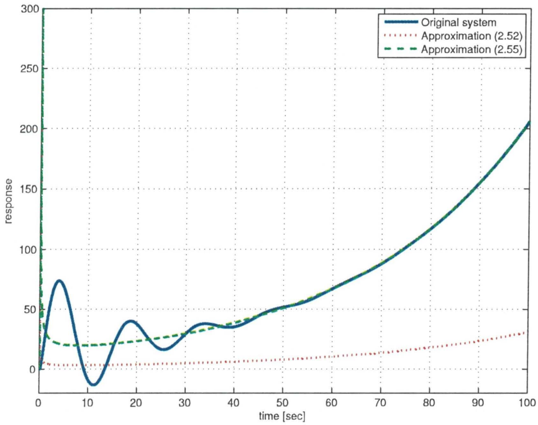

FIGURE 1.8 Responses of: (i) the original model (1.48) (blue solid line), (ii) the approximating model (1.52) (red dotted line), and (iii) the approximating model (1.55) retaining the asymptotic component of the response (green dashed line)

The previous examples show that the suggested response decomposition can be applied satisfactorily in many approximation problems. On the other hand, as outlined in the next section, some difficulties arise in achieving a simplification in terms of number of model parameters. A similar problem arises in the use of the suggested decomposition for solving the model–matching problem, strictly related to controller synthesis, as the example in the next section will show.

1.7 DISCUSSION AND EXTENSIONS

In the previous Section 1.5 the decomposition of the forced response has profitably been applied to the derivation of "simplified" models of fractional–order systems. It has been observed, in this regard, that the definition of model complexity is to some extent arbitrary. It may be related to the (finite) dimension of the integer–order models associated with the fractional–order systems via the variable transformation (1.3), or to the "compactness" of the fractional–order transfer functions, in particular, the number of non–zero parameters that appear in them, or to the maximum degree of the denominator of the transfer functions. The models obtained in the previous section can be considered simpler from all of these points of view. It should be observed, however, that the results strongly depend on the input whose asymptotic component of the forced response must be retained. If the minimum common denominator (mcd) of the fractional exponents of the input transform does not coincide with the mcd of fractional exponents of the original transfer function, forcing $U(w)$ and $G(w)$ to have a common q might entail a considerable increase of the order of the integer–order model obtained via (1.3) from the fractional–order transforms.

Also the reduction criterion adopted to approximate the system component of the forced response is rather arbitrary. Even if its choice is outside the scope of the present contribution, it should be noted that not all methods cannot be applied. In fact, most reduction methods suggested in the literature for integer–order systems are directly applicable only to (stable) systems with poles in the open LHP. To overcome this problem, it is sometimes suggested to preliminarily separate the stable and unstable parts of the systems with RHP poles and then apply the reduction procedure only to the first. Further difficulties arise in the case of fractional–order systems, because their stability is compatible with the presence of RHP poles in the integer–order function derived from the fractional one via (1.3), provided these poles are outside the instability sector.

In this book chapter attention has been focused on the model reduction problem, but the relevance of the response decomposition goes beyond model simplification. Suffice it to recall, in this regard, the interpolation problem, strictly related to the model matching problem (Doyle *et al.*, 1992) or, more generally, the moment matching problem (Astolfi, 2010). Indeed, forcing the coincidence of the input components of two different systems in the response to a given input entails interpolating the values taken by the transfer function of either system at the roots of the denominator of the input transform. To clarify this, consider two systems whose respective transfer functions are $G_1(w) = B_1(w)/A_1(w)$ and $G_2(w) = B_2(w)/A_2(w)$. Under suitable coprimeness assumptions, their responses to $U(w) = D(w)/C(w)$ can be decomposed as

$$Y_1(w) = \frac{B_1(w)}{A_1(w)}\frac{D(w)}{C(w)} = \frac{X_{A_1}(w)}{A_1(w)} + \frac{X_{C_1}(w)}{C(w)} \tag{1.56}$$

and

$$Y_2(w) = \frac{B_2(w)}{A_2(w)}\frac{D(w)}{C(w)} = \frac{X_{A_2}(w)}{A_2(w)} + \frac{X_{C_2}(w)}{C(w)}. \tag{1.57}$$

For $X_{C_1}(w) = X_{C_2}(w) = X_C(w)$ (equality of the input–dependent components), from (1.56) and (1.57) we get

$$B_1(w)D(w) = X_{A_1}(w)C(w) + X_C(w)A_1(w), \tag{1.58}$$

$$B_2(w)D(w) = X_{A_2}(w)C(w) + X_C(w)A_2(w), \tag{1.59}$$

so that, at the roots of $C(w)$, i.e., for $C(w) = 0$, we have

$$G_2(w) = \frac{B_2(w)}{A_2(w)} = \frac{B_1(w)}{A_1(w)} = G_1(w), \tag{1.60}$$

which means that $G_2(w)$ interpolates $G_1(w)$ at the poles of $U(w)$. As is well known, if $G_2(w)$ and $G_1(w)$ are realised in a unity–feedback fashion, this means, in turn, that their forward paths include an internal model of the (common) input (Francis & Wonham, 1976).

The previous considerations have obvious implications on the so–called *direct* or *analytic* synthesis of control systems (Ferrante *et al.*, 2000) whose first step consists in choosing an overall, or total, or complementary sensitivity, system transfer function $T(w)$ that satisfies the specifications, the next step being its realisation, possibly by means of a feedback structure with controller $G_c(w)$ and plant $G_p(w)$ located in the forward path so that

$$T(w) = \frac{G_c(w)G_p(w)}{1 + G_c(w)G_p(w)} \tag{1.61}$$

and

$$G_c(w) = \frac{1}{G_p(w)} \frac{T(w)}{1 - T(w)}. \tag{1.62}$$

In the case of fractional–order systems, to profit by the efficient techniques developed for integer–order systems, the rational function $T(w)$ will be obtained, via (1.3), from an original fractional–order transfer function. To facilitate the synthesis procedure, it is convenient to choose the least common denominator (lcd) of the fractional powers in this function equal to the lcd of the powers in the fractional–order process transfer function. Correspondingly, also the lcd of the powers in the resulting fractional–order controller transfer function obtained from (1.62) via (1.3) will be the same.

1.7.1 Example 4

Let the fractional–order transfer function of a given process be

$$\widehat{G}_p(s) = \frac{1}{1 + 10\, s^{0.8}} \tag{1.63}$$

whose response to the input $\hat{u}(t)$ with Laplace transform

$$\widehat{U}(s) = 1/s^{0.8} \tag{1.64}$$

(see Fig. 1.4) is shown in Fig. 1.9 (solid line). Assume that it is desired to speed up the response by resorting to a unity–feedback control system. To this purpose, the complementary sensitivity function of such a feedback system is chosen to be

$$\widehat{T}(s) = \frac{1}{1 + s^{0.8}} \tag{1.65}$$

whose response to $\hat{u}(t)$ is compared to that of $\widehat{G}_p(s)$ in Fig. 1.9.

This choice also ensures that the response to the aforementioned input contains an input-dependent component equal to the input itself because $\widehat{Y}(s) = \widehat{T}(s)\widehat{U}(s)$ can be decomposed as

$$\widehat{Y}(s) = -\frac{1}{1 + s^{0.8}} + \frac{1}{s^{0.8}},$$

so that it tends quickly to the chosen reference input.

By the change of variable $w = s^{0.1}$, $\widehat{G}_p(s)$ is transformed into

$$G_p(w) = \frac{1}{1 + 10w^8} \tag{1.66}$$

and $\widehat{T}(s)$ into

$$T(w) = \frac{1}{1 + w^8} \tag{1.67}$$

so that, according to (1.62), the controller transfer function in w-domain turns out to be

$$G_c(w) = \frac{1 + 10w^8}{w^8}, \tag{1.68}$$

whence

$$\widehat{G}_c(s) = \frac{1 + 10\, s^{0.8}}{s^{0.8}}. \tag{1.69}$$

As is expected, this controller contains an internal model of the input transform (Francis & Wonham, 1976).

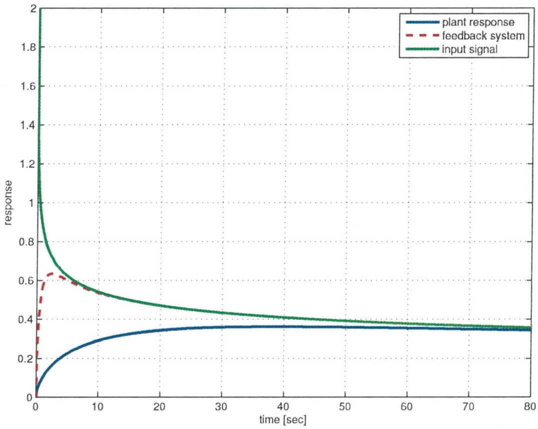

FIGURE 1.9 Responses to input (1.64) of: (i) the plant (1.63) (blue lower solid line), and (ii) the feedback control system (1.65) (red dashed line). The input (1.64) is represented by the green upper solid line.

1.8 CONCLUSIONS

It has been shown that the forced response of a fractional–order system to an input belonging to a very numerous class can uniquely be decomposed into a system component and an input component, as is the case also for integer–order systems. The first is characterised by the same modes as the system and the second by the same modes as the input. Therefore, if the system is asymptotically stable and the input is persistent, the input component corresponds to the steady–state or asymptotic response to the selected input whereas the system component corresponds to the transient response.

To ascertain whether the fractional–order system is BIBO stable without computing numerically the system poles, resort can be made to the Routh–Hurwitz criteria for complex polynomials which allow us to determine the root distribution with respect to any straight line of the complex plane. On the basis of these criteria, simple stability and instability conditions have been provided that partly extend previous results of the same kind presented in the literature.

The response decomposition can be used in various contexts, ranging from system approximation to system analysis and synthesis. In particular, it has been applied to find a simplified model that retains the asymptotic behaviour of the original system in the response to characteristic inputs, a result that most model–reduction techniques do not ensure even for integer–order systems. As shown by some examples, in most cases the approximation of the corresponding system component is not appreciably affected by the requirement of steady–state retention, which only entails a usually small increase of the reduced model order depending on the input complexity.

Future research directions along the same lines include: (i) the characterisation of the transient output component in the response to suitable inputs, (ii) the synthesis of feedback controllers that ensure the desired asymptotic behaviour, and (iii) the extension of the decomposition procedure to systems of non–rational order.

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