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HYBRID LEVEL–SET PHASE FIELD METHOD IN SHAPE OPTIMIZATION

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Abstract.
The paper deals with the analysis and numerical solution of the topology optimization of system governed by the variational inequalities using the combined level set and phase field rather than standard level set approach. Standard level set method allows to evolve a given sharp interface but is not capable to genetrate holes unless the topological derivative is used. The phase field method indicates the position of the interface in a blurry way but is flexible in hole generation. In the paper two-phase topology optimization problem is formulated in terms of the modified level set method and regularized using Cahn-Hilliard based interfacial energy term rather than the standard perimeter term. The derivative formulae of the cost functional with respect to the level set function is calculated. Modified reaction-diffusion equation updating the level set function is derived. The necessary optimality condition for this optimization problem is formulated. The finite element and finite difference methods are used to solve the state and adjoint systems. Numerical examples are provided and discussed.

Keywords: topology optimization, unilateral problems, level set approach, phase field method

MSC 2010: 35J86, 49K20, 49Q10, 49Q12, 74N20, 74P10

1. INTRODUCTION

Shape or topology optimization problems of systems governed by PDEs arise in many applications. Examples include different branches of industry, biology or image processing [1, 2, 11, 12]. The paper is concerned with the topology and/or shape optimization problem for an elastic body in unilateral contact with a rigid foundation. The contact phenomenon with Tresca friction is governed by the second order elliptic variational inequality [3, 15, 30]. The structural optimization problem consists in finding such material distribution in a given design domain occupied by the body and/or the shape of its boundary that the normal contact stress along the boundary of the body is minimized.

Shape and topology optimization problems are studied in literature both from analytical point of view as well as numerical. Topology optimization problems are usually ill-posed and require regularization [6, 8, 21, 27, 28, 30, 33]. The existence results for this class of optimization problems may be found in [6, 7]. The material derivative [30] or topology derivative methods [29] are employed to calculate the derivatives of the cost functional with respect to the shape boundary variations or to the inserting or removing a void (hole) from the material of the body, respectively and to formulate a necessary optimality condition.

Many successful numerical methods have been proposed to solve shape and topology optimization problems. For the review of these methods see [11, 12]. Especially, Simple Isotropic Material Penalization metod, Evolutionary Structural Optimization approach [10] or topology derivative method [29] are the main methods used to solve topology optimization problems. Recently the use of the level set methods [26] and the phase field methods [13, 16, 19, 20, 25] has been proposed to solve the topology optimization problems [3, 4, 5, 6, 7, 8, 12, 21, 24, 27, 31, 32, 33]. In numerical algorithms of structural optimization the level set method is employed for capturing the evolution of the domain boundary on a fixed mesh and finding an optimal domain [1, 2]. The level set method is a simple and versatile method to compute and analyze the motion of an interface in two or three dimensions. It is based on the implicit representation of the boundaries of the optimized structure. It introduces a continuous auxiliary function over the whole global domain and embedds the optimized domain interface as the zero level set of this higher dimensional function, i.e. the position of the domain boundary is described as an isocountour of a scalar function of a higher dimensionality. In standard level set approach the evolution of the domain boundary is governed by Hamilton - Jacobi equation. The speed vector field driving the propagation of the level set function is given by the Eulerian derivative of the cost functional with respect to the variations of the free boundary. Therefore the interface is propagated exactly as the zero level set of the level set function. Since the standard level set method is not capable to generate voids, requires reinitialization and the use of Heaviside and Dirac functions it has been generalized to reduce these drawbacks. Especially, binary and piecewise constant level set functions have been introduced. Using two phase formulation of the original topology optimization problem and the radial [22] or the piecewise constant level set approach the evolution of the domain boundary is governed by the gradient flow equation.

While level-set methods have become an accepted tool in structural topology optimization the use of phase field methods in this field has not yet become popular. The topology optimization problem in multiphase setting can be transformed further into a phase field problem where the optimal topology is characterized as the steady state of the phase transition. Phase field models in the form of Cahn-Hilliard or

Allen-Hillard equations [6, 7, 10, 14, 19, 32] have been first introduced in metalurgy to describe phase separation in binary alloy systems. Next these approaches have been used to provide mathematical models in different areas, including crack propagation, image processing, tumor growth. Phase field models have many similarities with the level set approach. The basic concept of the phase field model is the representation of two fluid or material phases by two minima of a double-well potential with a smooth transition region representing the interface. The form and width of the transition region between the two phases gives rise to the surface tension forces. The phase field method can be considered as a physically motivated level set method. The evolution equations for the smooth fields corresponding to the phase field variable are obtained using a variational approach associated to searching minimum of the corresponding free energy or entropy.

The paper is concerned with the analysis and numerical solution of the topology optimization of system governed by the elliptic variational inequalities modelling the elastic unilateral contact problem with Tresca friction. The aim of the optimization problem is to find such distribution of the material of the body in unilateral contact with the rigid foundation to minimize normal contact stress. The combined level set and phase field rather than standard level set approach is used. Two-phase topology optimization problem is formulated in terms of the modified level set function. This problem is regularized using Cahn-Hilliard interface energy term rather than the perimeter term. Derivatives formulae of the cost functional with respect to the level set function are calculated. Interface evolution is governed by the modified gradient flow equation of reaction-diffusion type. The necessary optimality condition for this optimization problem is formulated. The numerical implementation issues are described. Numerical examples are provided and discussed.

2. Problem Formulation

Consider deformations of an elastic body occupying two–dimensional domain Ω with the smooth boundary Γ (see Fig. 1). Assume $\Omega \subset D$ where D is a bounded smooth hold–all subset of R^2. The body is subject to body forces $f(x) = (f_1(x), f_2(x))$, $x \in \Omega$. Moreover, surface tractions $p(x) = (p_1(x), p_2(x))$, $x \in \Gamma$, are applied to a portion Γ_1 of the boundary Γ. We assume, that the body is clamped along the portion Γ_0 of the boundary Γ, and that the contact conditions are prescribed on the portion Γ_2, where $\Gamma_i \cap \Gamma_j = \emptyset$, $i \neq j$, $i, j = 0, 1, 2$, $\Gamma = \bar{\Gamma}_0 \cup \bar{\Gamma}_1 \cup \bar{\Gamma}_2$.

Let $\rho = \rho(x) : \Omega \to R$ denote the material density function at any generic point x in a design domain Ω. It is a phase field variable taking value close to 1 in the presence of material, while $\rho = 0$ corresponds to regions of Ω where the material is absent, i.e. there is a void. In the phase field approach the interface between material

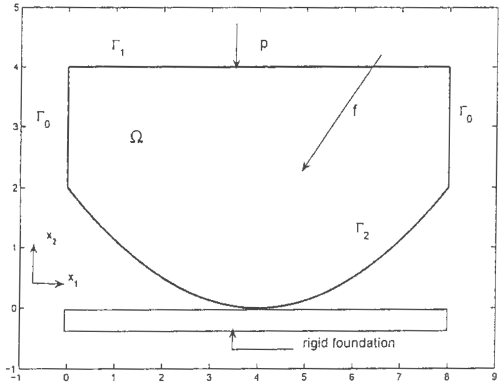

FIGURE 1. Initial domain Ω.

and void is described by a diffusive interfacial layer of a thickness proportional to a small lenght scale parameter $\epsilon > 0$ and at the interface the phase field ρ rapidly but smoothly changes its value [10]. We require that $0 \leq \rho \leq 1$. The ρ values outside this range do not seem to correspond to admissible material distributions. The elastic tensor $\mathcal{A}$ of the material body is assumed to be a function depending on density function ρ:

$$\mathcal{A} = g(\rho)\mathcal{A}_0, \quad \mathcal{A}_0 = \{a_{ijkl}\}_{i,j,k,l=1}^{2} \tag{2.1}$$

and $g(\rho)$ is a suitable chosen function [3, 6, 10, 29].

We denote by $u = (u_1, u_2)$, $u = u(x)$, $x \in \Omega$, the displacement of the body and by $\sigma(x) = \{\sigma_{ij}(u(x))\}$, $i, j = 1, 2$, the stress field in the body. Consider elastic bodies obeying Hooke's law, i.e., for $x \in \Omega$ and $i, j, k, l = 1, 2$

$$\sigma_{ij}(u(x)) = g(\rho)a_{ijkl}(x)e_{kl}(u(x)). \tag{2.2}$$

We use here and throughout the paper the summation convention over repeated indices [15]. The strain $e_{kl}(u(x))$, $k, l = 1, 2$, is defined by:

$$e_{kl}(u(x)) = \frac{1}{2}(u_{k,l}(x) + u_{l,k}(x)), \tag{2.3}$$

where $u_{k,l}(x) = \frac{\partial u_k(x)}{\partial x_l}$. The stress field σ satisfies the system of equations in the domain Ω [15]

$$-\sigma_{ij}(x)_{,j} = f_i(x) \quad x \in \Omega, i, j = 1, 2, \tag{2.4}$$

where $\sigma_{ij}(x)_{,j} = \frac{\partial \sigma_{ij}(x)}{\partial x_j}$, $i, j = 1, 2$. The following boundary conditions are imposed on the boundary $\partial\Omega$

$$u_i(x) = 0 \quad \text{on} \quad \Gamma_0, \quad i = 1, 2, \tag{2.5}$$

$$\sigma_{ij}(x) n_j = p_i \quad \text{on} \quad \Gamma_1, \quad i, j = 1, 2, \tag{2.6}$$

$$u_N \leq 0, \quad \sigma_N \leq 0, \quad u_N \sigma_N = 0 \quad \text{on } \Gamma_2, \tag{2.7}$$

$$| \sigma_T | \leq 1, \quad u_T \sigma_T + | u_T | = 0 \quad \text{on } \Gamma_2, \tag{2.8}$$

where $n = (n_1, n_2)$ is the unit outward versor to the boundary Γ. Here $u_N = u_i n_i$ and $\sigma_N = \sigma_{ij} n_i n_j$, $i, j = 1, 2$, represent the normal components of displacement u and stress σ, respectively. The tangential components of displacement u and stress σ are given by $(u_T)_i = u_i - u_N n_i$ and $(\sigma_T)_i = \sigma_{ij} n_j - \sigma_N n_i$, $i, j = 1, 2$, respectively. $| u_T |$ denotes the Euclidean norm in R^2 of the tangent vector u_T. The results concerning the existence of unique solutions to (2.4)-(2.8) can be found in [15, 30].

2.1. **Variational Formulation of Contact Problem.** Let us formulate contact problem (2.4)-(2.8) in the variational form. Denote by V_{sp} and K the space and the set of kinematically admissible displacements:

$$V_{sp} = \{z \in [H^1(\Omega)]^2 \; : \; z_i = 0 \text{ on } \Gamma_0, \; i = 1, 2\}, \tag{2.9}$$

$$K = \{z \in V_{sp} \; : \; z_N \leq 0 \;\; \text{on} \;\; \Gamma_2\}. \tag{2.10}$$

$H^1(\Omega)$ denotes Sobolev space of square integrable functions and their first derivatives [15, 30]. $[H^1(\Omega)]^2 = H^1(\Omega) \times H^1(\Omega)$. Denote also by Λ the set

$$\Lambda = \{\zeta \in L^2(\Gamma_2) \; : | \zeta | \leq 1\}.$$

Variational formulation of problem (2.4)-(2.8) has the form: *find a pair* $(u, \lambda) \in K \times \Lambda$ *satisfying*

$$\int_\Omega g(\rho) a_{ijkl} e_{ij}(u) e_{kl}(\varphi - u) dx - \int_\Omega f_i(\varphi_i - u_i) dx -$$

$$\int_{\Gamma_1} p_i(\varphi_i - u_i) ds + \int_{\Gamma_2} \lambda(\varphi_T - u_T) ds \geq 0 \;\; \forall \varphi \in K, \tag{2.11}$$

$$\int_{\Gamma_2} (\zeta - \lambda) u_T ds \leq 0 \quad \forall \zeta \in \Lambda, \tag{2.12}$$

$i, j, k, l = 1, 2$. Function λ is interpreted as a Lagrange multiplier corresponding to term $| u_T |$ in equality constraint in (2.8) [15]. This function is equal to tangent stress along the boundary Γ_2, i.e., $\lambda = \sigma_{T|\Gamma_2}$. Function λ belongs to the space $H^{-1/2}(\Gamma_2)$, i.e., the space of traces on the boundary Γ_2 of functions from the space $H^1(\Omega)$. Here following [15] function λ is assumed to be more regular, i.e., $\lambda \in L^2(\Gamma_2)$. The results concerning the existence of solutions to system (2.11)-(2.12) under the introduced assumptions can be found, among others, in [15, 30], i.e,

Theorem 2.1. *There exists a unique solution $(u, \lambda) \in K \times \Lambda$ to system (2.11)-(2.12).*

2.2. **Topology Optimization Problem.** Before formulating a structural optimization problem for (2.11)-(2.12) let us introduce the set U_{ad} of admissible domains. Denote by $Vol(\Omega)$ the volume of the domain Ω equal to

$$Vol(\Omega) = \int_\Omega \rho(x)dx. \tag{2.13}$$

Domain Ω is assumed to satisfy the volume constraint of the form

$$Vol(\Omega) - Vol^{giv} \leq 0, \tag{2.14}$$

where the constant $Vol^{giv} = const_0 > 0$ is given. In a case of shape optimization of problem (2.11) - (2.12) the optimized domain Ω is assumed to satisfy equality volume condition, i.e., (2.14) is assumed to be satisfied as equality. In a case of topology optimization Vol^{giv} is assumed to be the initial domain volume and (2.14) is satisfied in the form $Vol(\Omega) = r_{fr} Vol^{giv}$ with $r_{fr} \in (0,1)$ [29]. The set U_{ad} has the following form

$$\begin{aligned} U_{ad} = \{\Omega \ : \ E \subset \Omega \subset D \subset R^2 \ : \\ \Omega \text{ is Lipschitz continuous, } \Omega \text{ satisfies condition (2.14)}\}, \end{aligned} \tag{2.15}$$

where $E \subset R^2$ is a given domain such that Ω as well as all perturbations of it satisfy $E \subset \Omega$. The constant $const_1 > 0$ is assumed to exist. The set U_{ad} is assumed to be nonempty. In order to define a cost functional we shall also need the following set M^{st} of auxiliary functions

$$\begin{aligned} M^{st} = \{\eta = (\eta_1, \eta_2) \in [H^1(D)]^2 : \eta_i \leq 0 \text{ on } D,\ i = 1, 2, \\ \| \eta \|_{[H^1(D)]^2} \leq 1\}, \end{aligned} \tag{2.16}$$

where the norm $\| \eta \|_{[H^1(D)]^2} = (\sum_{i=1}^2 \| \eta_i \|^2_{H^1(D)})^{1/2}$. Recall from [21, 22, 24] the cost functional approximating the normal contact stress on the contact boundary

$$J_\eta(u(\Omega)) = \int_{\Gamma_2} \sigma_N(u)\eta_N(x)ds, \tag{2.17}$$

depending on the auxiliary given bounded function $\eta(x) \in M^{st}$. σ_N and η_N are the normal components of the stress field σ corresponding to a solution u satisfying system (2.11)-(2.12) and the function η, respectively.

Consider the following structural optimization problem: *for a given function $\eta \in M^{st}$, find a domain $\Omega^\star \in U_{ad}$ such that*

$$J_\eta(u(\Omega^\star)) = \min_{\Omega \in U_{ad}} J_\eta(u(\Omega)). \tag{2.18}$$

Adding to (2.15) a perimeter constraint $P_D(\Omega) \leq const_1$, where $P_D(\Omega) = \int_\Gamma dx$ is a perimeter of a domain Ω in D [7, 21, 30] and $const_1 > 0$ is a given constant the existence of an optimal domain $\Omega^\star \in U_{ad}$ to the problem (2.18) is ensured (see [6, 7, 30]).

Theorem 2.2. *Assume the number of connected components of the complement set Ω^c of domain Ω with respect to $D \subset R^2$ is bounded. There exists a solution to $\hat{\Omega} \subset U_{ad}$ to the problem (2.18).*

Proof. The class of admissible domains is endowed with the complementary Hausdorff topology that guarantees the class itself to be compact. The existence of an optimal domain $\Omega^\star \in U_{ad}$ to the topology optimization problem (2.18) follows from Šverák theorem and arguments provided in [7]. □

3. HYBRID APPROACH TO TYOPOLOGY OPTIMIZATION

3.1. **Level set based topology optimization.** In [1, 12, 21] the standard level set method [26] is employed to solve numerically problem (2.18). Consider the evolution of a domain Ω under a velocity field V. Let $t > 0$ denote the artificial time variable and I an identity operator. Under the suitable regular mapping $T(t, V)$ we have

$$\Omega_t = T(t, V)(\Omega) = (I + tV)(\Omega), \quad t > 0.$$

By Ω_t^- we denote the interior of the domain Ω_t and by Ω_t^+ we denote the outside of the domain Ω_t. The domain Ω_t and its boundary $\partial\Omega_t$ are defined by a function $\phi = \phi(x,t)$: $R^2 \times [0, t_0) \to R$ satisfying:

$$\begin{aligned} \phi(x,t) &= 0, \quad \text{if } x \in \partial\Omega_t, \\ \phi(x,t) &< 0, \quad \text{if } x \in \Omega_t^-, \\ \phi(x,t) &> 0, \quad \text{if } x \in \Omega_t^+. \end{aligned} \tag{3.1}$$

Function ϕ satisfying (3.1) is called the level set function. Recall [26], the gradient of the implicit function is defined as $\nabla\phi = (\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2})$, the local unit outward normal n to the boundary is equal to $n = \frac{\nabla\phi}{|\nabla\phi|}$, the mean curvature $\kappa = \nabla \cdot n$. In the level set approach Heaviside function $H(\phi)$ and Dirac function $\delta(\phi)$ are used to transform integrals from domain Ω into domain D [3]. These functions are defined as

$$H(\phi) = 1 \text{ if } \phi \geq 0, \quad H(\phi) = 0 \text{ if } \phi < 0, \tag{3.2}$$

$$\delta(\phi) = H'(\phi), \quad \delta(x) = \delta(\phi(x)) \mid \nabla\phi(x) \mid, \quad x \in D. \tag{3.3}$$

$$\begin{aligned} \int_\Omega f(x)dx &= \int_D f(x)H(\phi)dx \\ \int_{\partial\Omega} f(x)ds &= \int_D f(x)\delta(\phi) \mid \nabla\phi \mid ds \end{aligned} \tag{3.4}$$

Assume that velocity field $V = V(x,t)$ is known for every point x lying on the boundary $\partial\Omega_t$, i.e., such that $\phi(x,t) = 0$. Therefore the equation governing the evolution of the interface $\partial\Omega_t$ in $D \times [0, t_0]$, known as Hamilton-Jacobi equation, has the form [1, 26]

$$\frac{\partial\phi(x,t)}{\partial t} + V(x,t) \cdot \nabla_x\phi(x,t) = 0. \tag{3.5}$$

$$\phi(x,0) = \phi_0(x), \tag{3.6}$$

where $\phi_0(x)$ is a given signed distance function of the set Ω_t. Velocity field V is chosen as the shape derivative of the cost functional (2.18) with respect to the boundary variations of the domain. Topology derivative of this cost functional has to be used to indicate the areas of the weak material. The shape and topology derivatives of the cost functional (2.17) are provided as well as a necessary optimality condition is shown in [21].

3.2. **Phase field based topology optimization.** Consider a two phase problem. Let material density function $0 \leq \rho \leq 1$ be a variable describing the concentraation of one of the phases in the domain Ω. The other phase is obtained as $(1-\rho)$. This variable is used to describe the phase transition [10]. To indicate the evolution of the material density function ρ let us assume this function depends not only on $x \in \Omega$ but also on the artificial time variable $t \in [0,T)$, $T > 0$ given, i.e. $\rho = \rho(t,x)$. Let us introduce the regularized cost functional $J(\rho, u)$ in the form:

$$J(\rho, u) = J_\eta(u) + E(\rho), \tag{3.7}$$

where the functionals $J_\eta(u)$ is given by (2.17) and Ginzburg-Landau free energy term $E(\rho)$ [3, 4, 6, 7, 10, 12, 32] satisfies:

$$E(\rho) = \int_\Omega \psi(\rho)d\Omega, \tag{3.8}$$

with the total free energy function $\psi(\rho)$ in the form

$$\psi(\rho) = \frac{\gamma\epsilon}{2} \mid \nabla\rho \mid^2 + \frac{\gamma}{\epsilon}\psi_B(\rho), \tag{3.9}$$

where $\gamma > 0$ is a constant, $\epsilon > 0$ is a parameter related to the interfacial energy density and $\psi_B(\rho)$ is a double-well potential which characterizes the two phases [3, 10]. Usually it is taken as an even-order polynomial of the form [19]

$$\psi_B(\rho) = \rho^2(1-\rho^2). \tag{3.10}$$

The first term in (3.9) is called the interface energy. It represents [10] a measure of the perimeter of the interfaces between the phases and in this sense it is the relaxed version of the global perimeter constraint. The term (3.10) is called the bulk energy. It is a non-convex smooth function attaining minimum in the pure phases $\rho = 0$ and $\rho = 1$. The values assumed by $\psi_B(\rho)$ for intermediate values of ρ are larger than for pure phases and are not prefered in the optimization process. Parameter ϵ measures the width of the transition zone.

The structural optimization problem (2.18) takes the form: *find* $\rho^\star \in U^\rho_{ad}$ *such that*

$$J(\rho^\star, u^\star) = \min_{\rho\in U^\rho_{ad}} J(\rho, u), \tag{3.11}$$

where $u^\star = u(\rho^\star)$ denotes a solution to the state system (2.11)-(2.12) depending on $\rho^\star$ and $U^\rho_{ad} = \{\rho : Vol(\Omega) = Vol^{giv}\}$ denotes the set of admissible material density functions.

The definition of the phase transition model is based on the concept of the flow of the gradient ∇L of the Lagrangian of the problem(3.11) with respect to ρ in the norm of a suitable chosen Hilbert space H:

$$\frac{\partial\rho}{\partial t}(t,x) = -\frac{\partial L}{\partial\rho}(\rho) \quad \text{in} \quad \Omega, \ \ \forall t \in [0,T), \tag{3.12}$$

and the initial condition

$$\rho(0,x) = \rho_0(x) \ \ t = 0 \ \ x \in \Omega. \tag{3.13}$$

Selecting the space H [19] as the subspace of a space $[H^1(\Omega)]'$ dual to $H^1(\Omega)$ (3.12)-(3.13) leads to a necessary optimality condition in the form of the modified Cahn-Hilliard equation. For details see [24].

4. Hybrid approach to topology optimization

Hybrid interface tracking models, joining the level set approach and the phase field approach, are used in fluid dynamics governed by Navier-Stokes equations [16] or in modelling surface tension interface [19]. Among others, in [17, 18] to avoid singularities at the contact point between the fluid and the wall hybrid interface evolution model has been used combining convective transport equation in the bulk domain and Cahn-Hilliard equation in the vicinity of the interface. The numerical tests indicate the computational efficiency of the hybrid model compared to plain phase field one.

The relation between level set and phase field approaches are studied among others in [3, 5, 9, 10, 28, 33]. Based on application of both models in Mumford-Shah functional minimization for image registration and segmentation [9], it is stated that these models are well-known due to their topological flexibility. Both approaches are very flexible and allow a wide range of extensions for model-based matching, registration and segmentation, optical flow with discontinuities, fluid flow. In these methodologies the process of splitting a curve into several curves is a smooth one. However these two approaches differes significantly in the representation of the discontinuity set. The level set method allows to represent, trace and evolve a given sharp interface. This fits very well to the framework of the calculus of shape derivatives in which the current interface is given precisely. On the other hand the phase field function is capable to indicate the position of a inteface in a blurry way only determined by the order of a grid size. The classical level set framework is restricted to closed curves and thus it does not allow to represent crack tips or to generate a hole using a single level set function. Topological derivative is used to generate holes in the framework of the level set method[7]. On the other hand the phase field method appears to be more flexible and practicable for the aforemntioned applications. The phase field representation is global by definition and respects the features of the topology in the entire domain occupied by a structure without requiring any initialization.

As far as it concerns algorithmic implementation of these approaches [9], the phase field method, especially in the form of Allen-Cahn equation, seems to be easier to implement. The phase field method can be implemented by solving parabolic equations with coefficients dependent on spatial variables. Such problems are standard and can be solved with PDE toolboxes. Since the interface is represented by a smooth phase field function the solution of Helmholtz problems in the domains divided by free discontinuity is straightforward and does not require any additional effort to take care of free boundaries. The sharp interface approach requires to evaluate the velocity along the interface.

Structural optimization problems with a level set function and different phase field like gradient flow equations are considered in [8, 28, 31, 33]. The relation between phase field and sharp interface tracking models in optimal control problems is considered in [3]. Using the method of the matched asymptotic expansions it is shown that for the compliance topology optimization problem in linear elasticity the sharp interface limit of the necessary optimality condition for the phase field model when the interface width parameter is passing to zero coincides with the necessary optimality condition for this optimization problem obtained by the shape calculus [1].

4.1. **Hybrid optimization problem formulation.** Consider slightly modified level set function ϕ compare to the standard one (3.1),

$$
\begin{aligned}
0 < \phi(x) \leq 1 \quad \text{for} \quad x \in \Omega \setminus \partial\Omega,\\
\phi(x) = 0 \quad \text{for} \quad x \in \partial\Omega,\\
-1 \leq \phi(x) < 0 \quad \text{for} \quad x \in D \setminus \Omega.
\end{aligned} \tag{4.1}
$$

Remark the level set function (4.1) is close to the phase field variable governing the evolution of phases in the phase field method or to the so-called binary level set method [10]. This function is bounded and takes values close to +1 or -1 in regions sufficiently distant from the interfaces. Consider the regularized cost functional (2.17):

$$
J_R(\phi) = J_\eta(u(\phi)) + E_R(\phi), \quad E_R(\phi) = \frac{1}{2}\tau \int_D \mid \nabla\phi \mid^2 d\Omega, \tag{4.2}
$$

$\tau > 0$ is a regularization parameter. The structural optimization problem (2.18) takes the form: find $\phi \in U^{\phi}_{ad}$ such that:

$$
\min_{\phi \in U^{\phi}_{ad}} J_R(\phi), \tag{4.3}
$$

where the admissible set U^{ϕ}_{ad} (2.15) in terms of ϕ has the form:

$$
U^{\phi}_{ad} = \{\phi \in H^1(D) : Vol(\phi) = \int_D H(\phi)dx - Vol^{giv} \leq 0\}. \tag{4.4}
$$

$(u, \lambda) \in K \times \Lambda$ solves the state system (2.11)-(2.12) in the domain D rather than Ω:

$$
\begin{aligned}
\int_D H(\phi)a_{ijkl}e_{ij}(u)e_{kl}(\varphi - u)dx - \int_D H(\phi)f_i(\varphi_i - u_i)dx - \\
\int_{\Gamma_1} p_i(\varphi_i - u_i)ds + \int_{\Gamma_2} \lambda(\varphi_T - u_T)ds \geq 0 \quad \forall\varphi \in K,
\end{aligned} \tag{4.5}
$$

$$
\int_{\Gamma_2} (\zeta - \lambda)u_T ds \leq 0 \quad \forall\zeta \in \Lambda. \tag{4.6}
$$

The existence of a unique solution to (4.5)-(4.6) follows from Theorem 2.1.

5. Necessary optimality condition

Let us formulate the necessary optimality condition for problem (4.3)-(4.6). In order to do it we introduce the Lagrangian $L(\phi, \tilde{\lambda}) : H^1(D) \times R \to R$

$$
\begin{aligned}
L(\phi, \tilde{\lambda}) = L(\phi, u_\epsilon, \lambda_\epsilon, p^a, q^a, \tilde{\lambda}) = J_R(\phi) + \\
\int_D H(\phi) a_{ijkl} e_{ij}(u_\epsilon) e_{kl}(p^a) dx - \int_D H(\phi) f_i(p_i^a)) dx - \\
\int_{\Gamma_1} p_i p_i^a ds + \int_{\Gamma_2} \lambda_\epsilon (p_T^a) ds + \\
\int_{\Gamma_2} q^a u_{\epsilon T} ds + \tilde{\lambda} c(\phi) + \frac{1}{2\mu} c^2(\phi),
\end{aligned} \tag{5.1}
$$

where $\tilde{\lambda} \in R$, $c(\phi) = [Vol(\phi)]$, $\mu > 0$ is a given real. By $(p^a, q^a) \in K_1 \times \Lambda_1$ we denote an adjoint state. Using the results on differentiability of varaitional inequalities [30] we obtain [21] the adjoint state satisfies:

$$
\int_D H(\phi) a_{ijkl} e_{ij}(\eta + p^a) e_{kl}(\varphi) dx + \int_{\Gamma_2} q^a \varphi_T ds = 0 \quad \forall \varphi \in K_1, \tag{5.2}
$$

and

$$
\int_{\Gamma_2} \zeta (p_T^a + \eta_T) ds = 0 \ \ \forall \zeta \in \Lambda_1. \tag{5.3}
$$

The sets K_1 and Λ_1 are given by

$$
K_1 = \{\xi \in V_{sp} \ : \ \xi_N = 0 \ \text{ on } A^{st} \}, \tag{5.4}
$$

$$
\Lambda_1 = \{\zeta \in \Lambda \ : \ \zeta(x) = 0 \text{ on } B_1 \cup B_2 \cup B_1^+ \cup B_2^+ \}, \tag{5.5}
$$

while the coincidence set $A^{st} = \{x \in \Gamma_2 \ : \ u_N + v = 0\}$. Moreover $B_1 = \{x \in \Gamma_2 : \lambda(x) = -1\}$, $B_2 = \{x \in \Gamma_2 : \lambda(x) = +1\}$, $\tilde{B}_i = \{x \in B_i : u_N(x) + v = 0\}$, $i = 1, 2$, $B_i^+ = B_i \setminus \tilde{B}_i$, $i = 1, 2$.

Using (5.2)-(5.5) we can calculate the derivative of the Lagrangian L with rescpect to ϕ:

$$
\begin{aligned}
\int_D \frac{\partial L}{\partial \phi}(\phi, \tilde{\lambda}) \zeta dx = \int_D [H(\phi)(a_{ijkl} e_{ij}(u_\epsilon) e_{kl}(p^a + \eta) - \\
f(p^a + \eta)) + \tau \bigtriangleup \phi] \zeta dx + \int_D (\tilde{\lambda} + \frac{1}{\mu} c(\phi)) \zeta dx \ \ \forall \zeta \in H,
\end{aligned} \tag{5.6}
$$

The necessary optimality condition for problem (4.3)-(4.6) follows from standard arguments [15, 30]:

Theorem 5.1. *If $(\hat{\phi}, \tilde{\lambda}^*) \in U_{ad}^{\phi} \times R$ is an optimal solution to problem (4.3)-(4.6) than:*

$$L(\hat{\phi}, \tilde{\lambda}) \leq L(\hat{\phi}, \tilde{\lambda}^*) \leq L(\phi, \tilde{\lambda}^*) \ \ \forall(\phi, \tilde{\lambda}) \in U_{ad}^{\phi} \times R, \tag{5.7}$$

with $\tilde{\lambda} \geq 0$.

(5.7) implies [15, 30] that for all ϕ and $\tilde{\lambda}$

$$\frac{\partial L(\hat{\phi}, \tilde{\lambda})}{\partial \phi} \geq 0 \ \text{ and } \ \frac{\partial L(\phi, \tilde{\lambda}^*)}{\partial \tilde{\lambda}} \leq 0 \tag{5.8}$$

6. Implementation issues

Uzawa type algorithm is employed to solve numerically optimization problem (4.3). First as in (3.1) we assume that due to the evolution of the subdomains ϕ is also time dependent. The minimization of the Lagrangian $L(\phi, \tilde{\lambda})$ with respect to ϕ is realized by solving the time dependent PDE [26]

$$\begin{aligned} \frac{\partial \phi(x,t)}{\partial t} &= \nabla_{\phi} L(\phi, \tilde{\lambda}) \ \text{ in } \ D \times (0, \infty), \\ \phi(x,0) &= \phi_0(x) \ \text{ in } \ D, \quad \nabla \phi \cdot n = 0 \text{ on } \partial D \end{aligned} \tag{6.1}$$

to reach the steady state $\frac{\partial \phi}{\partial t} = 0$. It implies gradient $\nabla_{\phi} L(\phi, \tilde{\lambda})$ given by (5.6) equals to zero. $\phi_0(x)$ is a given function. The explicit Euler scheme [2] is used to solve numerically the equation (6.1), i.e.,

$$\phi^{n+1} = \phi^n + \triangle t^n \frac{\partial L(\phi^n, \tilde{\lambda}^n)}{\partial \phi}, \tag{6.2}$$

where $\phi^n = \phi(x, t^n)$, $\triangle t^n$ denotes the n-th time step and $\frac{\partial L(\phi^n, \tilde{\lambda}^n)}{\partial \phi}$ is given by (5.6). To satisfy CFL stability condition the stepsize $\triangle t^n$ is assumed to satisfy [26]

$$\triangle t^n = \alpha h / \max_{x \in D} | \frac{\partial L(\phi^n(x), \tilde{\lambda}^n)}{\partial \phi} |, \tag{6.3}$$

where α is a suitable given number and h is the uniform mesh size. The updating scheme for the Lagrange multiplier $\tilde{\lambda}$ is as follows:

$$\tilde{\lambda}^{n+1} = \tilde{\lambda}^n + \frac{1}{\mu^n} Vol(\phi), \tag{6.4}$$

with the penalty parameter $\mu^{n+1} \in (0, \mu^n)$, $\mu^0 > 0$ given.

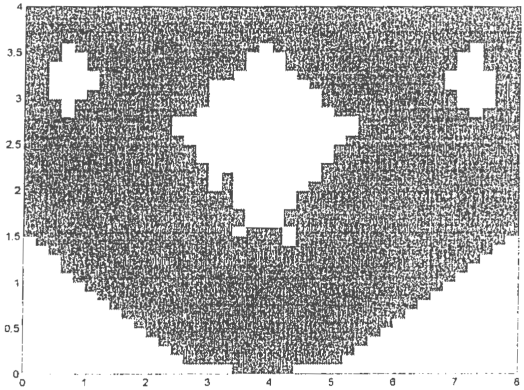

FIGURE 2. Optimal domain $\Omega^\star$.

6.1. **Numerical example.** The discretized topology optimization problem (4.3) - (4.6) is solved numerically. As an example a body occupying 2D domain

$$\Omega = \{(x_1, x_2) \in R^2 : 0 \leq x_1 \leq 8 \ \wedge \ 0 < v(x_1) \leq x_2 \leq 4\}, \tag{6.5}$$

is considered. The boundary Γ of the domain Ω is divided into three pieces

$$\begin{aligned} \Gamma_0 &= \{(x_1, x_2) \in R^2 : x_1 = 0, 8 \ \wedge \ 0 < v(x_1) \leq x_2 \leq 4\}, \\ \Gamma_1 &= \{(x_1, x_2) \in R^2 : 0 \leq x_1 \leq 8 \ \wedge \ x_2 = 4\}, \\ \Gamma_2 &= \{(x_1, x_2) \in R^2 : 0 \leq x_1 \leq 8 \ \wedge \ v(x_1) = x_2\}. \end{aligned} \tag{6.6}$$

The domain Ω and the boundary Γ_2 depend on the function v. The initial position of the boundary Γ_2 is given as in Fig. 1. The computations are carried out for the elastic body characterized by the Poisson's ratio $\nu = 0.29$, the Young modulus $E = 2.1 \cdot 10^{11} N/m^2$. The body is loaded by boundary traction $p_1 = 0$, $p_2 = -5.6 \cdot 10^6 N$ along Γ_1, body forces $f_i = 0$, $i = 1, 2$. Auxiliary function η is selected as piecewise constant (or linear) on D and is aproximated by a piecewise constant (or bilinear) functions. The computational domain $D = [0, 8] \times [0, 4]$ is selected. Domain D is discretized with a fixed rectangular mesh of 80 $\times$ 40.

Fig. 6.1 presents the optimal domain obtained by solving structural optimization problem (4.3) in the computational domain D using Uzawa type algorithm and employing the optimality condition (5.7). The areas with low values of density function appeare in the central part of the body and near the fixed edges. The obtained normal contact stress is almost constant along the optimal shape boundary and has been significantly reduced comparing to the initial one.

7. CONCLUDING REMARKS

The topology optimization problem for elastic contact problem with the prescribed friction is analyzed and solved numerically in the paper. The level set approach combined with the phase field approach are used. The friction term complicates both the form of the gradients of the cost functional as well as numerical process. Obtained numerical results seems to be in accordance with physical reasoning. They indicate that the proposed method allows for significant improvements of the structure from one iteration to the next and is more efficient than the algorithms based on standard level set approach. Comparing to the standard level set approach the proposed approach do not require to solve Hamilton - Jacobi equation and to perform the reinitialization process of the signed distance function. The proposed method has also a hole nucleation capabilieties as topological gradient based methods.

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