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Multiple Criteria Analysis of Discrete Alternatives with a Simple Preference Specification: Pairwise-Outperformance-Based Approaches

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Abstract

Many methods have been developed for multiple criteria analysis and/or ranking of discrete alternatives. Most require a complex specification of preferences. They are, therefore, not applicable for problems with numerous alternatives and/or criteria, where the decision maker cannot specify her/his preferences in a way acceptable for small problems, e.g., through pairwise comparisons. In this paper we propose new methods built on combining existing concepts with the developed outperformance aggregations that take into account inter-alternative factors. The methods have been applied for analyzing real-life problems like multiple-criteria analysis of future energy technologies, energy and climate policy and controlling a space robot. Analysis of the first two cases involved large numbers of alternatives, the first case also large number of criteria. Moreover, the analysis was conducted by a large number of stakeholders without experience in analytical methods. This is why, a simple method for interactive preference specification was a precondition for the analysis. A comparison of the developed methods is presented, and experience of using them is summarized.

Keywords: Multiple criteria analysis, Decision analysis, OR in energy

1. Introduction

Multiple criteria analysis is a well established area of applied science, which was developed in response to a need for problem analysis that could not be met by single criteria optimization methods. A sample of diverse approaches and the corresponding tools can be found in [1, 3, 5, 7, 12, 16, 35, 38, 39, 42, 43]. One could, therefore, ask why new methods still need to be developed. To answer this question, we begin the paper by summarizing in Section 2 an application that was intended to be supported by one of the existing methods. However, the requirement analysis made it clear that none of the existing methods could meet the requirements. Thus, the reported research was motivated by real needs.

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The structure of the remaining part of the paper is as follows. Basic terminology and the specification of preferences are discussed in Section 3. The fundamental requirement for the methods designed was simplicity of the preference specification, and this motivated use of the criteria relative importance for representation of preferences. Because of the strong demand for organizing the large number of criteria into the three pillars of sustainable development, the corresponding hierarchy of the criteria was implemented. The main scientific result is presented in Section 4, which presents the proposed Pairwise Outperformance Measure that is based on differences of the compared achievements, as well as on their values. Next, in Section 5 we define the Ordered Pairwise Outperformance Aggregation and show its applicability; this aggregation is based on comparison of pairs of achievements ordered (for each criterion) from the worst to the best. The transitivity property of the methods developed, and the Net-Flow approaches are discussed in Section 6. Section 7 summarizes the extensive experiments with the methods developed. Section 8 concludes.

2. Motivation

Multicriteria analysis was needed to support a large number of diversified stakeholders in individual analyses of preferences for diverse future energy technologies developed within the European Integrated Project NEEDS.[1] The over 3,000 stakeholders invited to carry on the analysis came from different backgrounds and typically had rather limited mathematical skills. Because of the number of stakeholders and their geographical dispersion as well as the limited time available, the analysis was conducted using the popular Web browsers. Moreover, the users typically had little time to become familiar with the tool supporting the analysis, and also to complete the analysis. The Web-based tool for multicriteria analysis thus had to be easy to use; in particular, specification of preferences had to be intuitive, and the corresponding multicriteria analysis method needed to be able to support an effective analysis of a large number of Pareto-efficient alternatives characterized by a large number of criteria organized in a hierarchical structure. Earlier approaches to this class of problems had been limited to the weighted additive value function [13].

A concerted effort on the part of European researchers resulted in over 20 technologies being defined in each of the four European countries analyzed. The set of 26 energy generation technologies includes 2 nuclear, 16 fossil (10 coal and lignite, 6 natural gas), and 8 renewable (biomass, solar, and wind). Each technology is characterized by approximately 40 attributes.[2]

From a modeling point of view, for each of the four countries a multicriteria analysis was conducted for a set of over 20 alternatives, each characterized by 61 criteria (composed of attributes, three top-level criteria, and intermediate criteria) organized in a hierarchical structure forming an unbalanced criteria tree.

The criteria were organized in a hierarchical structure composed of three subsets of criteria following the concept of sustainable development (i.e., environmental, economic, and social) criteria.

[1] Information about the NEEDS Project is available at `http://www.needs-project.org/2009/`, and e.g., [20, 36].

[2] The description of technologies is available at: `http://www.iiasa.ac.at/~marek/mca_doc/pdf_needs/tech_en.pdf`. The corresponding database report is available at: `http://www.iiasa.ac.at/~marek/mca_doc/pdf_needs/db_rep.pdf`. Full description of the criteria and indicators is available at: `http://www.iiasa.ac.at/~marek/mca_doc/pdf_needs/hier_en.pdf`.

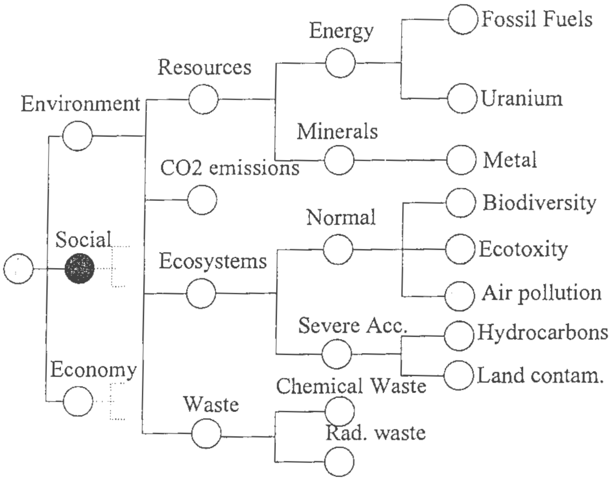


Figure 1: Illustration of the criteria hierarchy used in the NEEDS case study.

Figure 1 illustrates part[3] of the criteria hierarchy.

A detailed requirement analysis of the problem [20] showed that there were no multicriteria analysis methods that would meet all these requirements [8], although various approaches to multicriteria analysis in energy have been used (see e.g., [4, 10]). To provide adequate support for analysis of the class of discrete decision problems with large number of alternatives, numerous and structured criteria, and simultaneously involving multiple stakeholders, we developed and tested new methods.

The analysis was anonymous and involved a large number of participants; thus no support for individual elicitation of preferences was possible. Moreover, it was clear that most users would not be willing to devote time needed for a non-trivial specification of preferences, or to study the underlying methodological background. Therefore, a simple specification of preferences was the necessary condition for actual use of the desired multiple-criteria analysis.

To meet those challenges, the authors had many discussions with colleagues experienced in multiple-criteria analysis involving practitioners from the energy domain. The conclusion was that relative criteria importance would be the best way for specification of preferences. About 30 versions of methods based on relative criteria importance were developed, and about 10 of them were blind-tested[4] by colleagues experienced in multiple-criteria analysis of energy technologies. The goal of these tests was to select the method most suitable to the planned analysis in terms

[3] Only one branch from the top level criteria in shown. Therefore criteria belonging to the Economy and Social criteria sets are not displayed.

[4] The testers experimented with each method in row, without knowing its theoretical background.

of correspondence between changes in preference specification and the resulting change of Pareto alternative. As the result, the POA method with non-linear mapping of relative importance was recommended. More extensive tests done later motivated further development of the corresponding family of methods presented in this paper. Other methods are described in [18] and [41]. They were not used in the case studies discussed in this paper, but are still subject to further development and will be presented in other publications.

3. Problem definition

3.1. *Preference model*

In this paper we focus on the problem of analysis of a discrete set of alternatives (objects) o_j, $j \in J = \{1, 2, \dots, m\}$. The set of all alternatives is referred to as $Q = \{o_j : j \in J\}$. Objects o_j are described by numerical attributes (or criteria, selected outcomes) c_i, $i \in I = \{1, 2, \dots, n\}$. Attribute values are denoted by $q_{ij} = c_i(o_j)$ specified for each pair $\{i, j\}$.

In the process of problem analysis the user selects some of the attributes as criteria and decides on each criterion type (minimization or maximization). Optionally, the user can define the hierarchical structure of criteria forming a tree, in which the leaves are the criteria defined by the selected attributes, and the higher-level criteria are defined to aggregate lower-level criteria; see [19] for details.

There are three basic types of multicriteria analysis:

- Choice: select the most preferred object,
- Ranking: order all objects from the most to the least preferred,
- Sorting: partition of the set of alternatives into several ordered categories.

The essence of multiple criteria analysis is to help the user to find a solution (either an object, or ranking, or sorting) that best fits his/her preferences. The basic function of multicriteria analysis is to support the user in an interactive modification of his preferences when the corresponding solutions are being analyzed. This approach substantially differs from the classical (single-objective) optimization which requires a prior specification of one objective function (optimization criterion).

To facilitate the discussion we recall here the basic concepts of Pareto efficiency (Pareto-optimal solutions) and preference models.

An alternative is called Pareto-optimal, if no other alternative has: (1) at least one criterion with a better value, and (2) no criterion with a worse value. In other words (and assuming for the following definition that all criteria are maximized) alternative $o_l \in Q$ is Pareto-optimal if and only if:

$$\neg \exists o_j \in Q : \{c_i(o_j) \geq c_i(o_l) \;\; \forall i \in I \;\text{ and } \exists k \in I : c_k(o_j) > c_k(o_l)\;\}. \tag{1}$$

If such an alternative o_j exists, then we say that it dominates o_l. A Pareto-optimal alternative is also called an *efficient* or non-dominated one. A Pareto-optimal set is composed of all Pareto-optimal alternatives. A Pareto-optimal outcome vector is composed of values of all criteria for a corresponding Pareto-optimal alternative.

It is clear that a dominated alternative is not a rational choice. Therefore, it is rational to analyze trade-offs between non-dominated alternatives only. Thus the purpose of multicriteria analysis is help the user to analyze the Pareto set to find either a Pareto efficient solution or a ranking of alternatives.

Let us now consider a pairwise comparison, i.e., decide which of two (say o_1 and o_2) selected alternatives (or corresponding outcomes) is preferred. Two situations can be distinguished:

- One of these alternatives dominates the other; in this case the dominating outcome is clearly preferred.[5]
- If the alternatives do not dominate each other, then it cannot be objectively decided which one is better than the other; however, the user either (subjectively) prefers one of them, or cannot decide which he prefers.

Generally, it is clear that if one outcome dominates another, then it is better than the other. In truly multicriteria problems, however, no alternative dominates all other alternatives. In other words, the best (in terms of strict mathematical relations) alternative cannot be distinguished because the nondominated outcomes are incomparable on the basis of the specified set of criteria. However, a user usually has preferences that help him to select an alternative that best fits these preferences.

A preference structure [32] (which can be used in definition of advanced preference models) is a collection of binary relations defined on the set of alternatives Q such that exactly one relation is satisfied. The simplest preference model assumes that when two different elements of the set Q are being compared only two situations can be distinguished: preference of one element to the other (relation $\succ$), or indifference of one element to the other (relation $\sim$). Note that $\succ$ is asymmetric while $\sim$ is reflexive and symmetric. Such a simple preference model can be defined by a *preference structure* composed of two disjoint binary relations on $Q \times Q$:

$$\langle \succ, \sim \rangle. \tag{2}$$

The preference model (2) is called complete, if for any pair of alternatives (o_1, o_2) either $o_1 \succ o_2$ or $o_2 \succ o_1$, or $o_1 \sim o_2$. The preference model (2) is called transitive, if for any three alternatives o_1, o_2, o_3 the following implications hold:

- if $o_1 \succ o_2$ and $o_2 \succ o_3$ then $o_1 \succ o_3$, and
- if $o_1 \sim o_2$ and $o_2 \sim o_3$ then $o_1 \sim o_3$.

By extending the properties of the binary relations, various more specific preference structures called *orders* (e.g., total, weak, semi-order, interval) can be defined. For example, outranking methods are based on preference structures called partial and quasi order. The details of various preference structures can be found, for example, in [32].

In multiple criteria analysis it is assumed that preferences depend only on the evaluation of attributes included in the outcome vectors. This implies that the preference structure $\langle \succ, \sim \rangle$ is equivalently defined on the corresponding outcome vectors.

For any pair of alternatives (o_1, o_2) with outcome vectors $\boldsymbol{q}^1$ and $\boldsymbol{q}^2$

$$o_1 \succ o_2 \Leftrightarrow \boldsymbol{q}^1 \succ \boldsymbol{q}^2 \quad \text{and} \quad o_1 \sim o_2 \Leftrightarrow \boldsymbol{q}^1 \sim \boldsymbol{q}^2$$

We will use both of them interchangeably. Note that the preference model must fulfill:

$$q^1 = q^2 \quad \Rightarrow \quad q^1 \sim q^2.$$

[5] This does not mean that the dominated alternative is a bad one. Actually, the implemented ranking procedure assigns dominated (but "close") alternatives a next (or even the same) ranking position, i.e., will rank them above other (originally non-dominated) alternatives.

The preference model for any pair of outcome vectors q^1 and q^2 (corresponding to alternatives (o_1, o_2))

$$q^1 \geq q^2 \quad \Rightarrow \quad q^1 \succ q^2 \text{ or } q^1 \sim q^2. \tag{3}$$

It is called *strictly monotone*, if additionally

$$q^1 \geq q^2 \text{ and } q^1 \neq q^2 \quad \Rightarrow \quad q^1 \succ q^2. \tag{4}$$

Note that the monotonicity properties of the preference model are crucial for its consistency with the Pareto-optimality principle. Actually, within the monotonic preference model no Pareto-dominated alternative can be strictly preferred to a dominating one. The strict monotonicity guarantees that any alternative maximal according to the preference relation is Pareto-optimal.

The preference models can also have numerical representations. The most common numerical representations of preference models is a value function $v : Q \to R$ defined for each alternative. In such cases while considering a pair of two alternatives (o_1, o_2):

- Alternative o_1 is preferred to o_2 (i.e., $o_1 \succ o_2$), if and only if $v(o_1) > v(o_2)$;
- Alternatives o_1 and o_2 are indifferent (i.e., $o_1 \sim o_2$), if and only if $v(o_1) = v(o_2)$.

The preference model defined by a value function is obviously complete and transitive.

As the preference model is based on the outcome vectors, the value function also has to be defined on outcomes, thus representing some aggregation of the criteria. The scalarizing functions may have various constructions and properties depending on the specific approach to preference modeling applied in the corresponding methods. Nevertheless, most scalarizing functions or, more generally, preference model constructions, can be viewed as two-stage process:

- First, the individual outcomes are rescaled to some uniform measures of achievement with respect to several criteria and preference parameters. Thus, the individual achievement functions $a_i : R \to R$ are built to measure the actual achievement of each outcome in a uniform scale, say [0, 1]. We denote individual achievements for each alternative by $a_{ij} = a_i(q_{ij})$, and the entire achievement vector by $\boldsymbol{a}^j = (a_{1j}, a_{2j}, \ldots, a_{nj})$.
- Second, the outcomes transformed into a uniform scale of individual achievements are compared in order to build a preference model. Using the value function concept they are aggregated at the second stage to form a final scalarization. The aggregation may measure, for instance, the average or the worst individual achievement. Typically, the aggregation is impartial or symmetric with respect to the individual achievements. It thus treats all individual achievements as equally important as long as no criteria importance is introduced.

While building the preference model over the uniformly scaled and equally important achievements, a small improvement of the worst achievement value is usually preferred over a worsening of the much better achievement by the same value. To illustrate this feature let us consider three equally important criteria, and alternatives o_1 and o_2, with the corresponding achievements shown in Table I.

Typically, o_2 is preferred to o_1 (although the sum of differences in achievement values is equal to 0) because the improvement of the worst value of a_1 is usually preferred over the worsening of the much better performing a_2 by the same value. This can be formalised by the classical (Pigou-Dalton) principle of transfers (see e.g., [14] and the references therein):

$$a_{i'} > a_{i''} \quad \Rightarrow \quad \boldsymbol{a} - \varepsilon \boldsymbol{e}_{i'} + \varepsilon \boldsymbol{e}_{i''} \succ \boldsymbol{a} \quad \text{for } 0 < \varepsilon \leq (a_{i'} - a_{i''})/2 \tag{5}$$

	alternatives			
achievements	o_1	o_2	...	o_m
a_1	0.0	0.1	...	1
a_2	1.0	0.9	...	1
a_3	0.5	0.5	...	0

Table 1: Sample achievement vectors

where e_i denotes the i–th unit vector in the criterion space. We say that the preference fulfills the *achievement equitability* if it complies with the principle of transfers (5). Note that the principle of transfers is equivalent to the property of decreasing marginal achievement utility

$$a_{i'} > a_{i''} \quad \Rightarrow \quad a + \varepsilon e_{i''} \succ a + \varepsilon e_{i'} \quad \text{for } \varepsilon > 0. \tag{6}$$

Further, preference models may treat the uniformly scaled and equally important achievements impartially thus focusing on the distribution of achievement values while ignoring their ordering. This means that the preference model is based on a set of achievement values that do not take into account which outcome is taking a specific value. Such a preference model is called (achievement) *impartial* (anonymous, symmetric), if it fulfills the following property:

$$(a_{\pi(1)}, a_{\pi(2)}, \dots, a_{\pi(m)}) \sim (a_1, a_2, \dots, a_n) \quad \text{for any permutation } \pi \text{ of } I \tag{7}$$

which means that any permuted achievement vector is indifferent in terms of the preference relation.

3.2. Specification and aggregation of preferences

Analysis of Pareto-optimal alternatives aims at finding the alternative that has the best (in terms of the user preferences) trade-offs between the criteria values. The corresponding analysis support is composed of two concerted mechanizms. First, an effective way for specification of user preferences; second, an aggregation of the preferences in a way that results in finding a Pareto-alternative that possibly well fits the user preferences.

Preference information is generally considered in two categories:

- Information between the criteria (e.g., relative importance of criteria);
- Information within each criterion (e.g., satisfaction/utility levels for different values of a criterion).

Because of the requirements explained in Section 2, the methods developed had to have a very simple representation of preference specification that is also suitable for users without analytical skills. Although we refrain from detailed specification of preferences within each criterion, the inter-criteria preferences model needs to be specified. For the criteria types (maximized or minimized), and very diverse orders of the criteria value magnitudes to be dealt with rationally, all criteria values are linearly mapped into the $[0, 1]$ interval of achievements, where 0 and 1 correspond to the worst and best value, respectively. Moreover, the lack of specification by the user of intra-criterion preferences is to some extent compensated for by the pairwise outperformance measures presented in Section 4.

During extensive discussions with practitioners it was agreed that specification of the relative importance of each criterion using the importance categories was the most suitable way of specifying preferences. The preferences for each criterion are therefore specified interactively through selection of one of eight levels which are interpreted as the corresponding value of $ri_i, i \in I$ as follows:

- $ri_i = 4$ denotes average importance;
- ri_i values 5 through 7: more, much more, vastly more, important than average, respectively;
- ri_i values 3 through 1: less, much less, vastly less, important than average, respectively;
- $ri_i = 0$ stands for temporally ignoring the criterion (and is children in the criteria hierarchy, if one is specified).

The non-zero values of $\boldsymbol{ri}$ are mapped into weights $w_i, i \in I$ in one of the following ways (depending on the method selected).

The first is the simplest linear (standard) mapping defined by:

$$\varrho_i = ri_i/7, \qquad i = 1, \ldots, n. \tag{8}$$

The second is the multiplicative mapping which, though is less popular than the linear one, has a number of advantages (see e.g., [16]); it is therefore used by all methods described in this paper. The multiplicative mapping is defined by:

$$\varrho_i = (\sqrt{2})^{2(ri_i-4)} = (2)^{ri_i-4}, \qquad i = 1, \ldots, n. \tag{9}$$

In other words, the values of weights are selected from the ordered set according to the position defined by the relative importance ri.

For both methods the vector ϱ is normalized to obtain

$$\bar{w}_i = \varrho_i / \sum_{i=1}^{n} \varrho_i, \qquad i = 1, \ldots, n. \tag{10}$$

They are treated as the final criteria weights $w_i = \bar{w}_i$, if no criteria hierarchy is considered.

If a criteria hierarchy is defined, then the following procedure is applied:

1. Compute weights $\bar{\boldsymbol{w}}$ defined by (10).
2. Define sets $S_k, k = 1, \ldots, K$ composed of siblings (i.e., nodes having a common parent node) of criteria.
3. Normalize subsets of siblings:

$$\hat{w}_l = \bar{w}_l / \sum_{l=1}^{L_k} \bar{w}_l, \qquad l \in S_k, k = 1, \ldots, K \tag{11}$$

   where L_k is the number of elements in S_k.
4. For each leaf-criterion define

$$w_i = \prod_{k \in M_i} \hat{w}_k, \qquad i = 1, \ldots, n \tag{12}$$

   where set M_i is composed of indices of the following criteria: i-th leaf criterion, intermediate-level criteria belonging to the branch of the active criteria tree leading to the i-th criterion.

Note that weights $\boldsymbol{w}$ generated by the above procedure are rational numbers, and are normalized, i.e.,

$$\sum_{i=1}^{n} w_i = 1.$$

The weights representing criteria importance can be introduced into methods either within the aggregation level or within the individual achievement model. We now outline both approaches used for the same three pairwise outperformance measures. In other words, we present six methods organized into two sets characterized by the way in which the weights are used for aggregation of preferences. These two sets of pairwise outperformance methods are presented in Sections 4 and 5, respectively.

The traditional weighted sum aggregation

$$s(\boldsymbol{a}^j) = \sum_{i=1}^{n} w_i a_{ij}, \qquad j \in J \tag{13}$$

is one of the oldest approaches to multicriteria analysis. The weights in (13) are typically interpreted in terms of a tradeoff preference model. This implies an additional scaling of individual achievements introduced in order to transform them into equally important units, while the aggregation itself remains impartial (symmetric). Depending on the method (or aggregation applied later), the individual achievements are multiplied either by w_i or by $1/w_i$. This approach is still popular because it is believed to be simple, intuitive, and reliable. Actually, however, the weights applied in the form of (13) offer only poor support to analysis of Pareto sets, and are often contra-intuitive. A discussion of this approach is beyond the scope of this paper, but it can be found, for example, in [17, 21, 26, 27].

Formula (13) may also be interpreted as the weighted average achievement with importance weights introduced at the aggregation level. This interpretation follows the rule that the importance weights w_i define a repetition measure within the distribution (population) of achievement values while the impartial aggregation takes into account this repetition measure. For example, let us consider two symmetric achievement vectors $\boldsymbol{a}^1 = (0, 1)$ and $\boldsymbol{a}^2 = (1, 0)$; introducing importance weights $w_1 = 0.75$ and $w_2 = 0.25$ we replace $\boldsymbol{a}^1 = (0, 1)$ with the distribution taking value 0 with the repetition measure 0.75, and taking value 1 with the repetition measure 0.25; while $\boldsymbol{a}^2 = (1, 0)$ is replaced with the distribution taking value 0 with the repetition measure 0.25, and taking value 1 with the repetition measure 0.75. In this specific case, the distributions can easily be equivalently interpreted in terms of four-dimensional space of equally important achievements (measure 1/4 each) where the original first achievement has been triplicated, thus $\bar{\boldsymbol{a}}^1 = (0, 0, 0, 1)$ and $\bar{\boldsymbol{a}}^2 = (1, 1, 1, 0)$.

Certainly, different interpretations of the weighted sum aggregation do not change its properties. This shows, however, how the importance weights can be utilised in more complicated aggregations. We will use this approach in Section 5 to exploit the importance weights to define ordered achievements.

Note that in the presence of importance weights, the preference model may still fulfill the property of *equitability* with respect to equally important achievements. However, the preference model needs then to comply with the importance weighted generalization [29] of the transfers' principle (5):

$$a_{i'} > a_{i''} \quad \Rightarrow \quad a - \varepsilon w_{i''} e_{i'} + \varepsilon w_{i'} e_{i''} \succ a \quad \text{for } 0 < \varepsilon \leq \frac{a_{i'} - a_{i''}}{w_{i'} + w_{i''}}. \tag{14}$$

4. Pairwise outperformance aggregation (POA)

4.1. Motivation and basic features

We present here the background, motivation, and implementation of three methods based on the pairwise outperformance aggregation approach. Further on, we assume the achievements are normalized to [0, 1], where 0 and 1 correspond to the worst and best values, respectively. This assumption serves only to simplify the presentation: the approach is applicable to criteria that are either minimized or maximized, and that have any range of values.

A natural improvement of the weighted sum aggregation is to transform individual achievements by a nonlinear (utility) function. The scalarizing function is then defined by:

$$s(a^j) = \sum_{i=1}^{n} w_i u(a_{ij}), \qquad j \in J. \tag{15}$$

The utility function $u(a_{ij})$ may be used to amplify the impact of increasing weak values (much) more than that of good values. A concave increasing utility function guarantees that an improvement of a smaller value may result in a larger satisfaction increase than the same improvement (in terms of the criterion value) of a larger value. Further, standard (user-defined) importance weights w_i are applied at the aggregation level. Thus, the entire scalarization may be viewed as the weighted average of nonlinear utilities.

As already mentioned, such a scalarizing function can be used for defining outperformance aggregation. Let us consider two alternatives o_j and o_l, and apply a nonlinear aggregation to a simple preference model, for example:

$$o_j \succ o_l \quad \Leftrightarrow \quad \sum_{i=1}^{n} w_i[u(a_{ij}) - u(a_{il})] > 0 \quad \text{and} \quad o_j \sim o_l \quad \Leftrightarrow \quad \sum_{i=1}^{n} w_i[u(a_{ij}) - u(a_{il})] = 0.$$

In the case of a strictly increasing and strictly concave utility function, the resulting preference model is monotonic and equitable with respect to the equally important achievements in the sense of (14).

Such a preference model is based on scalarizing functions defined for each alternative separately, and therefore does not take into account inter-alternative factors. However, the inter-alternative factors are strongly desired for pairwise comparisons, and this observation has motivated the authors to develop a new approach to pairwise outperformance aggregation. This approach is especially useful for problems for which the user cannot make pairwise comparisons directly because of a large number of alternatives or criteria.

Various approaches to aggregation of preference-relations are discussed in [3]. One of these is the widely used outranking procedure. Pirlot presented in [33] a common framework for defining outranking procedures. These procedures use pairwise comparisons instead of attempting numerical evaluation of each alternative using a common scale. The ELECTRE methods are examples of outranking procedures and belong to the class of weighted majority relation with veto. In these procedures the statement *alternative* o_i *outranks* o_j is equivalent to the statement that it is at least as good as o_j. The procedure of checking if one alternative outranks another is based on semiorder S_i and veto relation V_i. The semiorder S_i is determined by:

$$q_{ij} \; S_i \; q_{il} \Leftrightarrow q_{ij} \geq q_{il},$$

and the veto relation V_i is defined as:

$$q_{il} \; V_i \; q_{ij} \Leftrightarrow q_{il} > q_{ij} + \nu_i$$

where ν_i is a given value representing the maximum acceptable tolerance for worsening the i-th criterion value.

Then, o_j outranks o_l (denoted by $o_j \succeq_o o_l$), if the following condition is fulfilled:

$$\sum_{i \in I:\; q_{ij} S_i q_{il}} w_i \geq \delta \quad \text{and} \quad \text{there is no } i \text{ on which} \quad q_{il} \; V_i \; q_{ij}$$

where w_i denotes normalized weights and $\delta \in [0.5, \; 1]$ stands for the majority threshold. The above formula means that the sum of weights of the criteria having better values with respect to S_i is greater than a given threshold δ, and that there is no veto (V_i) on worsening any other criterion. This outranking relation is used in ELECTRE I. There are more advanced definitions of the outranking relation $\mathcal{S}$, for example, used in ELECTRE II and III, as well as in PROMETHEE I and II. However, these outranking procedures are not applicable to problems with many criteria; as pointed out in [6], the ELECTRE methods are suitable for decision models with more than five criteria, and preferably fewer than thirteen. Moreover, the methods based on pairwise comparisons are not really suitable for problems with more than six alternatives.

For pairwise comparison it is desirable to evaluate i-th achievements from the perspective of both compared alternatives, and then to aggregate these evaluations for all criteria. Let us consider two alternatives o_j and o_l. While evaluating the i-th achievement value of alternative o_l from the perspective of alternative o_j we consider the difference of the values relative to a_{ij}:

$$dc'_{jli} = \beta(a_{ij})(a_{ij} - a_{il}) \qquad i = 1, 2, \ldots, n \tag{16}$$

where $\beta(\cdot)$ is a convex, decreasing, and positive internal scaling function.

The role of $\beta(\cdot)$ is to amplify differently the impact of a given difference between criterion values for both alternatives. The amplification for weak achievements (values close to 0) is stronger than for strong ones (i.e., values close to 1), thus implementing the equitable preferences.

Returning to comparing alternatives o_j and o_l, we also consider the comparison from the perspective of alternative o_l. Symmetrically to (16), we define

$$dc''_{jli} = \beta(a_{il})(a_{il} - a_{ij}) \qquad i = 1, 2, \ldots, n. \tag{17}$$

By aggregating both comparisons we define for each criterion the following components dc_{jli} of the outperformance aggregation:

$$dc_{jli} = dc'_{jli} - dc''_{jli} = (\beta(a_{ij}) + \beta(a_{il}))(a_{ij} - a_{il}) \qquad i = 1, 2, \ldots, n. \tag{18}$$

Thus, the two factors of components dc_{jli} have the following roles:

- Factor $(a_{ij} - a_{il})$ is a difference between i-th criterion values of both alternatives compared.
- Factor $(\beta(a_{ij}) + \beta(a_{il}))$ averages the amplification of the difference of the achievements compared. The amplification depends on both achievements' values under comparison, and thus averages the scaling of the difference of the achievements in order to treat both alternatives equally.

One should also note the following properties of (18):

- For a large absolute value of $(a_{ij} - a_{il})$ one element of $(\beta(a_{ij}) + \beta(a_{il}))$ is also large, and thus the value of dc_{jli} is large.
- For a small absolute value of $(a_{ij} - a_{il})$ the value of dc_{jli} depends on whether the corresponding achievements are weak (small) or strong (large).

Based on the above discussion we aggregate the components dc_{jli} defined for each criterion by (18) into the following Pairwise Outperformance Aggregation (POA) measure $POA(o_j, o_l)$ for use in comparing alternatives (o_j, o_l):

$$POA(o_j, o_l) = \overline{POA}(a^j, a^l) = \sum_{i=1}^{n} w_i dc_{jli} = \sum_{i=1}^{n} w_i(\beta(a_{ij}) + \beta(a_{il}))(a_{ij} - a_{il}). \tag{19}$$

In other words, we define the POA preference model as:

$$POA(o_j, o_l) > 0 \quad \Rightarrow \quad o_j \succ o_l \quad \text{and} \quad POA(o_j, o_l) = 0 \quad \Rightarrow \quad o_j \sim o_l. \tag{20}$$

The aggregated outperformance measure (19) allows us to build the correspondingly valued preference relation. Note that the values of the component measures dc_{jlk} and dc_{ljk} have different signs but equal absolute values. Similarly, $d_{jl} = -d_{lj}$, where

$$d_{jl} = POA(o_j, o_l). \tag{21}$$

Hence, we can define the preference model (20). We will refer to this preference model as the *outperformance relation* and will say that alternative o_j weakly outperforms alternative o_l ($o_j \succeq o_l$) if $d_{jl} \geq 0$.

Such a weak outperformance relation is quite different from the commonly used outranking relations. However, it is similar in the sense that it is a binary relation defined on $Q \times Q$ such that $o_j \succeq o_l$, if there are enough arguments to decide that o_j is at least as good as o_l, while there is no essential reason to reject that statement [34, 39]. To account for both the differences and the similarity, we use the slightly different name, that is, outperformance instead of outranking.

The outperformance relation can be lexicographically enhanced by comparison of the original differences once the scaled lead to equal results, i.e.

$$\begin{aligned} o_j \succ_e o_l \quad &\Leftrightarrow \quad d_{jl} > 0 \quad \text{or} \quad \left(d_{jl} = 0 \quad \text{and} \quad \sum_{i=1}^{n} w_i(a_{ij} - a_{il}) > 0 \right) \\ o_j \sim_e o_l \quad &\Leftrightarrow \quad d_{jl} = 0 \quad \text{and} \quad \sum_{i=1}^{n} w_i(a_{ij} - a_{il}) = 0. \end{aligned} \tag{22}$$

Note that while the enhancement narrows the indifference relation, it does not affect the weak outperformance relation, as

$$o_j \succeq_e o_l \quad \Leftrightarrow \quad d_{jl} \geq 0 \quad \Leftrightarrow \quad o_j \succeq o_l.$$

4.2. Properties of POA

In this section we analyze the dependence between the form and parameter of functions $\beta(\cdot)$ defined above, the concavity and monotonicity properties of POA, as well as their relationship with the corresponding valued preference relations.

The properties of $POA(\cdot)$ depend on the choice of $\beta(\cdot)$. The following two forms of the function $\beta(x)$ have been analyzed and implemented:

$$\beta(x) = \lambda^{-x} \tag{23}$$

$$\beta(x) = \frac{\lambda - 1}{1 + (\lambda - 1)x} \tag{24}$$

where $x \in [0, 1]$ stands for normalized values of criteria and the parameter $\lambda > 1$.

The choice of the form of $\beta(x)$ and its parameter λ not only implies the analytical properties of the $POA(\cdot)$ but also the behavior of the corresponding multicriteria method. The following two elements are important from the implementation point of view:

- The ratio λ of values of $\beta(\cdot)$ for the worst and best values of normalized criteria:

$$\lambda = \beta(0)/\beta(1) \tag{25}$$

  which characterizes the amplification depending on the performance (weakness or strength) of the corresponding criterion. Note that for $\beta(x)$ defined by either (23) or (24) the ratio λ is actually equal to the parameter λ. Experiments show that values of λ of about 10 are satisfactory. However, advanced users should have a means of controlling the value of λ.
- Consistency of the aggregation (19) in the sense of monotonicity with respect to the Pareto dominance relation (4), i.e.:

$$\boldsymbol{a}^j \geq \boldsymbol{a}^l \text{ and } \boldsymbol{a}^j \neq \boldsymbol{a}^l \quad \Rightarrow \quad POA(o_j, o_l) > 0. \tag{26}$$

  If (26) does not hold, then the application of (20) does not guarantee that a non-dominated alternative will be selected. To avoid such situations, a preprocessing of alternatives is needed to filter-out the dominated alternatives before the pairwise outperformance aggregation (19) is applied. Such preprocessing is very easy for discrete alternatives problems but cannot be applied for MCA of mathematical models (where an auxiliary parametric optimization problem is generated for each specification of preferences).

For brevity's sake we use in this section a simplified notation for POA. For any alternative o_j we consider a relative outperformance function comparing any vector $\boldsymbol{y} = (y_1, y_2, \dots, y_n)$ in the achievement space (any possible achievement vector $\boldsymbol{a}$ even if not attainable for any alternative) with the achievement vector $\boldsymbol{a}^j$ defined by the alternative o_j, and denote:

$$P_j(\boldsymbol{y}) = \overline{POA}(\boldsymbol{y}, \boldsymbol{a}^j) = \sum_{i=1}^{n} w_i P_{ji}(y_i) \quad \text{where} \quad P_{ji}(y_i) = (\beta(y_i) + \beta(a_{ij}))(y_i - a_{ij}). \tag{27}$$

First of all, note that the POA preference model is monotonic and equitable. and that the following statement is valid.

Proposition 1. *For any positive and decreasing internal scaling function $\beta(\cdot)$ the POA preference relation (20) is strictly monotonic and equitable with respect to equally important achievements (14).*

Proof. Let us consider an achievement vector $\mathbf{a}^j$ and a vector $\mathbf{a} \neq \mathbf{a}^j$, $\mathbf{a} \geq \mathbf{a}^j$. Obviously, $\mathbf{a} = \mathbf{a}^j + \sum_{i=1}^n \varepsilon_i \mathbf{e}_i$ with $\varepsilon_i \geq 0$ for all $i \in I$ and at least one positive among them and computing value $P_j(\mathbf{a})$ we get

$$P_j(\mathbf{a}^j + \sum_{i=1}^{n} \varepsilon_i \mathbf{e}_i) = \sum_{i=1}^{n} \varepsilon_i(\beta(a_{ij} + \varepsilon_i) + \beta(a_{ij})) > 0.$$

Hence, $\overline{POA}(\mathbf{a}, \mathbf{a}^j) > 0$, and $\mathbf{a} \succ \mathbf{a}^j$, which justifies the strict monotonicity.

Let us further consider an achievement vector $\mathbf{a}^j$ with $a_{i'j} > a_{i''j}$, and compute

$$\begin{aligned} P_j(\mathbf{a}^j &- \varepsilon w_{i''} \mathbf{e}_{i'} + \varepsilon w_{i'} \mathbf{e}_{i''}) = \\ &- w_{i'}(\beta(a_{i'j} - \varepsilon w_{i''}) + \beta(a_{i'j}))\varepsilon w_{i''} + w_{i''}(\beta(a_{i''j} + \varepsilon w_{i'}) + \beta(a_{i''j}))\varepsilon w_{i'} \end{aligned}$$

where $\mathbf{e}_i$ denotes the i–th unit vector in the achievement space. Note that

$$P_j(\mathbf{a}^j - \varepsilon w_{i''} \mathbf{e}_{i'} + \varepsilon w_{i'} \mathbf{e}_{i''}) > 0$$

since

$$0 < \varepsilon \leq \frac{a_{i'j} - a_{i''j}}{w_{i'} + w_{i''}}$$

implies

$$\beta(a_{i''j}) > \beta(a_{i''j} + \varepsilon w_{i'}) \geq \beta(a_{i'j} - \varepsilon w_{i''}) > \beta(a_{i'j}).$$

Hence,

$$\overline{POA}(\mathbf{a}^j - \varepsilon w_{i''} \mathbf{e}_{i'} + \varepsilon w_{i'} \mathbf{e}_{i''}, \mathbf{a}^j) > 0$$

and

$$\mathbf{a}^j - \varepsilon w_{i''} \mathbf{e}_{i'} + \varepsilon w_{i'} \mathbf{e}_{i''} \succ \mathbf{a}^j,$$

which completes the proof. □

Let us further note that the outperformance function $P_j(\mathbf{y})$ is strictly increasing whenever all partial functions P_{ji} are strictly increasing and concave whenever all partial functions are concave. The two propositions below deal with the concavity and monotonicity properties of POA for $\beta(\cdot)$ defined by (23) and (24), respectively.

Proposition 2. *For any alternative (achievement vector $\mathbf{a}^j$) the corresponding relative outperformance functions P_{ji} are concave and strictly increasing with respect to each achievement y_i whenever $\beta(x) = \lambda^{-x}$ with $1 \leq \lambda \leq e$.*

Proof. Calculating the derivative of function $P_{ji}(y_i)$ we get

$$P'_{ji}(y_i) = (1 - \mu y_i + a_{ij}\mu)\lambda^{-y_i} + \lambda^{-a_{ij}}], \quad i = 1, 2, \dots, n$$

where $\mu = \ln\lambda$. If $1 \leq \lambda \leq e$, then $0 \leq \mu \leq 1$ and $1 - \mu y_i + a_{ij}\mu \geq 0$ for any $0 < y_i < 1$ and $0 \leq a_{ij} \leq 1$. Therefore, $P'_{ji}(y_i) > 0$ for all $0 < y_i < 1$.

Further, calculating the second derivative we get

$$P''_{ji}(y_i) = (\mu^2 y_i - 2\mu + a_{ij}\mu^2)\lambda^{-y_i}, \quad i = 1, 2, \dots, n.$$

If $1 \leq \lambda \leq e$, then $0 \leq \mu \leq 1$ and $(y_i - a_{ij})\mu \leq 2$ for any $0 < y_i < 1$ and $0 \leq a_{ij} \leq 1$. Therefore, $P''_{ji}(y_i) \leq 0$ for all $0 < y_i < 1$, thus guaranteeing the concavity properties. □

Corollary 1. *For any alternative (achievement vector a^j) the corresponding relative outperformance functions P_j is concave and strictly increasing with respect to each achievement y_i whenever $\beta(x) = \lambda^{-x}$ with $1 \leq \lambda \leq e$.*

To sum up, the POA defined by (19) with $\beta(\cdot)$ defined by (23) is concave and strictly increasing for $\lambda \in [1, e]$. Such a rather small range of values of λ results is a rather small amplification of weak criteria values. Thus, the corresponding method may either have undesired behavior for some problems (if applied with $\lambda < e$) or does not guarantee that a Pareto-efficient solution for $\lambda > e$ will be found.

Although the latter problem may be effectively addressed by filtering-out dominated alternatives in the preprocessing phase of multicriteria analysis, we have found an alternative form of $\beta(\cdot)$, which guarantees concavity and monotonicity of POA for any $\lambda > 1$.

To demonstrate this, let us now consider $\beta(\cdot)$ defined by (24). By applying $\beta(\cdot)$ defined (24) to (16) one obtains

$$dc'_{jli} = \frac{\lambda - 1}{1 + (\lambda - 1)a_{ij}}(a_{ij} - a_{il}) = \frac{1}{\tilde{a}_{ij}}(\tilde{a}_{ij} - \tilde{a}_{il}) \tag{28}$$

where

$$\tilde{a}_{ij} = \frac{1}{\lambda} + (1 - \frac{1}{\lambda})a_{ij}. \tag{29}$$

In other words, the criteria values are rescaled by (29) from [0, 1] to $[\frac{1}{\lambda}, 1]$, which in turn allows application of the standard inverse-proportional scaling.

Similarly,

$$dc''_{jli} = dc'_{lji} = \frac{1}{\tilde{a}_{il}}(\tilde{a}_{il} - \tilde{a}_{ij}) \qquad i = 1, 2, \dots, n \tag{30}$$

and

$$dc_{jli} = dc'_{jli} - dc''_{jli} = (\frac{1}{\tilde{a}_{ij}} + \frac{1}{\tilde{a}_{il}})(\tilde{a}_{ij} - \tilde{a}_{il}) = \frac{\tilde{a}_{ij}}{\tilde{a}_{il}} - \frac{\tilde{a}_{il}}{\tilde{a}_{ij}} \qquad i = 1, 2, \dots, n. \tag{31}$$

The corresponding relative outperformance function (27) comparing any achievement vector $y = (y_1, y_2, \dots, y_n)$ with achievements of o_j is then based on scalar functions

$$P_{ji}(y_i) = \frac{\tilde{y}_i}{\tilde{a}_{il}} - \frac{\tilde{a}_{il}}{\tilde{y}_i} = \frac{1 + (\lambda - 1)y_i}{1 + (\lambda - 1)a_{il}} - \frac{1 + (\lambda - 1)a_{il}}{1 + (\lambda - 1)y_i}). \tag{32}$$

Proposition 3. *For any alternative (achievement vector a^j) the corresponding partial relative outperformance functions (32) are concave and strictly increasing with respect to each achievement y_i whenever $\beta(\cdot)$ is defined by (24) with $\lambda > 1$.*

Proof. Calculating the derivative of function $P_{ji}(y_i)$ we obtain

$$P'_{ji}(y_i)\frac{(\lambda-1)(1+(\lambda-1)a_{ij})}{(1+(\lambda-1)y_i)^2}+\frac{\lambda-1}{1+(\lambda-1)a_{ij}}, \quad i=1,2,\dots,n.$$

If $\lambda > 1$, then $P'_{ji}(y_i) > 0$ for all $0 < y_i < 1$.

Further, calculating the second derivative we get

$$P''_{ji}(y_i)=\frac{-2(\lambda-1)^2(1+(\lambda-1)a_{ij})}{(1+(\lambda-1)y_i)^3}, \quad i=1,2,\dots,n.$$

If $\lambda > 1$, then $P''_{ji}(y_i) \leq 0$ for all $0 < y_i < 1$, thus guaranteeing the concavity properties. □

Corollary 2. *For any alternative (achievement vector a^j) the corresponding relative outperformance function P_j with partial functions (32) is concave and strictly increasing with respect to each achievement y_i whenever $\beta(\cdot)$ is defined by (24) with $\lambda > 1$.*

Note that by choosing a (very) large value of λ for $\beta(\cdot)$ defined by (24) the values of the achievements rescaled by (29) can be made very close to the original achievements, and the POA aggregation will be driven by improving the worst achievements' values. This is, in a sense, consistent with the Rawlsian approach (improve the weakest) which is a methodological justification for using the max-min scalarizing functions in the reference point approaches.

4.3. Illustration of POA properties

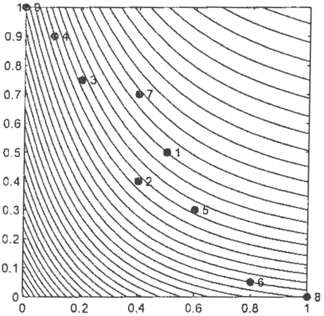
In this section we illustrate some properties of $P_j(y)$ using a sample problem with two criteria and nine alternatives. We focus our discussion on two pairs of alternatives (o_6, o_8) and (o_3, o_7) with achievement values as shown in Table 2. The criteria values are normalized: values of 0 and 1 correspond to the worst and best values of the corresponding criterion, respectively. Moreover, criteria are assumed to have equal relative importance.

	o_6	o_8	o_3	o_7	o_6 - o_8	o_3 - o_7
a_1	0.80	1.00	0.20	0.40	-0.20	-0.20
a_2	0.05	0.00	0.75	0.70	0.05	0.05

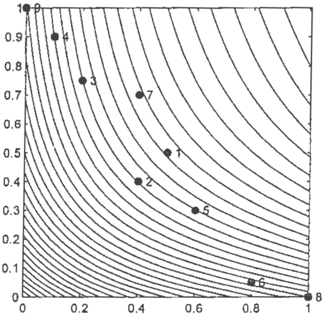
Table 2: Values of achievements a_1 and a_2 for alternatives o_6, o_8, o_3, o_7, and their differences for pairs (o_6, o_8) and (o_3, o_7).

Although these four alternatives differ substantially, they were defined in such a way that the pairs (o_6, o_8) and (o_3, o_7) have the same differences in achievement values for criterion 1 and criterion 2, respectively. We focus on two pairs of comparisons, namely (o_6, o_8) and (o_3, o_7). We observe that both alternatives of the pair (o_6, o_8) perform very well with respect to criterion 1, and very poorly for criterion 2; while alternatives (o_3, o_7) perform moderately on criterion 1 (20% to

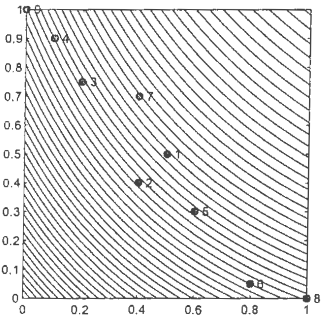
40% of the best value, respectively) but quite well on criterion 2 (75% to 70% of the best value, respectively). For both pairs, the trade-off (in terms of the difference between the achievement values) between the two corresponding alternatives is the same: 20% of improvement/worsening of achievement a_1 for 5% of worsening/improvement of achievement a_2. Thus, any method that does not take into account inter-criteria relations[6] will result in either $o_6 \succ o_8$ and $o_3 \succ o_7$, or $o_6 \prec o_8$ and $o_3 \prec o_7$.



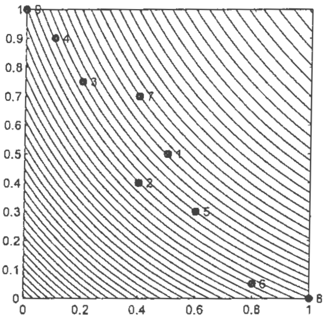
(a) Function $P_6(y)$, $\beta(x) = 10^{-x}$



(b) Function $P_3(y)$, $\beta(x) = 10^{-x}$



(c) Function $P_6(y)$, $\beta(x) = e^{-x}$



(d) Function $P_3(y)$, $\beta(x) = e^{-x}$

Figure 2: Isoline contours of functions $P_6(y)$ and $P_3(y)$; $\beta(x) = \lambda^{-x}$, for λ equal to 10 and e, respectively.

[6]Such methods use separable component achievement scalarizing functions, i.e., functions built for each criterion separately.

All alternatives are shown in Figure 2 as points marked with the corresponding numbers 1 through 9. The coordinates of the points correspond to the achievement values (achievement a_1 is shown on the horizontal axis). It is easy to see that all alternatives except o_2 are Pareto optimal. Figures 2(a) through 2(d) provide isoline contours for different functions $P_j(\cdot)$; namely, from the perspective of o_6 ($P_6(y)$ in Fig. 2(a) and 2(c)), and of o_3 ($P_3(y)$ in Fig. 2(b) and 2(d)), respectively. These two pairs of figures differ by the applied function $\beta(\cdot)$. The values of functions $P_6(\cdot)$ and $P_3(\cdot)$ are in the ranges: $P_6(y)$: $[-1.021,\ 0.993]$, and $[-1.258,\ 1.412]$, for λ equal to 10 and e, respectively. $P_3(y)$: $[-1.21,\ 0.654]$ and $[-1.469,\ 1.161]$, for λ equal to 10 and e, respectively. The contour lines are displayed for the values that differ by 0.05, and increase in the upward and right-hand directions. In other words, the outperformance relation can easily be seen by comparing the isolines corresponding to the respective alternatives.

Let us first consider alternatives a_6 and a_8. The ratio of improving (between alternative 6 and 8) the value of achievement a_1 (0.8 and 1, respectively) to compromising the value of achievement a_2 (0.05 and 0, respectively) is equal to 4. For both alternatives, a_1 performs quite well while a_2 performs very poorly. In other words, a preference of a_8 over a_6 ($a_6 \prec a_8$) means that a large improvement in the already well performing criterion is preferred over a small improvement of the very poor achievement. Conversely, a preference for improving the poor achievement implies $o_6 \succ o_8$. Such preferences are represented by different values of the λ parameter of the $\beta(\cdot)$ function that scales the achievements' differences in the relative outperformance function (27). To show the difference in the scaling effects caused by different values of λ, let us consider the isoline contours of function $P_6(y)$ for $\beta(x) = 10^{-x}$ and for $\beta(x) = e^{-x}$ shown in Fig. 2(a) and Fig. 2(c), respectively. From the isoline contours around alternatives 6 and 8 it can be seen that o_6 is preferred over o_8 for $\lambda = 10$, and o_8 is preferred over o_6 for $\lambda = e$. From the analytical point of view, to prefer the small improvement of the very weakly performing criterion 2 over the much larger improvement of the very well performing achievement a_1, the sum of the two functions $\beta(\cdot)$ in (27) for achievement a_2 needs to be more than four times larger than for achievement a_1. By easy calculations it can be shown that this is the case for $\lambda \geq 5$, and $o_6 \succ o_8$, while $\lambda \leq 4.9$ results in $o_6 \prec o_8$.

The pair of alternatives $\{a_3, a_7\}$ illustrates another situation; namely, that the moderately performing criterion 1 is improved much more than the clearly better performing criterion 2. In such situations (under the assumption of the equal relative importance of both criteria) a typical preference is that $o_7 \succ o_3$. Figures 2(b) and 2(d) show isoline contours of functions $P_3(y)$ for $\beta(\cdot) = 10^{-x}$, and $\beta(\cdot) = e^{-x}$, respectively. It is easy to see that in both cases $o_7 \succ o_3$. This shows the desired properties of the POA: achievement a_1 is clearly weaker than achievement a_2 for these two alternatives; therefore a larger (relative to compromising a_2) improvement of a_1 is preferred (let us recall that the relative importance of criteria is assumed to be equal).

The examples presented illustrate the useful property of the developed pairwise outperformance aggregation; namely, that it includes an effective mechanism for controlling the trade-offs between improvements and compromising the criteria values by taking into account the corresponding achievements. This is achieved by a rather simple internal scaling through the parameterized function $\beta(\cdot)$, where parameter λ has an intuitive interpretation that corresponds well to the practice of pairwise comparison. For problems with a large number of alternatives or criteria, methods based on pairwise comparison done directly by users are impracticable, but the POA approach is effective independently of the number of alternatives and criteria.

5. Ordered Pairwise Outperformance Aggregation (OPOA)

5.1. Background and basic features

Standard multiple criteria optimization problems with a general preference structure essentially assume the criteria to be incomparable (i.e., there exists no way for their comparison). Nevertheless, in our approach, as in many typical multiple criteria optimization methods, the individual achievement functions are built to measure the actual achievement of each outcome (a criterion value) with respect to the corresponding preference parameters. Thus, all the outcomes are transformed into a uniform scale of individual achievements within intervals $[0, 1]$. This enables comparison of the achievements of criteria with a possibly large ranges of values.

In the case of equally important attributes, the outperformance aggregation can easily be applied to the ordered achievement values, thus guaranteeing comparison of the worst results, the second-worst, etc. This can be formalized as follows. First, we introduce the ordering map $\Theta : R^n \rightarrow R^n$ such that $\Theta(y) = (\theta_1(y), \theta_2(y), \ldots, \theta_n(y))$, where $\theta_1(y) \leq \theta_2(y) \leq \cdots \leq \theta_n(y)$ and there exists a permutation τ of set I such that $\theta_i(y) = y_{\tau(i)}$ for $i \in I$. Next, we define the single criterion outperformance components in a similar way as in Section 4.1:

$$odc_{jlk} = (\beta(\theta_k(a^j)) + \beta(\theta_k(a^l)))(\theta_k(a^j) - \theta_k(a^l)) \qquad k = 1, 2, \ldots, n. \tag{33}$$

In particular the role of function $\beta(\cdot)$ is the same as discussed in Section 4.1; namely, to amplify the influence of improving weak achievements, and to lessen the impact of improving already good achievements.

The ordered pairwise outperformance relation is based on the aggregated quantities:

$$od_{jl} = \frac{1}{n}\sum_{k=1}^{n} odc_{jlk} = \frac{1}{n}\sum_{k=1}^{n}(\beta(\theta_k(a^j)) + \beta(\theta_k(a^l)))(\theta_k(a^j) - \theta_k(a^l)). \tag{34}$$

In the ordered outperformance aggregation (34) only distribution of the values of achievements is evaluated, and the resulting preference model is impartial in the sense of (7). When two alternatives o_j and o_l result in different achievement vectors a^j and a^l that are built of identically distributed achievement values, they lead to a zero value of the ordered outperformance value. Indeed, for two achievement vectors a^j and a^l which differ only in terms of the order of individual achievement values, one obtains $\Theta(a^j) = \Theta(a^l)$, and thereby $od_{jl} = od_{lj} = 0$. For instance, having $a^j = (0.1, 0.2, 0.3)$ and $a^l = (0.3, 0.1, 0.2)$ we get unordered outperformance measure $d_{jl} = 2\beta(-0.2) - \beta(-0.3) - \beta(-0.3)$ which is negative due to convexity of β, while obviously for the ordered measure $od_{jl} = od_{lj} = 0$.

The ordered outperformance aggregation (34) is built for equally important achievements. Importance weights of achievements can be introduced into the aggregation following the rule that importance weights w_i define a repetition measure within the distribution (population) of achievement values, similarly to [28, 31]. The outperformance components are then calculated within specific quantiles of this distribution that are small enough to guarantee the constant values of the ordered achievements for both alternatives. For instance, let us consider two symmetric achievement vectors $a^1 = (0, 1)$ and $a^2 = (1, 0)$; the ordered outperformance measure (34) for these achievements od_{12} is equal to 0. By introducing importance weights $w_1 = 0.75$ and $w_2 = 0.25$, we replace the achievement vector $a^1 = (0,1)$ by the distribution taking value 0 with the repetition measure 0.75, and taking value 1 with the repetition measure 0.25; similarly, $a^2 = (1,0)$ is replaced by the distribution taking value 1 with the repetition measure 0.75, and taking value 0 with the repetition measure 0.25. In this specific case, the distributions may easily be equivalently interpreted in terms of four dimensional space of equally important achievements (measure 1/4 each) where the original first achievement has been triplicated, thus $\bar{a}^1 = (0,0,0,1)$ and $\bar{a}^2 = (1,1,1,0)$. The ordered outperformance aggregation calculated for subsequent quantiles of size 1/4 results in:

$$\begin{aligned} od_{12} &= 0.25(1+1)(0-0) + 0.25(1+0.1)(0-1) + 0.25(1+0.1)(0-1) \\ &+ 0.25(0.1+0.1)(1-1) = -0.55. \end{aligned}$$

Certainly, not all the cases need to be transformed to equally important achievements so that the appropriate aggregation value can be calculated. The pairwise analysis may be split into (various size) quantile intervals of constant ordered achievements for both alternatives instead of quantile intervals of equal size. For the above straightforward example, this takes the following form:

$$od_{12} = 0.25(1+1)(0-0) + 0.5(1+0.1)(0-1) + 0.25(0.1+0.1)(1-1) = -0.55.$$

Independently of the importance weighting patterns, there are actually no more than $2n$ such quantile intervals to be analyzed. This approach can also be mathematically formalized as follows [31]. First, we introduce the right-continuous cumulative distribution function (cdf) of achievement values:

$$F_j(d) = \sum_{i=1}^{n} w_i \delta_{ij}(d) \tag{35}$$

where $\delta_{ij}(d) = 1$, if $a_{ij} \leq d$, and $\delta_{ij} = 0$, otherwise. Next, we introduce the quantile function $F_j^{(-1)}$ as the left-continuous inverse of the cumulative distribution function F_j, i.e., $F_j^{(-1)}(\xi) = \inf\{\eta : F_j(\eta) \geq \xi\}$ for $0 < \xi \leq 1$. Finally,

$$OPOA(o_j, o_l) = od_{jl} = \int_0^1 (\beta(F_j^{(-1)}(\xi)) + \beta(F_l^{(-1)}(\xi)))(F_j^{(-1)}(\xi) - F_l^{(-1)}(\xi))\, d\xi, \tag{36}$$

and the symmetry property holds, i.e.,

$$OPOA(o_j, o_l) = od_{jl} = -od_{lj} = -OPOA(o_l, o_j).$$

Note that in the case of equal weights $w_i = 1/n$, for any alternative oj one obtains the corresponding quantile function defined as a stepwise function with the same regular grid of breakpoints

$$F_j^{(-1)}(\xi) = \theta_k(a^j) \quad \text{for } \frac{k-1}{n} < \xi \leq \frac{k}{n}, \quad k = 1, 2, \ldots, n$$

thus allowing us to reduce formula (36) to the unweighted formula (34). However, not in all cases the formula (36) can be simplified in this way. Nevertheless, as both $F_j^{(-1)}(\xi)$ and $F_l^{(-1)}(\xi)$ are stepwise functions with n breakpoints, the entire integrated function is also stepwise with no more than $2n$ breakpoints. The ordered outperformance aggregation (36) can therefore simply be computed as a sum of $2n$ terms [9].

As with the unordered approach, we introduce the ordered outperformance preference model as:

$$o_j \succ o_l \quad \Leftrightarrow \quad od_{jl} > 0 \qquad \text{and} \qquad o_j \sim o_l \quad \Leftrightarrow \quad od_{jl} = 0. \tag{37}$$

The model can be lexicographically enhanced by comparing the unscaled ordered differences when the scaled ones lead to equal results. However, for unscaled ordered differences we get:

$$\int_0^1 (F_j^{(-1)}(\xi) - F_l^{(-1)}(\xi))\, d\xi = \int_0^1 F_j^{(-1)}(\xi)\, d\xi - \int_0^1 F_l^{(-1)}(\xi)\, d\xi = \sum_{i=1}^n w_i a_{ij} - \sum_{i=1}^n w_i a_{il}.$$

Hence, the comparison of the unscaled ordered differences is equivalent to the comparison of the average achievements, and the enhanced preference model can be formalized as follows:

$$\begin{aligned} o_j \succ_e o_l \quad &\Leftrightarrow \quad od_{jl} > 0 \quad \text{or} \quad \left(od_{jl} = 0 \quad \text{and} \quad \sum_{i=1}^n w_i (a_{ij} - a_{il}) > 0 \right) \\ o_j \sim_e o_l \quad &\Leftrightarrow \quad od_{jl} = 0 \quad \text{and} \quad \sum_{i=1}^n w_i (a_{ij} - a_{il}) = 0. \end{aligned} \tag{38}$$

5.2. Properties

The ordered outperformance aggregation (36) retains the property that only a distribution of the achievements' values is evaluated. In the presence of importance weights, the impartiality property (7) means that two alternatives are indifferent, if they lead to the same cumulative distribution function of achievements:

$$F_{a'} = F_{a''} \quad \Rightarrow \quad a' \sim a''. \tag{39}$$

When two alternatives o_j and o_l, both built of identically distributed achievement values taking into account the importance weights w_i, result in different achievement vectors $\boldsymbol{a}^j$ and $\boldsymbol{a}^l$, then they lead to the zero value of the ordered outperformance value. Indeed, two achievement vectors $\boldsymbol{a}^j$ and $\boldsymbol{a}^l$ then result in the same cumulative distribution functions F_j and F_l. Therefore, $F_j^{(-1)} = F_l^{(-1)}$ and thereby $od_{jl} = od_{lj} = 0$.

Moreover, the ordered outperformance aggregation (36) has the properties of monotonicity and equitability [14, 15]. Note that for the preference model based only on distribution of achievements (39), the equitability with respect to equally important achievements (14) can also be expressed through the cumulative distribution functions. The classical results of majorization theory [22], and the theory of stochastic orders [25] actually provide us with various alternative analytical characterizations of equitability with respect to equally important achievements (14). In particular, whenever $\boldsymbol{a}' = \boldsymbol{a} - \varepsilon w_{i''} \boldsymbol{e}_{i'} + \varepsilon w_{i'} \boldsymbol{e}_{i''}$ with some $0 < \varepsilon \leq (a_{i'} - a_{i''})/(w_{i'} + w_{i''})$, then

$$\int_0^\alpha F_{a'}^{(-1)}(\xi) d\xi \geq \int_0^\alpha F_a^{(-1)}(\xi) d\xi \tag{40}$$

for all $\alpha \in (0, 1]$, where at least one strict inequality holds. The latter is equivalent to the Second Stochastic Dominance (SSD) relation [25].

Proposition 4. *For any positive and decreasing internal scaling function $\beta(\cdot)$ the OPOA preference relation (37) is strictly monotonic and equitable.*

Proof. Suppose that $\boldsymbol{a}^j \geq \boldsymbol{a}^l$ and $\boldsymbol{a}^j \neq \boldsymbol{a}^l$. Then $F_j^{(-1)}(\xi) \geq F_l^{(-1)}(\xi)$ for all $0 < \xi \leq 1$ with a strict inequality holding for some subinterval $\alpha' \leq \xi \leq \alpha''$. Hence, the strict monotonicity follows simply from the fact that values of function $\beta(\cdot)$ are positive.

To prove the equitability, we need to reformulate the $OPOA(o_j, o_l)$ aggregation formula (36):

$$od_{jl} = \int_0^1 \beta(F_j^{(-1)}(\xi))(F_j^{(-1)}(\xi) - F_l^{(-1)}(\xi))\, d\xi + \int_0^1 \beta(F_l^{(-1)}(\xi))(F_j^{(-1)}(\xi) - F_l^{(-1)}(\xi))\, d\xi \tag{41}$$

where

$$F_j^{(-1)}(\xi) = \theta_k(\boldsymbol{a}^j) \quad \text{for } \xi_{k-1}^j < \xi \leq \xi_k^j, \quad k = 1, 2, \dots, n$$

with $\xi_0^j = 0$ and $\xi_k^j = \sum_{i=1}^k w_{\tau^j(i)}$ for $k = 1, 2, \dots, n$, and symmetrically for $F_l^{(-1)}(\xi)$. Hence,

$$\begin{aligned} od_{jl} &= \sum_{i=1}^n [\beta(\theta_k(\boldsymbol{a}^j)) \int_{\xi_{k-1}^j}^{\xi_k^j} (F_j^{(-1)}(\xi) - F_l^{(-1)}(\xi))\, d\xi] \\ &\quad + \sum_{i=1}^n [\beta(\theta_k(\boldsymbol{a}^l)) \int_{\xi_{k-1}^l}^{\xi_k^l} (F_j^{(-1)}(\xi) - F_l^{(-1)}(\xi))\, d\xi] \\ &= \sum_{i=1}^n [\beta_k^j \int_0^{\xi_k^j} (F_j^{(-1)}(\xi) - F_l^{(-1)}(\xi))\, d\xi] + \sum_{i=1}^n [\beta_k^l \int_0^{\xi_k^l} (F_j^{(-1)}(\xi) - F_l^{(-1)}(\xi))\, d\xi] \end{aligned} \tag{42}$$

where

$$\beta_n^j = \beta(\theta_n(\boldsymbol{a}^j)), \quad \beta_k^j = \beta(\theta_k(\boldsymbol{a}^j)) - \beta(\theta_{k+1}(\boldsymbol{a}^j)) \quad k = 1, 2, \dots, k-1$$

and accordingly defined β_k^l.

If $\boldsymbol{a}^j = \boldsymbol{a}^l - \varepsilon w_{i''} \boldsymbol{e}_{i'} + \varepsilon w_{i'} \boldsymbol{e}_{i''}$ with an $\varepsilon \in (0, (a_{i'}^l - a_{i''}^l)/(w_{i'} + w_{i''})]$ then, due to (40),

$$\int_0^\xi F_j^{(-1)}(\xi)\, d\xi \geq \int_0^\xi F_l^{(-1)}(\xi)\, d\xi$$

for all $0 < \xi \leq 1$. Note that for decreasing function $\beta(\cdot)$ all values β_k^j and β_k^l are nonnegative. This implies $od_{jl} \geq 0$ which proves equitability. □

For any alternative o_j let us consider a relative ordered outperformance function comparing any achievement vector $\boldsymbol{y} = (y_1, y_2, \dots, y_n)$ with the achievement vector $\boldsymbol{a}^j$ defined by the alternative o_j, and denote:

$$P_j(\boldsymbol{y}) = \overline{OPOA}(\boldsymbol{y}, a^j) = \int_0^1 (\beta(F_{\boldsymbol{y}}^{(-1)}(\xi)) + \beta(F_j^{(-1)}(\xi)))(F_{\boldsymbol{y}}^{(-1)}(\xi) - F_j^{(-1)}(\xi))\, d\xi. \tag{43}$$

For the POA aggregation we have shown that for our internal scaling functions $\beta(\cdot)$ the corresponding relative outperformance functions $P_j(y)$ are concave. While considering ordered arguments without importance weighting one obtains the so called Schur-concave functions (concave symmetric functions) [15, 22]. If achievements are considered with the importance weights, then the corresponding aggregation function may still fulfill equitability with respect to the equally important achievements. We say that function f is equitable if:

$$F_{a'} = F_{a''} \quad \Rightarrow \quad f(a') = f(a'') \tag{44}$$

$$a_{i'} > a_{i''} \quad \Rightarrow \quad f(a - \varepsilon w_{i''} e_{i'} + \varepsilon w_{i'} e_{i''}) > f(a) \quad \text{for } 0 < \varepsilon \le \frac{a_{i'} - a_{i''}}{w_{i'} + w_{i''}}. \tag{45}$$

The relative ordered outperformance function (43) depends on the application of POA relative outperformance functions $P_{ji}(y_i)$ defined by (27) to the ordered achievement values (inverse cdf). Therefore, as the theory of majorization [22] and stochastic orders [25] take advantage of rational weights [30], the monotonicity and concavity properties of the POA relative importance functions lead to monotonicity and equitability of the relative ordered outperformance function (43).

Proposition 5. *For any achievement vector a^j the corresponding relative ordered outperformance functions P_j (43) is equitable and strictly increasing with respect to each achievement y_i whenever $\beta(x) = \lambda^{-x}$ (Eq. (23)) with $1 \le \lambda \le e$ or $\beta(x) = (\lambda - 1)/(1 + (\lambda - 1)x)$ (Eq. (24)) with $\lambda > 1$.*

6. Transitivity property and Net-Flow approaches

6.1. Transitivity property of POA and OPOA

Unfortunately, neither standard nor enhanced preference models developed for POA and OPOA meet the transitivity requirement. This means that, although alternatives are pairwise comparable, there may not exist the best alternative (i.e., the one that (weakly) outperforms all others). This situation is also known as the Condorcet paradox (see e.g., [3]), and it can be illustrated by a simple example of three alternatives with two attributes.

Let us consider alternatives o_1, o_2 and o_3 with the corresponding achievement vectors $a^1 = (0.7255, 0.3110)$, $a^2 = (1.0, 0.2285)$ and $a^3 = (0.2230, 0.9992)$, respectively. Using the pairwise outperformance aggregation (19) with $\beta(x) = 10^{-x}$ we obtain: $POA(o_1, o_2) = 0.01$, $POA(o_2, o_3) = 0.0101$, and $POA(o_3, o_1) = 0.01$. Hence, $o_1 \succ o_2$, $o_2 \succ o_3$, and $o_3 \succ o_1$, which contradicts the transitivity. Indeed, alternatives o_1, o_2 and o_3 generate a cycle according to the pairwise outperformance aggregation (19), in which each alternative outperforms an alternative and is outperformed by another alternative.

The transitivity property is summarized by the following proposition.

Proposition 6. *Any alternative o_b selected by the linear search algorithm either weakly outperforms all the alternatives o_j ($j \in J$) or belongs to a cycle $o_b \succeq o_{j_1} \succeq o_{j_2} \succeq \ldots \succeq o_{j_k} \succeq o_b$ (with possible alternative repetitions $o_{j_{i'}} = o_{j_{i''}}$) such that for any alternative o_j ($j \in J$) there exists alternative $o_{j'}$ that belongs to the cycle and (weakly) outperforms o_j.*

Proof. If o_1 remains the selected alternative after the linear search algorithm then obviously $o_j \not\succ o_1$ for any $j = 2, 3, \ldots, m$ and thereby $o_1 \succeq o_j$ for any $j \in J$. Otherwise, the algorithm builds the sequence of subsequent outperforming alternatives $o_1 = o_{j_1} \prec o_{j_2} \prec \ldots \prec o_{j_p} = o_b$ and it identifies the following relations:

$$\begin{aligned}
&o_{j_1} \succeq o_{j_1+1}, o_{j_1} \succeq o_{j_1+2}, \ldots o_{j_1} \succeq o_{j_2-1}, \\
&o_{j_2} \succeq o_{j_2+1}, o_{j_2} \succeq o_{j_2+2}, \ldots o_{j_2} \succeq o_{j_3-1}, \\
&\ldots, \\
&o_{j_p} \succeq o_{j_p+1}, o_{j_p} \succeq o_{j_p+2}, \ldots o_{j_p} \succeq o_n.
\end{aligned}$$

If o_b does not outperform weakly all the alternatives then an alternative outperforming o_b exists. If alternative o_{j_i} outperforms o_b then we get cycle $o_b \prec o_{j_i} \prec o_{j_{i+1}} \prec \ldots \prec o_b$. If alternative o_{j_i+t} outperforms o_b then we get a longer cycle $o_b \prec o_{j_i} \prec o_{j_{i+1}} \prec \ldots \prec o_b$. If an alternative o_{j_k} ($k < i$) outperforms all alternatives of the cycle then we need to extend the cycle by additional cycle $o_{j_i} \prec o_{j_k} \prec o_{j_{k+1}} \prec \ldots \prec o_{j_i}$. If an alternative o_{j_k+t} ($k < i$) outperforms all alternatives of the cycle then we need to extend the cycle with additional cycle $o_{j_i} \prec o_{j_k+t} \preceq o_{j_k} \prec o_{j_{k+1}} \prec \ldots \prec o_{j_i}$. If needed, the cycle can be extended further to get finally a cycle with repetitions, such that for any alternative o_j ($j \in J$) there exists alternative $o_{j'}$ belonging to the cycle, and (weakly) outperforming o_j. □

Generally, for large problems it is difficult to either prove or disprove the transitivity property for pairwise outperformance methods (both POA and OPOA approaches). Indeed, for some methods, rather extensive tests (see Section 7) were needed to detect the Condorcet paradox. The use of these methods is thus risky because either dominated alternatives may be returned as Pareto-efficient or the algorithm may loop infinitely. On the other hand, the pairwise comparison methods are attractive because of their convincing background. Fortunately, it is possible to exploit the advantages of the pairwise outperformance method by applying the approach outlined below.

6.2. Net-Flow enhancement

The pairwise outperformance relations are built as the corresponding valued preference relations. To guarantee existence of the best alternative, one may thus use the standard way of obtaining a ranking method associated with valued preference relations, the so-called Net Flow Method. The Net Flow method is the only ranking method that is neutral, strongly monotonic, and independent of circuits [2].

For each alternative o_j we define the aggregate outperformance measure ds:

$$ds_j = 0.5 \sum_{l \in J} (d_{jl} - d_{lj}) \tag{46}$$

where, depending on the method

$$\text{either} \quad d_{jl} = POA(o_j, o_l) \quad \text{or} \quad d_{jl} = OPOA(o_j, o_l).$$

Note that due to the symmetry property of $POA(\cdot)$ defined by (19) and of $OPOA(\cdot)$ defined by (36), ds_j defined by (46) can be redefined as:

$$ds_j = \sum_{l \in J} d_{jl} = -\sum_{l \in J} d_{lj}. \tag{47}$$

Measure (47) assigns a real number to each alternative; it can thus be treated as a scalarizing (value) function for use in generating a complete ranking. Indeed, the preference model based on comparison of the measure values

$$(o_j \succ_n o_l \quad \Leftrightarrow \quad ds_j > ds_l) \qquad \text{and} \qquad (o_j \sim_n o_l \quad \Leftrightarrow \quad ds_j = ds_l) \tag{48}$$

is complete and transitive, thus allowing the best alternative to be identified. In particular, for the three alternative cycle defined in Section 6.1 we get $ds_1 = 0.02 - 0.02 = 0$, $ds_2 = -0.02 + 0.0202 + 0.0002$ and $ds_3 = 0.02 - 0.0202 = -0.0002$ and the final ranking $o_2 \succ_n o_1 \succ_n o_3$ with o_2 as the best alternative. Use of the linear search algorithm with relation $\succ_n$ allows us always to identify the best alternative o_b such that $o_b \succeq_n o_j$ for all $j \in J$.

Following (47), the value function defining the Net-Flow ordering can be expressed as the following function of achievement vectors:

$$v(y) = \sum_{j \in J} P_j(y), \tag{49}$$

where P_j are the corresponding relative outperformance functions defined according to the POA or OPOA methods. Hence, the following statements are valid.

Proposition 7. *If all outperformance functions P_j are strictly monotonic, then the Net-Flow value function (49) is strictly monotonic and the best solution selected according to the net flow ranking $\succeq_n$ is Pareto-optimal.*

Proposition 8. *If all outperformance functions P_j are concave, then the Net-Flow value function (49) is concave.*

Proposition 9. *If all outperformance functions P_j are equitable in the sense of (44)–(45), then the Net-Flow value function (49) is also equitable.*

In the case of the POA methods, it leads us to the separable value functions

$$v(y) = \sum_{i=1}^{n} w_i \sum_{j \in J} P_{ji}(y_i) \tag{50}$$

defined as a combination of partial outperformance functions P_{ji}.

Corollary 3. *The Net-Flow enhancement of the POA aggregation (19) is concave and strictly increasing with respect to each achievement y_i whenever $\beta(x) = \lambda^{-x}$ (Eq. (23)) with $1 \le \lambda \le e$ or $\beta(x) = (\lambda - 1)/(1 + (\lambda - 1)x)$ (Eq. (24)) with $\lambda > 1$.*

Note that the value function (50) for Net-Flow refined POA methods cannot, in general, be considered as the weighted utility function (15) because functions

$$u_i(y_i) = \sum_{j \in J} P_{ji}(y_i)$$

may differ due to possibly different sets of values $a_{i1}, \dots, a_{im}$ for various i. Therefore, despite the concave value function, such methods do not guarantee that the final ordering is equitable with respect to equally important achievements (14). The latter property is satisfied for the Net-Flow enhancement of the OPOA aggregation as the corresponding value function is equitable in the sense of (44)–(45), thus leading to the preference model which is equitable with respect to equally important achievements (14).

Corollary 4. *The Net-Flow enhancement of the OPOA aggregation (36) is equitable with respect to equally important achievements (14) and strictly increasing with respect to each achievement y_i whenever $\beta(x) = \lambda^{-x}$ (Eq. (23)) with $1 \le \lambda \le e$ or $\beta(x) = (\lambda - 1)/(1 + (\lambda - 1)x)$ (Eq. (24)) with $\lambda > 1$.*

7. Case studies: experience and results

7.1. Classification of the methods

The new methods described above have been developed and modified successively on the basis of analysis of their features and performance of earlier developed and/or modified methods. The methods have been implemented as the Web-based application called MCA,[7] using the client-server architecture [19]. We summarize here experience with the described methods to illustrate some of the methodological issues discussed above and to provide a justification of the methods selected for the public version of the MCA tool.

A key feature of each method is the mapping of the preferences (specified as relative importance of each criterion)[8] into the corresponding outperformance measure. We discuss here six different outperformance measures reflected by the corresponding method acronym: POA, POA-E, POA-Inv, OPOA, OPOA-E, and OPOA-Inv. Note that the first three methods are described in Section 4 and the next three in Section 5.

- POA uses the pairwise outperformance aggregation defined by (19) with $\beta(\cdot)$ defined by (23) for $\lambda = 10$. Thus alternative o_j dominates o_l, if

$$\sum_{i=1}^{n} w_i(10^{-a_{ij}} + 10^{-a_{il}})(a_{ij} - a_{il}) > 0.$$

  The safeguard defined by (22) is implemented in all POA methods to deal with the unlikely cases in which the above expression is equal to 0.
- POA-E differs from POA only by the value of $\lambda = e$. Thus alternative o_j dominates o_l, if

$$\sum_{i=1}^{n} w_i(e^{-a_{ij}} + e^{-a_{il}})(a_{ij} - a_{il}) > 0.$$

[7]The application is available free of charge for research and educational purposes at `http://www.iiasa.ac.at/~marek`.

[8]Relative criteria importance was the only preference information users submitted. This approach fits the requirement analysis for the methods described, which needed a simple way of specifying user preferences; the reasons are explained in Section 2.

- POA-Inv uses the pairwise outperformance aggregation defined by (19) with $\beta(\cdot)$ defined by (24) for $\lambda = 10$. Thus alternative o_j dominates o_l, if

$$\sum_{i=1}^{n} w_i\left(\frac{1+(\lambda-1)a_{ij}}{1+(\lambda-1)a_{il}} - \frac{1+(\lambda-1)a_{il}}{1+(\lambda-1)a_{ij}}\right) > 0.$$

- OPOA, OPOA-E, OPOA-INV differ from the corresponding POA, POA-E, POA-INV method by the applied outperformance method; instead of the pairwise outperformance aggregation defined by (19) they use the ordered pairwise outperformance aggregation (36).

There are two types of similarities between these methods:

- The first three methods (POA, POA-E, POA-Inv) use linear aggregations, while the other three (OPOA, OPOA-E, OPOA-Inv) use quantile aggregations. We will refer to these subsets of methods as LA (Linear Aggregation) and QA (Quantile Aggregation), respectively.
- Pairs of methods (POA, OPOA), (POA-E, OPOA-E), and (POA-Inv, OPOA-Inv) have the same representation of key elements of the corresponding outperformance measure.

All six methods enumerated belong to the group of *Local* (pairwise) methods where the scalarizing function uses only comparisons of pairs of alternatives. We have also built and analyzed the Global (Net-Flow enhanced) methods with the scalarizing function based on the Net-Flow approach (see Section 6). We thus consider POA-NF, POA-E-NF, POA-Inv-NF, OPOA-NF, OPOA-E-NF, OPOA-Inv-NF methods to be the Net-Flow enhancements of the corresponding local methods defined above (i.e., having acronyms without the -NF suffix).

7.2. Problems used for exploring the features of the methods

The features of the methods developed have been studied using the following five real-world problems of multicriteria analysis. The details of these problems are summarized in Table 3.[9]

	ch	de	fr	it	robot
Number of criteria	61	61	61	61	5
Number of alternatives	19	25	26	21	184
Number of unique preferences	235	96	179	60	32

Table 3: Summary of test problems.

Four problems are based on the analysis of the future energy technologies developed by the EU funded NEEDS project[10] for each of the four countries which in the follow-up discussion are denoted by the corresponding Internet code: ch - Switzerland, de - Germany, fr - France, it - Italy. Each of these problems has about 20 alternatives and 61 criteria organized in hierarchical structures. Over 3,000 stakeholders from several countries were invited to make an individual analysis using the Web-based application. In the end, 348 stakeholders initialized the analysis, and 162 actually completed it. Of all preferences specified by these 162 stakeholders, 570 were unique;

[9] The problems are documented (and available for further testing and use) through the dedicated application called MCA-NEEDS which is linked to `http://www.iiasa.ac.at/~marek`.

[10] Details are available at: `http://www.needs-project.org/`.

these preferences were extracted from the database, and used (with the corresponding problem) to explore the properties of all the methods described.

The number of stakeholders who completed the analysis compared with those who initialized it shows that the specifying preferences for 61 criteria was a challenging task, even for the implemented (simplest) way of doing the specification. This observation illustrates an open methodological issue with respect to public participation in decisionmaking that involves analysis of a complex problem; namely, the complexity of the underlying analysis that on one hand should be understood by the participants, and on the other hand should cover all the important aspects. In the NEEDS project, the set of 61 criteria was decided during a fairly comprehensive research process, in which each of the criteria (and their hierarchy) was first discussed in detail with specialists in the corresponding domain. The results of these discussions were then discussed with specialists in the other two domains. Moreover, for each attribute (i.e., the lowest-level criteria) a methodology for evaluating its value was also carefully researched, and corresponding studies were organized. The set of criteria, and the values of the attributes were therefore very well justified. It is, however, an open question whether a stakeholder analysis of a simpler problem (i.e., one defining a smaller number of criteria, and having less complex criteria hierarchy, or even no hierarchical structure at all) would provide more meaningful results, in the sense, that stakeholders would be more comfortable with specifying preferences for a small number of criteria.

The *Robot* acronym is used for the path-design problem for remote control of a partly autonomous space robot, see [11]. This is quite a complex engineering problem for which a large number of instances have been generated (each instance corresponding to a specific area of the asteroid to which the robot[11] was sent). The instance of this problem which was selected to compare the MCA methods developed has a different characteristic from the four future energy technologies problems, namely it has 183 alternatives and only five criteria. The multiple-criteria analysis of this instance was done by the researcher, who for the path-design problem specified 32 unique preferences during the corresponding analysis.

The approach outlined above provided large and diversified set of data for exploring the features of the methods developed. The data is composed of actual preferences of stakeholders having different backgrounds and preferences. The numbers of the unique preferences specified for each problem are presented in Table 3. Note that such an extensive sample of actual preferences is both very valuable and quite rarely available.

7.3. Transitivity properties for the designed methods

A common feature of the methods described here is the automated pairwise outperformance approach. Pairwise comparison had to be automated because of the number of alternatives involved which makes human pairwise comparison impractical. A natural requirement for such a procedure is to assure the transitivity of preferences (see Section 6), a lack of which is known as the Condorcet paradox (see e.g., [3]).

In Table 4 we summarize for each method and each of the problems the percentage of preferences for which the Condorcet paradox occurred. For each problem we provide two numbers:

- *All* which denotes all occurrences during the process of determining the ranking of alternatives. The ranking was based on an iterative procedure, in which the chosen Pareto alternative was

[11]The robot has the form of cube of 10cm size; it is a "jumping" robot, thus difficult to control.

method	ch		de		fr		it		robot	
	All %	Par. %	All %	Par. %	All %	Par. %	All %	Par. %	All %	Par. %
POA	27.7	18.3	50.0	31.3	46.4	22.9	35.0	23.3	100.0	75.0
POA-E	2.1	1.3	6.3	2.1	4.5	2.8	3.3	0.0	56.3	28.1
POA-Inv	71.9	50.2	93.8	76.0	85.5	63.1	76.7	63.3	100.0	84.4
OPOA	1.7	0.4	3.1	1.0	4.5	1.7	0.0	0.0	90.6	43.8
OPOA-E	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	25.0	9.4
OPOA-Inv	7.2	3.8	13.5	8.3	23.5	9.5	3.3	1.7	100.0	62.5

Table 4: Summary of experiments related to the Condorcet paradox.

removed from the set of alternatives, and the next best Pareto solutions (for the same preferences) were found from sets of the remaining alternatives.

- *Par.* which denotes the occurrences while the first Pareto alternative was searched for.

The results collected during the ranking procedure can be considered as generation of a large number of subproblems (over 10 times larger than the original problem) derived from each of original problems. However, we did not use these results for the comparison of methods because the preferences were specified by the users only for the full sets of alternatives. In Table 4 we do not report the methods belonging to the NF subset as they conform to the transitivity requirement by construction, and therefore the Condorcet paradox does not occur when they are used.

It is worth noting that is not easy to detect the Condorcet paradox for methods that do not conform to the transitivity requirement. In particular for the OPOA-E the transitivity problem was not detected for any of the 570 preferences (which is equivalent to the analysis of about 6,000 combinations of problems and preferences) of the four energy problems.

In our opinion the transitivity property is a necessary condition for any method being recommended for widescale use. The transitivity property obviously cannot be proven for the six methods belonging to the subset called *Local*. These methods have thus been shown for comparison with other approaches although the net-flow methods are preferred from the point of view of applications.

7.4. Pairwise comparisons of methods

For the reasons explained in Section 7.3, for actual use we recommend only the six methods that form the subset called *NF*. However, we found it interesting to explore the similarities (understood as the correspondence between the specified preferences and the resulting Pareto solution) between each of these methods with the corresponding method from the subset *Local* to provide indications of how often the results (for each of the considered 602 preferences) differ depending on which method from each of these pairs is used. The results of these comparisons are summarized in Table 5. It is interesting to note that the corresponding pairs of the methods are fairly likely (on average in more than 90% of cases) to provide the same Pareto alternative. Thus replacing a *Local* outperformance measure by its corresponding *NF* scalarizing function does not change the characteristics of the method.

Local method	Global method	ch	de	fr	it	robot	Average
POA	POA-NF	93.2	93.8	93.3	81.7	87.5	91.9
POA-E	POA-E-NF	87.7	83.3	81.6	95.0	68.8	84.9
POA-Inv	POA-Inv-NF	91.5	89.6	86.6	93.3	84.4	89.5
OPOA	OPOA-NF	95.3	93.8	93.3	93.3	87.5	93.9
OPOA-E	OPOA-E-NF	99.6	100.0	98.9	93.3	96.9	98.7
OPOA-Inv	OPOA-Inv-NF	94.9	96.9	91.1	93.3	93.8	93.9

Table 5: Consistency of global (Net-Flow) methods with their local origins. The numbers are percentages of 602 computations, for which the corresponding pair of the methods returned the same Pareto solution.

There is no clear recommendation for a choice of any of these six methods. Users with analytical skills may have personal preferences based on the methodological background of a particular method. However, from the point of view of mapping the preferences (specified as relative importance of criteria) into the selected Pareto solutions, most of the methods are similar (i.e., most providing the same Pareto solution for a given preferences). To justify this statement we summarize in Table 6 the results of 30 pairwise comparisons of the six net-flow-based methods with the corresponding *Local* method, each run for 602 unique preferences specified for the five problems used for analysis of the properties of the methods. It can be observed that two pairs of methods, namely (POA-NF, POA-E-NF) and (OPOA-NF and OPOA-Inv-NF), provide the same Pareto solution for a given preferences in 99.5% and 97.8% of preferences, respectively. On the other hand, the pair (POA-Inv-NF, OPOA-E-NF) has the smallest (72.6%) similarity.

	POA-E-NF	POA-Inv-NF	OPOA-NF	OPOA-E-NF	OPOA-Inv-NF
POA-NF	99.5	86.7	91.0	83.2	91.0
POA-E-NF		86.7	91.0	83.2	91.0
POA-Inv-NF			79.2	72.6	79.9
OPOA-NF				87.4	97.8
OPOA-E-NF					86.5

Table 6: Comparison of the Net-Flow methods: average solution consistency.

The methods have also been used for two large-scale analyses of energy scenarios; due to the space constraint we only summarize the research scope, and do not provide analysis of the corresponding results. The first was the integrated approach to energy sustainability based on multiple-criteria analysis of several thousands of energy-climate scenarios developed using the Message, an integrated assessment model. The analysis [23] showed synergies of possible energy portfolios and climate change policy providing potentially enormous costs savings: up to 600 billion US$ annually by 2030 in reduced pollution control and energy security expenditures. The second is the IIASA Energy Multicriteria Analysis (ENE-MCA) Policy tool [24][12] which provides an interactive analysis of the various synergies and trade-offs involved in attaching priorities to

[12] `http://www.iiasa.ac.at/web-apps/ene/GeaMCA`.

four of the main energy sustainability objectives: climate change, energy security, air pollution and health, and affordability. Given that policymakers and public often assign different priorities to this set of conflicting objectives, ENE-MCA helps users to explore how alternative world-views can lead to qualitatively different energy system futures.

All methods described in the report have also been tested on several other problems, including the two problems described in [41], and several small problems (in terms of numbers of both criteria and alternatives). All experiments performed show that the methods support analysis of all Pareto alternatives in an intuitive and easy way (in terms of criteria specification). This is especially important for problems with a large number of criteria and alternatives. Thus the methods conform to the basic necessary conditions of multicriteria analysis, namely, the requirements for an effective analysis of the whole Pareto set. An easy way of specifying preferences has clear advantages for users who do not have analytical skills and/or needs. However, analysts may prefer more advanced ways for specification of preferences, which provide also possibilities of more advanced explorations of certain parts of the Pareto set. Several such methods have also been developed and implemented in MCA, see [18, 19] for details.

8. Conclusions

The newly developed methods described in this paper support effective multiple criteria analysis of problems with many alternatives and many criteria. The specification of preferences is carried out in very simple way that is especially suitable for users who have limited analytical skills and/or time for the analysis, as well as for applications (like the two energy studies summarized above) which require precomputations of a Pareto-set representations. Yet the methods and the way they have been implemented support an effective analysis of the whole Pareto-set.

The methods exploit the advantages of approaches based on pairwise comparisons by modeling this type of comparison without asking the users to do it, which would be impracticable for problems with many alternatives.

The pairwise comparison methods generally do not possess the transitivity property needed to guarantee the uniqueness of the solution. The Net-Flow methods have thus been applied to various pairwise comparison techniques to guarantee a unique selection of the alternative corresponding best to the preferences specified. Extensive tests using a large and diverse sample of actual user preferences have shown that the behavior (in the sense of mapping the preferences into the corresponding Pareto solution) of the pairs of methods (defined as a given method implemented with and without the Net-Flow approach) is rather similar.

Within our implementation all dominated alternatives are eliminated in the preprocessing phase of the solver, thus always guaranteeing efficiency of the best alternative selected according to the Net-Flow method. If not eliminated, a method may select a dominated alternative if the corresponding pairwise outperformance relation is not (strictly) monotonic (Proposition 7). Ranking, however, involves sequential solver executions; therefore, all alternatives (including those dominated within a larger alternative set) are ranked.

In the case of unordered pairwise outperformance relations, note that their monotonicity is related with properties of the $\beta(\cdot)$ functions. For some $\beta(\cdot)$ functions, their parameter has to conform to quite strong restrictions in order to result in the strict monotonicity of the outperformance relation, while for other forms of $\beta(\cdot)$ there are no such restrictions. On the other hand, the ordered pairwise outperformance relations preserve monotonicity for any positive function $\beta(\cdot)$. Thus the properties of the developed methods that have been presented offer quite a lot of flexibility in modeling the amplification of differences within particular intervals of achievements.

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