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Abstract

In this paper we study an influence of the robustness measure, which is used in the robustness analysis for the generic combinatorial optimization problem, on the values of so-called robustness tolerances of weights. Two of such closely related measures are considered: the worst-case absolute regret and the worst-case relative regret. The robustness tolerances in the later case are studied in Libura [12], where a method of calculating them is provided. In this note we show, that if the worst-case absolute regret is used as a robustness measure, then the problem of finding the robustness tolerances becomes very simple. Namely, it is shown that in this case any weight in the problem may be perturbed individually by 100% of its initial value without destroying the robustness of an optimal solution.

Keywords: combinatorial optimization, robustness analysis, robustness measures, robustness tolerances.

1 Introduction

We consider a combinatorial optimization problem in the following generic form:

$$v(\mathcal{F}, c) = \min\{w(F, c) : \ F \in \mathcal{F}\}, \tag{1}$$

where the set of feasible solutions $\mathcal{F}$ is a family of nonempty subsets of a given ground set $E = \{e_1, \ldots, e_n\}$ and $c = (c(e_1), \ldots, c(e_n))^{\mathrm{T}} \in \mathbb{R}^n$ denotes the vector of weights of the elements of E. For $c \in \mathbb{R}^n$ and $F \in \mathcal{F}$, the objective function in (1) represents the total weight of this solution, i.e.,

$$w(F, c) = \sum_{e \in F} c(e).$$

Numerous discrete optimization problems, like e.g. the traveling salesman problem, the minimum spanning tree problem, the shortest path problem, the linear 0–1 programming problem, can be stated in this general form.

We will assume that the set of feasible solutions $\mathcal{F}$ in problem (1) is fixed but the vector of weights can change or it is given with errors. Let $\mathcal{C} \subseteq \mathbb{R}^n$ be a set of all possible realizations of the vector c, called the *scenarios.* Consider an initial scenario $c^o \in \mathcal{C}$ and let $\Omega(c^o) = \arg\min\{w(F, c^o) : \ F \in \mathcal{F}\}$ denote the set of optimal solutions in (1) for $c = c^o$.

Given an optimal solution $F^o \in \Omega(c^o)$ an important question concerns the stability of this solution on the set of possible scenarios $\mathcal{C}$. This question belongs to so-called *sensitivity* (*stability*) *analysis*, which is regarded an essential step of any optimization procedure (see e.g. Greenberg [5], Libura [9], Sotskov et al. [17]). The main goal of the sensitivity analysis for combinatorial optimization problems consists in finding a subset of scenarios, for which the solution F^o remains *optimal.* In Libura [11, 12, 13] a natural extension of the standard sensitivity analysis called the *robustness analysis* of initially optimal solutions was proposed. The main goal of this analysis consists in determining a subset of scenarios for which the solution F^o remains *robust.*

There are various concepts of the robustness of solutions in optimization and there are many possible robustness measures as well (see e.g. Averbakh [1], Ben-Tal and Nemirowski [2], Bertsimas and Sim [3], Kasperski [6], Kouvelis and Yu [7], Mulvey et al. [15], Roy [16], and the references therein). In this paper we will consider two frequently used measures of the robustness: the worst-case relative regret and the worst-case absolute regret. We study an influence of the choice of the robustness measure on the values of so-called robustness tolerances introduced in Libura [12].

2 Robustness measures

In the following we assume that for any $F \in \mathcal{F}$ and $c \in \mathcal{C}$ the inequality $w(F, c) > 0$ holds.

Consider a feasible solution $F \in \mathcal{F}$ and an initial scenario $c^o \in \mathcal{C}$. The quality of the solution F for the scenario c^o can be measured by its *relative regret* $\epsilon(F, c^o)$, where

$$\epsilon(F, c^o) = \max_{F' \in \mathcal{F}} \frac{w(F, c^o) - w(F', c^o)}{w(F', c^o)} = \frac{w(F, c^o) - v(\mathcal{F}, c^o)}{v(\mathcal{F}, c^o)}. \qquad (2)$$

A feasible solution $F^o \in \mathcal{F}$ is called an *optimal solution* for the scenario c^o if and only if $\epsilon(F^o, c^o) \leq \epsilon(F, c^o)$ for any $F \in \mathcal{F}$. Let $\Omega(c^o)$ denote the set of optimal solutions in problem (1) for the scenario c^o. From (2) we have immediately that $\epsilon(F, c^o) \geq 0$, and that $\epsilon(F, c^o) = 0$ if and only if $F \in \Omega(c^o)$.

There are various other measures of the quality of a given feasible solution $F \in \mathcal{F}$ for a scenario c^o. The most frequently used is so-called *absolute regret*

$$\epsilon_a(F, c^o) = w(F, c^o) - v(\mathcal{F}, c^o). \qquad (3)$$

This measure leads to simpler models and it is more appropriate than the relative regret (2) when the absolute deviation from the optimality is more meaningful for a decision maker than the percentage deviation. On the other hand the relative regret leads usually to less conservative robustness approach. A comprehensive discussion of practical situations, where a particular choice of the quality measure is relevant, can be found in Kouvelis and Yu [7] or Roy [16].

The considered quality measures (2) and (3) lead now directly to the corresponding robustness measures of the feasible solution $F \in \mathcal{F}$. These measures are defined with respect to the set of possible scenarios $\mathcal{C}$ as the maximum values of the quality measures of this solution on the set of scenarios. We have therefore the *worst-case relative regret* of the solution F:

$$Z(F, \mathcal{C}) = \max_{c \in \mathcal{C}} \epsilon(F, c)$$

and the *worst-case absolute regret* of F:

$$Z_a(F, \mathcal{C}) = \max_{c \in \mathcal{C}} \epsilon_a(F, c).$$

A feasible solution $F \in \mathcal{F}$ is called a *robust solution* with respect to the set of scenarios $\mathcal{C}$ if and only if its robustness measure is the minimum among all the feasible solutions, i.e., $Z(F, \mathcal{C}) \leq Z(F', \mathcal{C})$ for any $F' \in \mathcal{F}$, or (respectively) $Z_a(F, \mathcal{C}) \leq Z_a(F', \mathcal{C})$ for any $F' \in \mathcal{F}$.

3 Robustness tolerances

The robustness tolerances were defined in Libura [12] for a particular set of scenarios. Namely it was assumed, that only a single weight $c(e)$ for some $e \in E$ can change and all the remaining weights are equal to their initial values given by the weight vector c^o. Moreover, this single weight may be increased or decreased by no more than $\delta \cdot 100\%$ of its initial value $c^o(e) > 0$, where $\delta \in [0, 1)$. For a given $e \in E$, $c^o \in \mathbb{R}$ and $\delta \in [0, 1)$ we will denote this particular family of scenarios by $T_\delta(e, c^o)$. Formally,

$$T_\delta(e, c^o) = \{c \in \mathbb{R}^n : \ |c(e) - c^o(e)| \leq \delta \cdot c^o(e), \ c(e') = c^o(e') \ \text{ for } \ e' \neq e\}.$$

Let F^o be an optimal solution of the problem (1) obtained for the initial vector of weights c^o. The *robustness tolerance* of the weight of element e is defined as the maximum value of the parameter δ for which the solution F^o remains robust with respect to any set of scenarios $T_{\delta'}(e, c^o)$ for any $\delta' \leq \delta$. In the following we will denote the robustness tolerance by $t^r(e)$ if the worst-case relative regret is used as a robustness measure, and – respectively – by $t^r_a(e)$ in the case of the worst-case absolute regret.

The robustness tolerances defined in this way are direct analogues of the standard sensitivity analysis tolerances of weights, which are considered in numerous papers (see e.g. Chakravarti and Wagelmans [4], Libura [8], Libura et al. [14], van Hoesel and Wagelmans [20], Sotskov et al. [17], Tarjan [18], Turkensteen et al. [19], Wendell [21]), and which provide the maximum increase and the maximum decrease of the weight of the considered element preserving the *optimality* of the solution F^o. It is easy to see that the optimality conditions corresponding to both quality measures: the relative regret and the absolute regret are the same. Consequently, the values of the standard tolerances do not depend on the fact, which one of these two measures is used in the sensitivity analysis.

The situation appears quite different in the framework of the robustness analysis. The case of the relative regret (and the corresponding worst-case relative robustness measure) is studied in Libura [12], where the following fact is obtained:

Theorem 1 *Let $\mathcal{F}^e = \{F \in \mathcal{F} : \ e \in F\}$. Then for $F^o \in \Omega(c^o)$,*

$$t^r(e) = \begin{cases} 1 & \text{if} \quad e \in F^o, \\ \min\left\{1, [\, v(\mathcal{F}^e, c^o)^2 - v(\mathcal{F}, c^o)^2\,]^{\frac{1}{2}} \cdot c^o(e)^{-1}\right\} & \text{if} \quad e \notin F^o. \end{cases} \tag{4}$$

Theorem 1 states that if the worst-case relative regret is used as the robustness measure, then the robustness tolerance of the weight of arbitrary element belonging to the optimal solution F^o is equal to 1. For any remaining element $e \in E \setminus F^o$ its robustness tolerance $t^r(e)$ can be calculated by finding the optimal value of an auxiliary optimization problem, which is obtained from the original problem (1) by requiring that e must belong to any feasible solution.

In the following we will show that if the worst-case absolute regret is used as the robustness measure, then there is no need to solve any auxiliary optimization problem, because all of the robustness tolerances are simply equal to 1. This means, that the weight of any single element $e \in E$ can be increased or decreased individually by 100% of its initial value without destroying the robustness of the solution F^o.

Before proving this result we will first calculate the worst-case absolute regret $Z_a(F, T_\delta(e, c^o))$ of the feasible solution $F \in \mathcal{F}$ for the particular set of scenarios $T_\delta(e, c^o)$. We have

$$\begin{aligned}
Z_a(F, T_\delta(e, c^o)) &= \max_{c \in T_\delta(e,c^o)} (w(F, c) - v(\mathcal{F}, c)) \\
&= \max_{c \in T_\delta(e,c^o)} \max_{F' \in \mathcal{F}} (w(F, c) - w(F', c)) \\
&= \max_{F' \in \mathcal{F}} \max_{c \in T_\delta(e,c^o)} (w(F, c) - w(F', c)) \\
&= \max_{F' \in \mathcal{F}} (w(F, c^o) - w(F', c^o) + \delta \cdot w((F \otimes F') \cap \{e\}, c^o)),
\end{aligned}$$

where $F \otimes F' = (F \setminus F') \cup (F' \setminus F)$.

Denote $\mathcal{F}^e = \{F \in \mathcal{F} : \ e \in F\}$, $\ \mathcal{F}_e = \{F \in \mathcal{F} : \ e \notin F\}$, and let

$$\begin{aligned}
a^e &= \max_{F' \in \mathcal{F}^e} (w(F, c^o) - w(F', c^o) + \delta \cdot w((F \otimes F') \cap \{e\}, c^o)) \\
&= w(F, c^o) - v(\mathcal{F}^e, c^o) + \delta \cdot (c^o(e) - w(F \cap \{e\}, c^o)), \qquad (5)
\end{aligned}$$

$$\begin{aligned}
a_e &= \max_{F' \in \mathcal{F}_e} (w(F, c^o) - w(F', c^o) + \delta \cdot w((F \otimes F') \cap \{e\}, c^o)) \\
&= w(F, c^o) - v(\mathcal{F}_e, c^o) + \delta \cdot w(F \cap \{e\}, c^o). \qquad (6)
\end{aligned}$$

We have therefore for $F \in \mathcal{F}$, $\delta \in [0, 1)$,

$$Z_a(F, T_\delta(e, c^o)) = \max\{a^e, \ a_e\}. \qquad (7)$$

It will be convenient to state a formula for calculating $Z_a(F, T_\delta(e, c^o))$ separately in both cases: $e \in F$ and $e \notin F$. From (5), (6) and (7) we obtain: For $F \in \mathcal{F}^e$ and $\delta \in [0, 1)$,

$$\begin{aligned} Z_a(F, T_\delta(e, c^o)) &= \max\{w(F, c^o) - v(\mathcal{F}^e, c^o), \\ &\qquad w(F, c^o) - v(\mathcal{F}_e, c^o) + \delta \cdot c^o(e)\}. \end{aligned} \tag{8}$$

For $F \in \mathcal{F}_e$ and $\delta \in [0, 1)$,

$$\begin{aligned} Z_a(F, T_\delta(e, c^o)) &= \max\{w(F, c^o) - v(\mathcal{F}^e, c^o) + \delta \cdot c^o(e), \\ &\qquad w(F, c^o) - v(\mathcal{F}_e, c^o)\}. \end{aligned} \tag{9}$$

The following result holds:

Theorem 2 *For $F^o \in \Omega(c^o)$ and for any $e \in E$, $t_a^r(e) = 1$.*

Proof Let $F^o \in \Omega(c^o)$ and $e \in E$. The robustness tolerance $t_a^r(e)$ of the weight of the element e was defined as the maximum value of the parameter δ for which the solution F^o remains robust with respect to any set of scenarios $T_{\delta'}(e, c^o)$ for any $\delta' \leq \delta$, i.e.,

$$\begin{aligned} t_a^r(e) &= \sup\{\delta \in [0, 1) : \ Z_a(F^o, T_{\delta'}(e, c^o)) \leq Z_a(F, T_{\delta'}(e, c^o)) \\ &\qquad \text{for any } \ F \in \mathcal{F} \ \text{ and } \ \delta' \leq \delta \ \}. \end{aligned}$$

(i) Consider first the case when $e \in F^o$, which implies that

$$w(F^o, c^o) = v(\mathcal{F}, c^o) = v(\mathcal{F}^e, c^o) \leq v(\mathcal{F}_e, c^o).$$

Using (8) we have,

$$Z_a(F^o, T_\delta(e, c^o)) = \max\{0, v(\mathcal{F}, c^o) - v(\mathcal{F}_e, c^o) + \delta \cdot c^o(e)\}.$$

Now it is easy to see that for any $F \in \mathcal{F}$ and $\delta \in [0, 1)$ the inequality $Z_a(F, T_\delta(e, c^o)) \geq Z_a(F^o, T_\delta(e, c^o))$ holds. Indeed, for any $F \in \mathcal{F}^e$ we have from (8),

$$\begin{aligned} Z_a(F, T_\delta(e, c^o)) &= \max\{w(F, c^o) - v(\mathcal{F}^e, c^o), w(F, c^o) - v(\mathcal{F}_e, c^o) + \delta \cdot c^o(e)\} \\ &\geq \max\{0, w(F, c^o) - v(\mathcal{F}_e, c^o) + \delta \cdot c^o(e)\} \\ &\geq \max\{0, v(\mathcal{F}, c^o) - v(\mathcal{F}_e, c^o) + \delta \cdot c^o(e)\} \\ &= Z_a(F^o, T_\delta(e, c^o)). \end{aligned}$$

Similarly, for $F \in \mathcal{F}_e$ we have from (9),

$$\begin{aligned}
Z_a(F, T_\delta(e, c^o)) &= \max\{w(F, c^o) - v(\mathcal{F}^e, c^o) + \delta \cdot c^o(e), w(F, c^o) - v(\mathcal{F}_e, c^o)\} \\
&\geq \max\{w(F, c^o) - v(\mathcal{F}^e, c^o) + \delta \cdot c^o(e),\ 0\} \\
&\geq \max\{w(F, c^o) - v(\mathcal{F}_e, c^o) + \delta \cdot c^o(e),\ 0\} \\
&\geq \max\{v(\mathcal{F}, c^o) - v(\mathcal{F}_e, c^o) + \delta \cdot c^o(e),\ 0\} \\
&= Z_a(F^o, T_\delta(e, c^o)).
\end{aligned}$$

(ii) Consider now the case when $F^o \in \mathcal{F}_e$, which implies that

$$w(F^o, c^o) = v(\mathcal{F}, c^o) = v(\mathcal{F}_e, c^o) \leq v(\mathcal{F}^e, c^o),$$

and from (9) it follows that

$$Z_a(F^o, T_\delta(e, c^o)) = \max\{v(\mathcal{F}, c^o) - v(\mathcal{F}^e, c^o) + \delta \cdot c^o(e),\ 0\}.$$

We will show that also in this case the inequality

$$Z_a(F, T_\delta(e, c^o)) \geq Z_a(F^o, T_\delta(e, c^o))$$

holds for any $F \in \mathcal{F}$ and $\delta \in [0, 1)$. Indeed, for $F \in \mathcal{F}_e$ we have from (9)

$$\begin{aligned}
Z_a(F, T_\delta(e, c^o)) &= \max\{w(F, c^o) - v(\mathcal{F}^e, c^o) + \delta \cdot c^o(e), w(F, c^o) - v(\mathcal{F}_e, c^o)\} \\
&\geq \max\{w(F, c^o) - v(\mathcal{F}^e, c^o) + \delta \cdot c^o(e),\ 0\} \\
&\geq \max\{v(\mathcal{F}, c^o) - v(\mathcal{F}^e, c^o) + \delta \cdot c^o(e),\ 0\} \\
&= Z_a(F^o, T_\delta(e, c^o)).
\end{aligned}$$

Similarly, for $F \in \mathcal{F}^e$ we have from (8),

$$\begin{aligned}
Z_a(F, T_\delta(e, c^o)) &= \max\{w(F, c^o) - v(\mathcal{F}^e, c^o), w(F, c^o) - v(\mathcal{F}_e, c^o) + \delta \cdot c^o(e)\} \\
&\geq \max\{0,\ w(F, c^o) - v(\mathcal{F}_e, c^o) + \delta \cdot c^o(e)\} \\
&\geq \max\{0,\ w(F, c^o) - v(\mathcal{F}^e, c^o) + \delta \cdot c^o(e)\} \\
&\geq \max\{0,\ v(\mathcal{F}, c^o) - v(\mathcal{F}^e, c^o) + \delta \cdot c^o(e)\} \\
&= Z_a(F^o, T_\delta(e, c^o)).
\end{aligned}$$

Consequently, given $F^o \in \Omega(c^o)$ and $e \in E$ we obtain that for arbitrary $F \in \mathcal{F}$ and for any $\delta \in [0, 1)$ the inequality

$$Z_a(F, T_\delta(e, c^o)) \geq Z_a(F^o, T_\delta(e, c^o))$$

holds, which implies that for arbitrary $e \in E$, $t_a^r(e) = 1$.

■

4 Conclusions

In this paper we show that if the worst-case absolute regret is used as the robustness measure, then any single weight in the generic combinatorial optimization problem may be increased or decreased by 100% of its initial value without destroying the robustness of an optimal solution obtained for these initial values of weights. This means that the robustness tolerance problem in this case has an immediate solution contrary to the previously considered case of the worst-case relative regret, when solving an additional optimization problem is necessary for calculating the tolerances of weights.

References

[1] I. Averbakh, Computing and minimizing the relative regret in combinatorial optimization with interval data. Discrete Optimization 2 (2005) 273-287.

[2] A. Ben-Tal, A. Nemirovski, Robust convex optimization. Mathematics of Operations Research 23 (1998) 769-805.

[3] D. Bertsimas, M. Sim, Robust discrete optimization and network flows. Mathematical Programming B 98 (2003) 43-71.

[4] N. Chakravarti, A.P.M. Wagelmans, Calculation of stability radii for combinatorial optimization problems. Operations Research Letters 23 (1998) 1-7.

[5] H. Greenberg, An annotated bibliography for post-solution analysis in mixed integer and combinatorial optimization. In: D. Woodruff (Ed.). Advances in Computational and Stochastic Optimization, Logic Programming and Heuristic Search. Dordrecht: Kluwer Academic Publishers, 1998. p. 97-148.

[6] A. Kasperski, Discrete Optimization with Interval Data. Minmax Regret and Fuzzy Approach. Springer-Verlag, Berlin, Heidelberg, 2008.

[7] P. Kouvelis, G. Yu, Robust Discrete Optimization and Its Applications. Kluver Academic Publishers, Boston 1997.

[8] M. Libura, Sensitivity analysis for minimum Hamiltonian path and traveling salesman problem. Discrete Applied Mathematisc 30 (1991) 197-211.

[9] M. Libura, Optimality conditions and sensitivity analysis for combinatorial optimization problems. Control and Cybernetics 25 (1996) 1165-1180.

[10] M. Libura, Quality of solutions for perturbed combinatorial optimization problems. Control and Cybernetics 29 (2000) 199-219.

[11] M. Libura, On the robustness of optimal solutions for combinatorial optimization problems. Control and Cybernetics 38 (2009) 671-685.

[12] M. Libura, A note on robustness tolerances for combinatorial optimization problems. Information Processing Letters 110 (2010) 725-729.

[13] M. Libura, Sensitivity and robustness analysis in combinatorial optimization. Systems Research Institute, Polish Academy of Sciences, RB/7/2010, p. 1-21.

[14] M. Libura, E.S. van der Poort, G. Sierksma, and J.A.A. van der Veen, Stability aspects of the traveling salesman problem based on k-best solutions. Discrete Applied Mathematics 87 (1998) 159-185.

[15] J.M. Mulvey, S.A. Vanderbei, S.A. Zenios, Robust optimization of large scale systems. Operations Research 43 (1995) 264-281.

[16] B. Roy, Robustness in operational research and decision aiding: A multifaceted issue. European Journal of Operational Research 200 (2010) 629-638.

[17] Y.N. Sotskov, E.N. Leontev, E.N. Gordeev, Some concepts of the stability analysis in combinatorial optimization. Discrete Applied Mathematics 58 (1995) 169-190.

[18] R.E. Tarjan, Sensitivity analysis of minimum spanning trees and minimum path trees. Information Processing Letters 14 (1982) 30-33.

[19] M. Turkensteen, D. Ghosh, B. Goldengorin, G. Sierksma, Tolerance-based Branch and Bound algorithms for the ATSP. European Journal of Operational Research 189 (2008) 775-788.

[20] C.P.M. van Hoesel, A.P.M. Wagelmans, On the complexity of postoptimality analysis of 0/1 programs. Discrete Applied Mathematics 91 (1999) 251-263.

[21] R.E. Wendell, Tolerances sensitivity and optimality bounds in linear programming. Management Science 50 (2005) 797-803.