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Sensitivity and robustness analysis in combinatorial optimization

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Abstract

We consider so-called *generic combinatorial optimization problem*, where the set of feasible solutions is some family of nonempty subsets of a finite ground set with specified positive initial weights of elements, and the objective function represents the total weight of elements of the feasible solution. We assume that the set of feasible solutions is fixed, but the weights of elements may be perturbed or are given with errors. All possible realizations of weights form the set of *scenarios*. A feasible solution, which for a given set of scenarios guarantees the minimum value of the worst-case relative regret among all the feasible solutions, is called a *robust solution*.

In this paper we deal with so-called *robustness analysis* for the generic combinatorial optimization problem. Its main goal consists in finding subsets of scenarios for which an initially optimal solution of the problem remains robust. Thus, the robustness analysis may be considered as a natural extension of the standard *sensitivity analysis* in combinatorial optimization. Main results of the paper concern the *robustness region*, the *robustness radius* and the *robustness tolerances*, which are introduced as direct analogues of the optimality region, the optimality radius and the weigts tolerances considered in the sensitivity analysis.

Keywords: combinatorial optimization, sensitivity analysis, robustness analysis, robustness region, robustness radius, robustness tolerances.

1 Introduction

We consider a combinatorial optimization problem in the following generic form:

$$v(\mathcal{F}, c) = \min\{w(F, c) : F \in \mathcal{F}\}, \tag{1}$$

where the set of feasible solutions $\mathcal{F}$ is a family of nonempty subsets of a given ground set $E = \{e_1, \ldots, e_n\}$ and $c = (c(e_1), \ldots, c(e_n))^T \in \mathbb{R}^n$ denotes the vector of weights of the elements of E. For $c \in \mathbb{R}^n$ and $F \in \mathcal{F}$, the objective function in (1) represents the total weight of this solution, i.e.,

$$w(F, c) = \sum_{e \in F} c(e).$$

Numerous discrete optimization problems, like e.g. the traveling salesman problem, the minimum spanning tree problem, the shortest path problem, the linear 0–1 programming problem, can be stated in this general form.

We will assume that the set of feasible solutions $\mathcal{F}$ in problem (1) is fixed but the vector of weights can change or it is given with errors. Let $\mathcal{C} \subseteq \mathbb{R}^n$ be a set of all possible realizations of the vector c, called the *scenarios*. Consider an initial scenario $c^o \in \mathcal{C}$ and let $\Omega(c^o) = \arg\min\{w(F, c^o) : F \in \mathcal{F}\}$ denote the set of optimal solutions in (1) for $c = c^o$.

Given an optimal solution $F^o \in \Omega(c^o)$ an important question concerns the stability of this solution on the set of possible scenarios $\mathcal{C}$. This question belongs to so-called *sensitivity (stability) analysis*, which is regarded an essential step of any optimization procedure (see e.g. Greenberg [5], Libura [9], Sotskov et al. [18]). The main goal of the sensitivity analysis for combinatorial optimization problems consists in finding a subset of scenarios, for which the solution F^o remains *optimal*.

In this paper we consider a natural extension of the standard sensitivity analysis, which we will call the *robustness analysis* of initially optimal solutions. Namely, as the main goal of this analysis, we will consider a problem of determining a subset of scenarios for which the solution F^o remains *robust*.

There are various concepts of the robustness of solutions in optimization and there are many possible robustness measures as well (see e.g. Averbakh [1], Ben-Tal and Nemirowski [2], Bertsimas and Sim [3], Kouvelis and Yu [7], Mulvey et al. [15], Roy [16], and the references therein). In this paper we will use as a robustness measure the *worst-case relative regret*, i.e., the *maximum relative error* of the solution considered over the set of all scenarios.

In standard sensitivity analysis one seeks for the inclusion-wise maximal subset of the weights vectors in problem (1) for which the solution F^o remains optimal. Such a set is called the *optimality region* (or – the *stability* region) of the solution F^o.

It is obvious that an arbitrary optimal solution $F^o \in \Omega(c^o)$ is a robust solution for its optimality region. But this solution may remain robust for significantly larger set of scenarios. This motivates studying a natural analogue of the stability region, which we will call the *robustness region* of the solution F^o. Formally, we will define the robustness region of the solution F^o as the inclusion-wise maximal subset of the scenarios for which this solution remains a robust solution. Thus, the robustness region of an initially optimal solution F^o provides a set of all the scenarios for which this solution guarentees the minimum value of the worst-case relative regret among all the feasible solutions of the optimization problem.

It is important to note that in case of multiple optimal solutions, i.e. in case $\Omega(c^o) > 1$, all of the solutions belonging to $\Omega(c^o)$ are indistinguishable from the *optimality* point of view, but they may appear quite different from the *robustness* point of view. The following simple example (see Libura [12]) illustrates this situation.

Example 1 Consider an undirected graph $G = (V, E)$ (see Fig. 1), where $V = \{1,2,3,4,5\}$, $E = \{\{1,2\},\{1,3\},\{1,4\},\{2,4\},\{3,4\},\{3,5\},\{4,5\}\} = \{e_1,\ldots,e_7\}$. Let $\mathcal{F}$ be a family of subsets of E corresponding to all spanning trees in G, and let $c^o = (2,2,2,2,1,2,2)^T$ be a vector of the initial weights of edges in G. Then the combinatorial optimization problem (1) for $c = c^o$ is the minimum spanning tree problem in the weighted graph G. We will use this optimization problem with different weights vectors in the following examples, and therefore in Fig. 2 all of the spanning trees in graph G are presented.

The set of optimal solutions in problem (1) for $c = c^o$ contains exactly ten spanning trees: $\Omega(c^o) = \{T_5, T_6, T_8, T_9, T_{11}, T_{12}, T_{16}, T_{17}, T_{19}, T_{20}\}$. All of them are, obviously, robust for the set of scenarios $C = \{c^o\}$. But when we allow simultaneous independent perturbations of weights for all edges of the

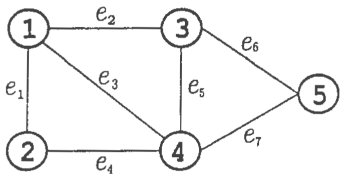

Figure 1: Weighted graph G from Example 1.

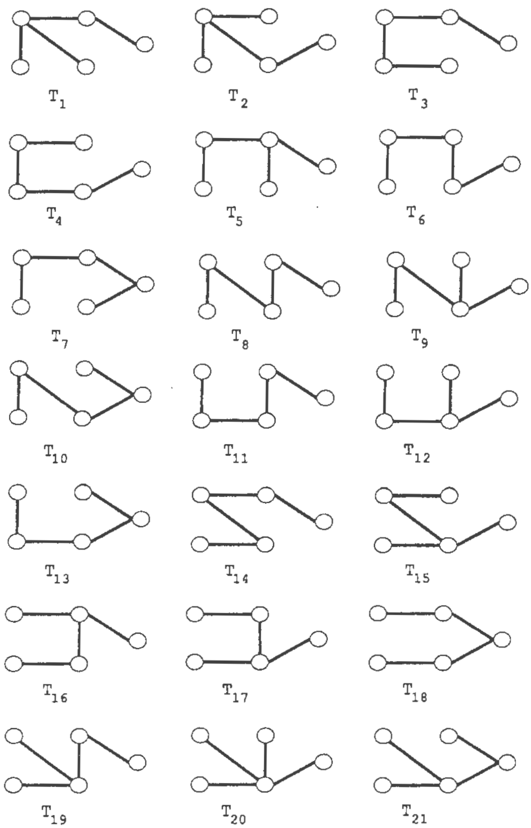

Figure 2: The feasible solutions in the minimum spanning tree problem.

graph G, then only two of them, namely T_{11} and T_{12}, appear robust for some nonempty neighborhood of the initial vector of weights c^o (see Example 2). For all the remaining optimal solutions we can construct arbitrarily small nonzero perturbations of the weights which destroy their robustness. Such a solutions may be regarded unacceptable from the robustness point of view.

It is therefore important to ask for a method of selecting the optimal solutions which preserve its robustness in a neighborhood of the initial vector of weights. In the following we will call such solutions the *robust optimal* solutions. A characterization of robust optimal solutions was obtained in Libura [12]; Section 3.2 describes a generalization of that result. □

2 Optimality and robustness regions

In the following we will assume that for any $F \in \mathcal{F}$ and $c \in C$ the inequality $w(F, c) > 0$ holds.

Consider a feasible solution $F \in \mathcal{F}$ and an initial scenario $c^o \in C$. The quality of the solution F for the scenario c^o can be measured by its *relative regret* $\epsilon(F, c^o)$, where

$$\epsilon(F, c^o) = \max_{F' \in \mathcal{F}} \frac{w(F, c^o) - w(F', c^o)}{w(F', c^o)} = \frac{w(F, c^o) - v(\mathcal{F}, c^o)}{v(\mathcal{F}, c^o)}. \quad (2)$$

A feasible solution $F^o \in \mathcal{F}$ is called an *optimal solution* for the scenario c^o if and only if $\epsilon(F^o, c^o) \leq \epsilon(F, c^o)$ for any $F \in \mathcal{F}$. Let $\Omega(c^o)$ denote the set of optimal solutions in problem (1) for the scenario c^o. From (2) we have imediately that $\epsilon(F, c^o) \geq 0$, and that $\epsilon(F, c^o) = 0$ if and only if $F \in \Omega(c^o)$.

There are various other measures of the quality of a given feasible solution $F \in \mathcal{F}$ for a scenario c^o. The most frequently used is so-called *absolute regret*

$$\epsilon_a(F, c^o) = w(F, c^o) - v(\mathcal{F}, c^o).$$

This measure leads to simpler models and it is more appropriate than the relative regret (2) when the absolute deviation from the optimality is more meaningful for a decision maker than the percentage deviation. On the other hand the relative regret leads usually to less conservative robustness approach. A comprehensive discussion of practical situations, where a particular choice of the quality measure is relevant, can be found in Kouvelis and Yu [7] or Roy [16]. In the following we will use the relative regret $\epsilon(F, c^o)$ as a measure of the quality of the feasible solution F for the scenario c^o.

Consider now a particular optimal solution $F^o \in \Omega(c^o)$. The main object studied in the sensitivity analysis for combinatorial optimization problems is so-called *optimality region* $S(F^o, \mathcal{C})$ of the solution F^o, defined as the inclusion-wise maximal subset of scenarios, for which this solution remains optimal, i.e.,

$$S(F^o, \mathcal{C}) = \{c \in \mathcal{C} : \ \epsilon(F^o, c) = 0\}.$$

Denote $S(F^o) = S(F^o, \mathbb{R}^n)$. We have immediately $S(F^o, \mathcal{C}) = S(F^o) \cap \mathcal{C}$. It is well known that the optimality region $S(F^o)$ is a convex polyhedral cone in $\mathbb{R}^n$ (see e.g. Libura [9]). This follows directly from the theory of linear programming. Namely, let $\xi(F) = (\xi_1(F), \ldots, \xi_n(F))^{\mathsf{T}}$, denote the characteristic vector of the subset $F \subseteq E$, i.e., for $i = 1, \ldots, n$, $\xi_i(F) = 1$ if $e_i \in F$, and $\xi_i(F) = 0$, otherwise. The generic combinatorial optimization problem (1) is now equivalent (see e.g. Schrijver [17]) to the following linear program:

$$v(\mathcal{F}, c) = \min\{c^{\mathsf{T}}x : \ x \in \Phi(\mathcal{F})\},$$

where $\Phi(\mathcal{F}) = \text{conv.hull}\ \{\xi(F) : F \in \mathcal{F}\}$.

The set of feasible solutions $\mathcal{F}$ is finite, which implies that $\Phi(\mathcal{F})$ is a polyhedral convex set and it can be – at least in principle – described by a system of linear inequalities

$$\Phi(\mathcal{F}) = \left\{x \in \mathbb{R}^n : \ h_i^{\mathsf{T}}x \leq \overline{h}_i, \quad i \in I\right\} \tag{3}$$

for some $h_i \in \mathbb{R}^n$, $\overline{h}_i \in \mathbb{R}$, $i \in I$. Let $I(F^o) \subseteq I$ be a subset of inequalities binding at vertex $x^o = \xi(F^o)$, i.e., $h_i^{\mathsf{T}}x^o = \overline{h}_i$ for $i \in I^o$. Then

$$S(F^o) = -\,\text{cone}\,\{h_i, \ i \in I(F^o)\}. \tag{4}$$

Although a polyhedral description (3) of the set $\Phi(\mathcal{F})$ may contain very large number of facets, it can be exploited in various approximations of the optimality region $S(F^o)$ (see e.g. Libura et al. [14]), and it appears useful in the sensitivity analysis. In the following we will define an analogue of the optimality region $S(F^o)$ in the robustness analysis framework.

Let for a feasible solution $F \in \mathcal{F}$ and for a given set of scenarios $\mathcal{C} \subseteq \mathbb{R}^n$,

$$Z(F, \mathcal{C}) = \max_{c \in \mathcal{C}} \ \epsilon(F, c).$$

We will call $Z(F, \mathcal{C})$ the *worst-case relative regret* of the solution F over the set of scenarios $\mathcal{C}$ and we will use this concept in a definition of the *robust solutions* for problem (1).

Definition 1 *A feasible solution $F \in \mathcal{F}$ is a robust solution for the set of scenarios $C \subseteq \mathbb{R}^n$ if and only if the following inequalities hold:*

$$Z(F,C) \leq Z(F',C) \qquad \textit{for any} \quad F' \in \mathcal{F}. \tag{5}$$

Thus, a feasible solution F is robust for the set of scenarios C if it guarantees the minimum value of the worst-case relative regret on the set C among all the feasible solutions of the optimization problem (1).

Consider now an optimal solution $F^o \in \Omega(c^o)$. It is obvious that F^o is a robust solution for the set of scenarios $S(F^o, C)$. But it may happen that F^o remains robust for significantly larger set of scenarios. Actually, we will be interested in the inclusion-wise maximal subset of scenarios, for which the solution F^o is a robust solution. Such a subset will be denoted $R(F^o, C)$ and called the *robustness region* of the initially optimal solution F^o. Formally,

$$R(F^o,C) = \{c \in C : \ Z(F^o, R(F^o,C)) \leq Z(F, R(F^o,C)) \ \text{ for any } \ F \in \mathcal{F}\}.$$

For $C = \mathbb{R}^n$ we will use a simplified notation $R(F^o) = R(F^o, \mathbb{R}^n)$. Observe anyway that this time – in contrast to the optimality region – we cannot simply express $R(F^o, C)$ as an intersection of the sets $R(F^o)$ and C.

The above definition of the robustness region leads to significant difficulties with calculating this set for particular combinatorial optimization problems. It appears that there is no direct relation between $R(F^o)$ and a polyhedral description of the convex hull of the characteristic vectors of feasible solutions as in case of the set $S(F^o)$. Therefore, it is reasonable to consider various subsets of the robustness region, which may appear easier to analyze and – simultaneously – give some insight into the robustness properties of the solutions considered. The main role in such analysis is played by appropriate choice of a particular set of scenarios.

3 Scenarios

The set of scenarios C plays a crucial role in describing the uncertainty concerning the data of the optimization problem. In this paper we will use the same sets of scenarios in the sensitivity analysis and in the robustness analysis contexts although their interpretations in both cases will be actually different. Namely, in the sensitivity analysis – regarded as a part of the postoptimality analysis – the set C represents all possible data changes which would influence the quality of the solution, which has been already implemented. In the robustness analysis this set describes all possible perturbations of the data, which we want to be hedged against. Frequently, a choice of the set $\mathcal{C}$ will be determined by various simplifying assumptions we have to make in case of approximate analysis.

An appropriate choice of the set of scenarios will lead to definitions of such objects as the *tolerances of weights*, the *optimality radius*, the *accuracy radius*, the *robustness radius* etc. In the following we are discussing several particular choices of the set of scenarios and corresponding objects studied in the sensitivity analysis and in the robustness analysis.

3.1 Basic scenarios

In sensitivity analysis the set $\mathcal{C} = \mathbb{R}^n$ may be regarded as an initial set of scenarios and it is actually a starting point for any further analysis. The main object studied for this particular set of scenarios is the optimality region $S(F)$ of a feasible solution $F \in \mathcal{F}$. Nevertheless, sometimes it is necessary to avoid negative weights of elements, which may have no reasonable interpretation. In such a case we will consider a restricted set $\mathcal{C}_+ = \{c \in \mathbb{R}^n : \; c \geq 0\}$ of the scenarios.

In the sensitivity analysis context a choice of the set of scenarios corresponds mainly to various simplifications, which make the analysis possible, or it reflects particular restrictions on data perturbations implied by real-live problems. A standard approach here consists in an assumption that only the weights of elements belonging to some given subset $Q \subseteq E$ may be perturbed while all the remaining weights are equal to their initial values given by the vector $c^o \in \mathbb{R}^n$. This leads to the following set of scenarios, which we will consider a *basic set of scenarios*:

$$\mathcal{C}(Q, c^o) = \{c \in \mathbb{R}^n : \; c(e) = c^o(e) \;\; \text{for} \;\; e \notin Q\}.$$

The most frequently studied special case of this family of scenarios corresponds to an assumption that only the weight of a particular element $e \in E$ may be perturbed, i.e., $Q = \{e\}$. This leads to so-called (optimality) *tolerances* of weights, which are considered in numerous papers (see e.g. Chakravarti and Wagelmans [4], Libura [8], Libura et al. [14], van Hoesel and Wagelmans [21], Sotskov et al. [18], Tarjan [19], Turkensteen et al. [20], Wendell [22]).

Let $F^o \in \Omega(c^o)$. From the convexity of the set $S(F^o)$ it follows directly that

$$\begin{aligned} S\left(F^o, \mathcal{C}(\{e\}, c^o)\right) = \{c \in \mathbb{R}^n \;\; : \;\; & c(e') = c^o(e') \;\; \text{for} \;\; e' \neq e, \\ & c^o - t^-(e) \leq c(e) \leq c^o + t^+(e)\}, \end{aligned}$$

where $t^+(e), t^-(e) \in \mathbb{R} \cup \{\infty\}$ denote so-called *upper* and *lower tolerances* of the weight $c(e)$. Thus, $t^+(e)$, $t^-(e)$ provide, respectively, the maximum increase and the maximum decrease of the initial weight $c^o(e)$ which will preserve the optimality of the solution F^o in problem (1) under the assumption that all the remaining weights are unchanged.

Lets $\mathcal{F}^e = \{F \in \mathcal{F} : e \in F\}$ and $\mathcal{F}_e = \{F \in \mathcal{F} : e \notin F\}$. It is well known (see e.g. Libura [8, 9], Sotskov et al. [18]), that the following facts hold:

Proposition 1 *If* $e \in X^o$, *then* $t^-(e) = \infty$, $t^+(e) = v(\mathcal{F}_e, c^o) - v(\mathcal{F}, c^o)$. *If* $e \notin X^o$, *then* $t^+(e) = \infty$, $t^-(e) = v(\mathcal{F}^e, c^o) - v(\mathcal{F}, c^o)$.

According to standard conventions, we take $v(\mathcal{F}_e, c^o) = \infty$ or $v(\mathcal{F}^e, c^o) = \infty$ if $\mathcal{F}_e = \emptyset$ or $\mathcal{F}^e = \emptyset$, respectively. Observe that given an algorithm for solving problem (1) for arbitrary $c \in \mathbb{R}^n$ and $\mathcal{F} \subseteq 2^E$, we may use it also to calculate values $v(\mathcal{F}_e, c^o)$ and $v(\mathcal{F}^e, c^o)$. From Proposition 1 it follows therefore that if the optimization problem (1) is polynomially solvable, then also the tolerances $t^+(e)$, $t^-(e)$ for $e \in E$, can be computed in polynomial time. Moreover, the opposite implication also holds under some mild assumptions (see Chakravarti, Wagelmans [4], van Hoesel, Wagelmans [21]).

In Libura [13] similar results are obtained in the robustness analysis context. We will present them after describing an important family of scenarios, which form a subset of the basic set of scenarios $\mathcal{C}(Q, c^o)$.

3.2 Family of scenarios $T_\delta(Q, c^o)$

In the basic set of scenarios $\mathcal{C}(Q, c^o)$ we allow arbitrary perturbations of the weights for all elements belonging to the subset $Q \subseteq E$. It appears interesting to consider some restrictions of these changes and – simultaneously – to allow additionally a simple parametrization of the perturbations. This leads to various families of scenarios, which were considered in Libura [10, 11, 12]. In this approach the parametrization corresponds to appropriately chosen norm of perturbations considered. In this paper we will concentrate on a particular family of these scenarios, which we will denote $T_\delta(Q, c^o)$ and define for a scalar parameter $\delta \in [0, 1)$ in the following way:

$$T_\delta(Q, c^o) = \{c \in \mathcal{C}(Q, c^o) : |c(e) - c^o(e)| \leq \delta \cdot c^o(e) \quad \text{for } e \in Q\}. \qquad (6)$$

This means that we are interested in *percentage* perturbations of the weights of elements, and for a given value of the parameter δ we allow simultaneous and independent increases or decreases of the weight of any element belonging to the subset Q, which do not exceed $\delta \cdot 100\%$ of their initial values given by the vector c^o. This family of scenarios may be alternatively introduced in the framework of so-called tolerance approach (see Wendell [22]).

The family of scenarios $T_\delta(Q, c^o)$ leads directly to the concept of the *accuracy function* introduced in Libura [10, 11], which will be exploited in the following. Namely, the accuracy function is defined for a given feasible solution $F \in \mathcal{F}$, and for a given subset of elements $Q \subseteq E$. Its value $a(F, Q, \delta)$ for an argument $\delta \in [0, 1)$ is equal to the worst-case relative regret of the solution F over the set $T_\delta(Q, c^o)$, i.e.,

$$a(F, Q, \delta) = \max_{c \in T_\delta(Q, c^o)} \epsilon(F, c). \tag{7}$$

Thus, for $F \in \mathcal{F}$, $Q \subseteq E$, and $\delta \in [0, 1)$,

$$a(F, Q, \delta) = Z(F, T_\delta(Q, c^o)).$$

Denote for $S', S'' \subseteq E$, $S' \otimes S'' = (S' \setminus S'') \cup (S'' \setminus S')$. In Libura [11] the following general formula for computing the accuracy function is given:

Theorem 1 *For $F \in \mathcal{F}$, $Q \subseteq E$, and $\delta \in [0, 1)$,*

$$a(F, Q, \delta) = \max_{F' \in \mathcal{F}} \frac{w(F, c^o) - w(F', c^o) + \delta \cdot w((F \otimes F') \cap Q), c^o)}{w(F', c^o) - \delta \cdot w(F' \cap Q, c^o)}. \tag{8}$$

The above formula can hardly be regarded as an efficient tool for calculate values of the accuracy function, because computing the value $a(F, Q, \delta)$ for a given argument δ requires finding an optimal value of an auxiliary combinatorial optimization problem on the same set of feasible solutions as problem (1), but with fractional objective function. Nevertheless, Theorem 1 appears usefull in studying properties of these functions (see Libura [11]) and provides an initial step for further analysis in this paper.

Example 2 Consider again the minimum spanning tree problem described in Example 1, but now with the following initial wector of weights $c^o = (12, 13, 11, 15, 14, 12, 13)^T$. In this case there are exactly two optimal spanning trees T_1 and T_{10}. Assume that $Q = \{e_1, e_2, e_5\}$, i.e., we allow changes of weights for only tree edges of the graph G: e_1, e_2 and e_5. Fig. 3 presents the accuracy functions on the interval $\delta = [0, 0.7]$ for all of the spanning trees shown in Fig. 2. To allow a detailed analysis, in Fig. 4 the accuracy functions for three selected spanning trees are presented: two minimum spanning trees T_1, T_{10}, and one feasible spanning tree T_5, which appears crucial for the robustness of the optimal spanning trees.

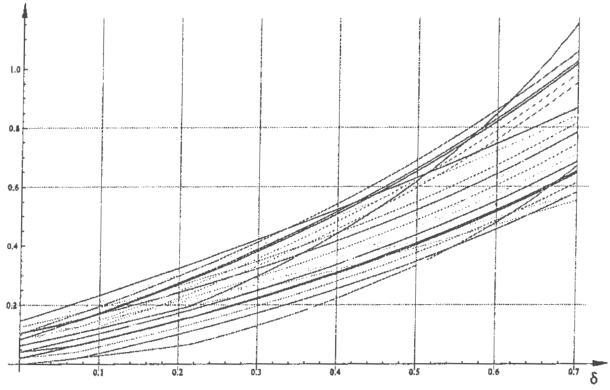

Figure 3: Accuracy functions for all solutions from Example 2.

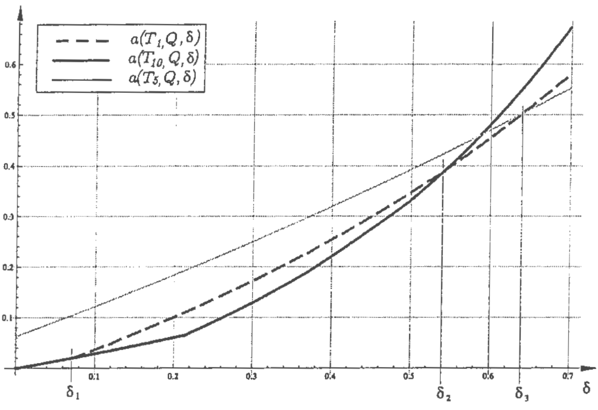

Figure 4: Accuracy functions for spanning trees T_1, T_{10} and T_5.

Figure 4 illustrates quite complicated aspects of the robustness of optimal solutions T_1 and T_{10}. It appears, that for $\delta \in [0, \delta_1)$ both of them remain robust solutions. But for $\delta \in [\delta_1, \delta_2)$ only T_{10} is a robust solution. The situation changes in δ_2: for larger δ the solution T_{10} is no longer robust and T_1 again becomes a single robust solution on the interval $[\delta_2, \delta_3)$. For δ larger than δ_3 we have a new robust solution: this time T_5, which initially, i.e., for $\delta = 0$, was not optimal one.

□

Consider now an initially optimal solution $F^o \in \Omega(c^o)$. Obviously, we have $a(F^o, Q, 0) = 0$. It is of special interest to know the maximum value of δ for which $a(F^o, Q, \delta) = 0$. This value is called the *accuracy radius* of the solution F^o with respect to the set Q and is denoted by $r^a(F^o, Q)$. Formally,

$$r^a(F^o, Q) = \sup\{\delta \in [0, 1) : a(F^o, Q, \delta) = 0\}. \tag{9}$$

A practical importance of the accuracy radius consists in the fact, that given the value $r \leq r^a(F^o, Q)$ we know, that the weight of any element e belonging to the set Q may be perturbed (increased or decreased) arbitrarily by $r \cdot 100\%$ of its initial value $c^o(e)$ without destroying the optimality of F^o. This implies for example that if the weights of elements in Q are estimated with the accuracy $r \cdot 100\%$, then we can guarantee that the solution F^o, calculated for the estimated vector of weights c^o, remains optimal for the actual vector of weights.

The following theorem (see Libura [11]) gives a general formula which allows for calculating for an initially optimal solution F^o its accuracy radius with respect to the subset $Q \subseteq E$.

Theorem 2 *For $F^o \in \Omega(c^o)$ and $Q \subseteq E$,*

$$r^a(F^o, Q) = \min\left\{ 1, \min_{F \in \mathcal{F}_Q} \frac{w(F, c^o) - w(F^o, c^o)}{w((F^o \otimes F) \cap Q), c^o)} \right\}, \tag{10}$$

where

$$\mathcal{F}_Q = \{F \in \mathcal{F} : w((F^o \otimes F) \cap Q, c^o) \neq 0\}.$$

Analogous concept can be introduced in the framework of the robustness analysis. Namely, instead of studying the maximum – in a sense of some norm – perturbations for which a given initial solution remains *optimal*, we may seek for the maximum perturbations preserving the *robustness* of this solution. In particular, in Libura [12] an analogue of the accuracy radius – called the *robustness radius* is considered.

Let $F^o \in \Omega(c^o)$. The robustness radius of F^o is denoted $r^r(F^o, Q)$, and it is defined as the maximum value of the parameter δ for which the solution F^o is a robust solution under the set of scenarios $T_{\delta'}(Q, c^o)$ for any $\delta' \leq \delta$. Formally,

$$\begin{aligned} r^r(F^o, Q) \;=\; \sup\{\delta \in [0,1) : \; & Z(F^o, T_{\delta'}(Q, c^o)) \leq Z(F, T_{\delta'}(Q, c^o)) \\ & \text{for any} \quad F \in \mathcal{F}, \; \delta' \leq \delta \; \}. \end{aligned}$$

Immediately from the definitions of the accuracy radius and the robustness radius we have the following inequality:

$$r^a(F^o, Q) \leq r^r(F^o, Q).$$

It frequently happens that the accuracy radius of the considered optimal solution is equal to zero. This fact is well known in the sensitivity analysis (see e.g. Libura et al. [14], Sotskov et al. [18]). In particular, this always happens for $Q = E$ when there are multiple solutions of the optimization problem (1) for $c = c^o$. In the general case, i.e. for arbitrary $Q \subseteq E$, a characterization of optimal solutions, for which the accuracy radius is positive, can be obtained directly from Theorem 2. Namely, the following fact holds:

Corollary 1 *For* $F^o \in \mathcal{F}, \; Q \subseteq E, \; \delta \in [0,1)$,

$$r^a(F^o, Q) > 0 \;\Leftrightarrow\; w((F^o \otimes F) \cap Q, c^o) = 0 \textit{ for any } F \in \Omega(c^o).$$

Proof Assume first that $w((F^o \otimes F) \cap Q, c^o) = 0$ for any $F \in \Omega(c^o)$. If now $F_Q \neq \emptyset$ then this means that there exists $F' \in F_Q \setminus \Omega(c^o)$ such that $w((F^o \otimes F') \cap Q, c^o) \neq 0$ and $w(F', c^o) - w(F^o, c^o) > 0$, which implies $r^a(F^o, Q) > 0$. Otherwise, for $F_Q = \emptyset$, from (10) we have $r^a(F^o, Q) = 1$.

To prove the oposite implication, assume that there exists $F'' \in \Omega(c^o)$ such that $w((F^o \otimes F'') \cap Q, c^o) \neq 0$. Then $F'' \in F_Q$, $w(F'', c^o) = w(F^o, c^o)$, and from (10) we have immediately $r^a(F^o, Q) = 0$. ∎

It is important to have analogous characterization of initially optimal solutions, for which the robustness radius is positive, which means that they remain robust in some nonempty neigborhood of the initial vector of weights c^o. We will call these solutions *robust optimal solutions* in $c = c^o$, and we will denote their subset by $\Omega_r(Q, c^o)$. In Libura [12] a characterization of solutions belonging to $\Omega_r(E, c^o)$ is given. The following theorem generalizes this characterization for arbitrary subset of elements $Q \subseteq E$. The main drawback of all these characterizations is that they require a knowledge of the whole set of optimal solutions of the considered optimization problem. It is an open question, whether this can be avoided.

For a given $\mathcal{F}$ and c^o we will use the folowing notation:

$$a = \min_{F \in \mathcal{F} \setminus \Omega(c^o)} \frac{w(F, c^o) - v(\mathcal{F}, c^o)}{v(\mathcal{F}, c^o)}. \tag{11}$$

Observe that a is the smallest positive value of the relative regret among all the feasible solutions of the optimization problem for $c = c^o$.

Theorem 3 *Let for $Q \subseteq E$,*

$$b(Q) = \min_{F \in \Omega(c^o)} \max_{F' \in \Omega(c^o)} w\left((F \otimes F') \cap Q, c^o\right). \tag{12}$$

Then

$$\Omega_r(Q, c^o) = \left\{ F \in \Omega(c^o) : \max_{F' \in \Omega(c^o)} w\left((F \otimes F') \cap Q, c^o\right) = b(Q) \right\}. \tag{13}$$

Proof According to (8), for $F^o \in \Omega(c^o)$, $Q \subseteq E$, and $\delta \in [0, 1)$,

$$a(F^o, Q, \delta) = \max_{F \in \mathcal{F}} \frac{w(F^o, c^o) - w(F, c^o) + \delta \cdot w((F^o \otimes F) \cap Q), c^o)}{w(F, c^o) - \delta \cdot w(F \cap Q, c^o)}.$$

It is easy to see, that there is a nonempty neighborhood of c^o in which the worst-case relative regret of the solution F^o is determined only by the elements of the set $\Omega(Q, c^o)$, i.e., all nonoptimal solutions in the above formula can be neglected. Indeed, for any $\delta \in [0, a)$ the following inequality holds

$$a(F^o, Q, \delta) = \max_{F \in \mathcal{F} \setminus \Omega(c^o)} \frac{w(F^o, c^o) - w(F, c^o) + \delta \cdot w((F^o \otimes F) \cap Q), c^o)}{w(F, c^o) - \delta \cdot w(F \cap Q, c^o)} < 0,$$

which means that then for arbitrary $F^o \in \Omega(c^o)$ and $Q \subseteq E$ we have

$$\begin{aligned} a(F^o, Q, \delta) &= \max_{F \in \Omega(c^o)} \frac{w(F^o, c^o) - w(F, c^o) + \delta \cdot w((F^o \otimes F) \cap Q), c^o)}{w(F, c^o) - \delta \cdot w(F \cap Q, c^o)} \\ &= \max_{F \in \Omega(c^o)} \frac{\delta \cdot w((F^o \otimes F) \cap Q), c^o)}{v(\mathcal{F}, c^o) - \delta \cdot w(F \cap Q, c^o)}. \end{aligned} \tag{14}$$

Denote for a given F^o and Q,

$$p_F(\delta) = \frac{\delta \cdot w((F^o \otimes F) \cap Q), c^o)}{v(\mathcal{F}, c^o) - \delta \cdot w(F \cap Q, c^o)}.$$

Thus, for $\delta \in [0, a)$,

$$a(F^o, Q, \delta) = \max_{F \in \Omega(c^o)} p_F(\delta).$$

For any $F \in \Omega(c^o)$, $p_F(\delta)$ is a continuous function of δ in the interval $[0, 1)$ and $p_F(0) = 0$. This implies that for small enough $\delta > 0$, $a(F^o, Q, \delta) = p_{\overline{F}}(\delta)$, where $\overline{F}$ is such an optimal solution, for which the value $\frac{\partial}{\partial\delta}\, p_F(\delta)|_{\delta=0}$ is the maximum among all optimal solutions. It is easy to show that

$$\frac{\partial}{\partial\delta} p_F(\delta)\bigg|_{\delta=0} = \frac{w((F^o \otimes F) \cap Q), c^o)}{v(\mathcal{F}, c^o)}.$$

Consequently, the value of $\frac{\partial}{\partial\delta}\, a(F^o, Q, \delta)|_{\delta=0}$ is equal to

$$\max_{F\in\Omega(c^o)} \frac{w((F^o \otimes F) \cap Q), c^o)}{v(\mathcal{F}, c^o)} = \frac{1}{v(\mathcal{F}, c^o)} \max_{F\in\Omega(c^o)} w((F^o \otimes F) \cap Q), c^o).$$

For any optimal solution F^o we have $a(F^o, Q, 0) = 0$. The set of robust optimal solutions $\Omega_r(Q, c^o)$ contains therefore all these optimal solutions $F \in \Omega(c^o)$, for which the minimum value of $\frac{\partial}{\partial\delta}\, a(F, Q, \delta)|_{\delta=0}$ is achieved, i.e., for which

$$\max_{F'\in\Omega(c^o)} w\left((F \otimes F') \cap Q, c^o\right) = \min_{F\in\Omega(c^o)} \max_{F'\in\Omega(c^o)} w\left((F \otimes F') \cap Q, c^o\right) = b(Q).$$

■

As a simple corollary of Theorem 3 we obtain a characterization of robust optimal solutions formulated in Libura [12] for the case $Q = E$. Observe that then

$$\begin{aligned} b(E) &= \min_{F\in\Omega(c^o)} \max_{F'\in\Omega(c^o)} w\left((F \otimes F') \cap E, c^o\right) \\ &= \min_{F\in\Omega(c^o)} \max_{F'\in\Omega(c^o)} \left(2 \cdot v(\mathcal{F}, c^o) - w(F \cap F', c^o)\right) \\ &= 2 \cdot v(\mathcal{F}, c^o) - \max_{F\in\Omega(c^o)} \min_{F'\in\Omega(c^o)} w(F \cap F', c^o) \end{aligned}$$

and

$$\max_{F'\in\Omega(c^o)} w\left((F \otimes F') \cap E, c^o\right) = 2 \cdot v(\mathcal{F}, c^o) - \min_{F'\in\Omega(c^o)} w(F \cap F', c^o).$$

Thus, from Theorem 3 we have:

Corollary 2 *$F^o \in \Omega(c^o)$ is a robust optimal solution for $Q = E$ if and only if*

$$\min_{F\in\Omega(c^o)} w(F^o \cap F, c^o) = \max_{F\in\Omega(c^o)} \min_{F'\in\Omega(c^o)} w(F \cap F', c^o).$$

Example 3

Consider again the minimum spanning tree problem from Example 1. For $c^o = (2,2,2,2,1,2,2)^T$ the set of optimal solutions $\Omega(c^o)$ contains the following ten spanning trees: $T_5, T_6, T_8, T_9, T_{11}, T_{12}, T_{16}, T_{17}, T_{19}, T_{20}$. But according to Corollary 2 only two of them, namely T_{11} and T_{12}, are robust optimal solutions for $Q = E$. Indeed, it is easy to see, that

$$\min_{T\in\Omega(c^o)} w(T_{11} \cap T, c^o) = \min_{T\in\Omega(c^o)} w(T_{12} \cap T, c^o) = 3,$$

whereas for any T_i, $i \neq 11, 12$,

$$\min_{T\in\Omega(c^o)} w(T_i \cap T, c^o) = 1.$$

Figure 5 shows the accuracy functions for all optimal solutions in problem (1) for $c^o = (2,2,2,2,1,2,2)^T$. It can be observed that the robust optimal solutions T_{11} and T_{12} guarantee smaller values of the worst-case relative regret than all remaining optimal solutions for any $\delta > 0$.

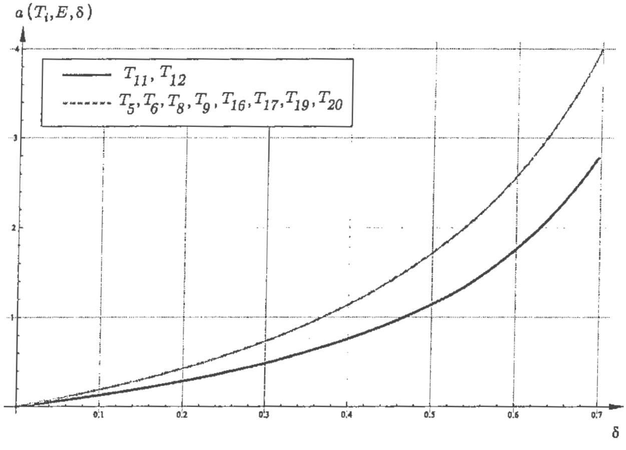

Figure 5: Accuracy functions for all optimal solutions from Example 3.

□

No formula like (10) is known for calculating the robustness radius $r^r(F^o, Q)$. In Libura [12] some evaluations of the robustness radius are given for the special case $Q = E$. Let

$$b = \max_{F \in \Omega(c^o)} \min_{F' \in \Omega(c^o)} \frac{w(F \cap F', c^o)}{v(\mathcal{F}, c^o)}.$$

The following facts hold:

Theorem 4 *If F^o is a single optimal solution of problem* (1) *for $c = c^o$, then*

$$r^r(F^o, E) \geq \begin{cases} \frac{a}{2-a} & if \quad a < 1, \\ 1 & otherwise. \end{cases} \tag{15}$$

Theorem 5 *If $F^o \in \Omega_r(E, c^o)$ and $a \geq \frac{b}{1-b}$, then*

$$r^r(F^o, E) \geq \begin{cases} \frac{a}{2(1-b)-a} & if \quad a < 1-b, \\ 1 & otherwise. \end{cases} \tag{16}$$

If $F^o \in \Omega_r(E, c^o)$ and $a < \frac{b}{1-b}$, then

$$r^r(F^o, E) \geq \begin{cases} \min\left\{\frac{a}{2(1-b)-a}, \frac{a}{2b+2ab-a}\right\} & if \quad a < 1-b, \\ \frac{a}{2b+2ab-a} & otherwise. \end{cases} \tag{17}$$

The situation simplifies significantly in case $Q = \{e\}$ for some $e \in E$. Then the robustness radius becomes an analogue of the tolerances of the weight of element e considered in the sensitivity analysis. Namely, for $e \in E$ and $F^o \in \Omega(c^o)$ we introduce so-called *robustness tolerance* of the weight $c(e)$, which we denote $t^r(e)$ and define it formally in the following way:

$$\begin{aligned} t^r(e) \quad = \quad & \sup\{\delta \in [0,1) : \; Z(F^o, T_{\delta'}(\{e\}, c^o)) \leq Z(F, T_{\delta'}(\{e\}, c^o)) \\ & \text{for any} \quad F \in \mathcal{F}, \; \delta' \leq \delta\}. \end{aligned}$$

Thus, $t^r(e)$ is the maximum value of the parameter δ, such that F^o remains robust for any set of scenarios $T_{\delta'}(\{e\}, c^o)$ where $\delta' < \delta$. This case we are able to show a result, which is a close analogue of Proposition 1. Namely, in Libura [13] it is proved that the following fact holds:

Theorem 6 *For $F^o \in \Omega(c^o)$,*

$$t^r(e) = \begin{cases} 1 & \text{if } \; e \in F^o, \\ \min\left\{1, [\,v(\mathcal{F}^e, c^o)^2 - v(\mathcal{F}, c^o)^2\,]^{\frac{1}{2}} \cdot c^o(e)^{-1}\right\} & \text{if } \; e \notin F^o. \end{cases} \tag{18}$$

Observe that this – as in case of the standard sensitivity analysis – leads to the polynomial solvability of the robustness tolerance problem provided that the original optimization problem is polynomially solvable itself.

Example 4

Consider again the graph G from Example 1. For the initial vector of weights $c^o = (14, 11, 14, 15, 13, 18, 17)^T$ the minimum spanning tree problem has the unique optimal solution $T_6 = \{e_1, e_2, e_5, e_7\}$.

From Theorem 6 it follows that the robustness tolerances of all the edges belonging to T_6 are equal to 1 which means that we can perturb individually the weights of these edges up to 100% of their initial values without destroying the robustness of the solution T_6. Consider therefore some edge from the set $E \setminus T_6$, e.g. the edge $e = e_3 = \{1, 4\}$, and the corresponding set of scenarios $T_\delta(\{e\}, c^o)$. We have $c^o(e) = 14$, $v(\mathcal{F}, c^o) = w(T_6, c^o) = 55$, $v(\mathcal{F}^e, c^o) = 56$. Calculating $t^r(e)$ from (18) we obtain:

$$t^r(e) = \frac{(\,56^2 - 55^2\,)^{\frac{1}{2}}}{14} \approx \frac{10.54}{14} \approx 0.75.$$

Thus, the spanning tree T_6 achieves the minimum value of the worst-case relative regret among all the spanning trees in G if the weight of the edge $e = \{1, 4\}$ is perturbed by no more than approximately 75%, and all the remaining weights are unchanged.

In Fig. 6 the worst case regret functions $Z\,(T, T_\delta(\{e\}, c^o))$ for all the feasible solutions $T \in \{T_1, \ldots, T_{21}\}$ in problem (1) are shown; bold line indicates the worst-case regret function for the spanning tree T_6. Observe that the solution T_6 guarantees, indeed, the minimum value of the worst-case regret among all the feasible solutions, i.e. it remains a *robust* spanning tree, provided $\delta \leq t^r(e) \approx 0.75$. It is interesting to note that in order to destroy the *optimality* of T_6 in problem (1) it is enough to increase the weight of edge e by approximately 7.14%, which corresponds to the first breakpoint $\delta' = 1/14$ of the worst case regret function $Z\,(T_6, T_\delta(\{e\}, c^o))$ in Fig. 6.

□

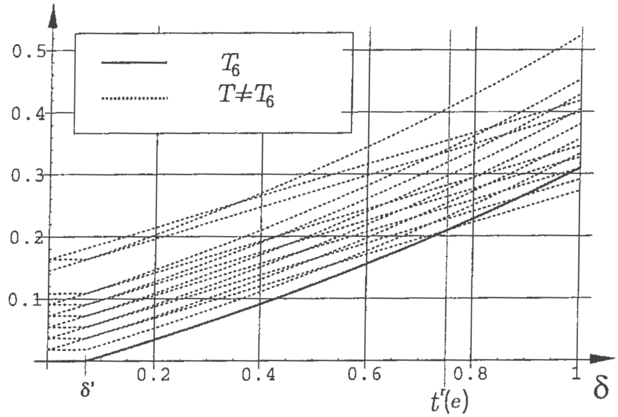

Figure 6: Worst-case regret functions of all spanning trees from Example 4.

4 Conclusions

This paper deals with the robustness analysis, regarded as a natural extension of the standard sensitivity analysis for combinatorial optimization problems. It is shown, that it is reasonably to define analogues of such objects as the stability region, the stability radius, the accuracy radius, the tolerances of weights. This leads to studying in the framework of the robustness analysis such objects as the robustness region, the robustness radius and the robustness tolerances. All of them have natural interpretations and give some insight in the quality of a given optimal solution from the robustness point of view.

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