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## A NONDERIVATIVE VERSION OF THE GRADIENT SAMPLING ALGORITHM FOR NONSMOOTH NONCONVEX OPTIMIZATION\*

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**Abstract.** We give a nonderivative version of the gradient sampling algorithm of Burke, Lewis and Overton for minimizing a locally Lipschitz function  $f$  on  $\mathbb{R}^n$  that is continuously differentiable on an open dense subset. Instead of gradients of  $f$ , we use estimates of gradients of the Steklov averages of  $f$  (obtained by convolution with mollifiers) which require  $f$ -values only. We show that the nonderivative version retains the convergence properties of the gradient sampling algorithm. In particular, with probability 1 it either drives the  $f$ -values to  $-\infty$ , or each of its cluster points is Clarke stationary for  $f$ .

**Key words.** generalized gradient, nonsmooth optimization, subgradient, averaged functions, gradient sampling, nonconvex

**AMS subject classifications.** 65K10, 90C26

**1. Introduction.** The *gradient sampling* (GS) algorithm of Burke, Lewis and Overton [BLO02b, BLO05] is designed for minimizing a locally Lipschitz function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  which is continuously differentiable on an open dense subset  $D$  of  $\mathbb{R}^n$ .

At each iteration, the GS algorithm computes the gradient of  $f$  at the current iterate and at  $m \geq n+1$  randomly generated nearby points. This bundle of gradients is used to find an approximate  $\epsilon$ -steepest descent direction as the solution of a quadratic program, where  $\epsilon$  is the sampling radius which may be fixed for all iterations or may be reduced dynamically. An Armijo line search along this direction produces a candidate for the next iterate, which is obtained by perturbing the candidate if necessary to stay in the set  $D$  where  $f$  is differentiable; here an additional condition of [Kiw07] on this perturbation may ensure stronger convergence results.

The GS algorithm is widely applicable and robust in practice [BHLO06, BLO02a, BLO04, BLO05, Lew05].

This paper presents a nonderivative version of the GS algorithm, called the *nondervative sampling* (NS) algorithm for short. Instead of gradients of  $f$ , it employs Gupal's [Gup77] estimates of gradients of the Steklov averages of  $f$ , which require  $f$ -values only (see [ENW95] and (2.3)–(2.6)). We show that the NS algorithm retains the convergence properties of the GS algorithm; e.g., with probability 1 it either drives the  $f$ -values to  $-\infty$ , or each of its cluster points is Clarke [Cla83] stationary for  $f$ .

At each iteration, the NS algorithm requires  $2mn$   $f$ -evaluations to sample the current bundle of size  $m$ , and several more in the line search. To save work, we give an *incremental* version with just  $2n+1$   $f$ -evaluations per iteration for augmenting the bundle with the next gradient estimate and testing a single step size; this may give descent before the bundle reaches its full size. In addition, the bundle may include some past gradient estimates within the current sampling region to speed convergence.

The NS algorithm is suited to applications where one has a blackbox oracle for computing  $f(x)$  at any given  $x \in \mathbb{R}^n$ , but where finding  $\nabla f(x)$  (for  $x \in D$ ) is impossible or too expensive. Thus its applicability area is similar to that of derivative-free direct search methods, especially *mesh adaptive direct search* (MADS) algorithms whose cluster points are Clarke stationary for  $f$  when  $f$  is locally Lipschitz and its

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initial level set is bounded; see, e.g., [ABLD08, AuD06, CDV08, CuV07]. Each MADS iteration typically requires  $n + 1$  or  $2n$   $f$ -evaluations (when no search step is considered), whereas the incremental NS iteration needs  $2n + 1$ ; these costs are similar, but the iteration numbers until approximate convergence may differ. Numerical comparisons of MADS and incremental NS are deferred to a future paper. For now, NS may be of independent theoretical interest, since it uses novel properties of Gupal's estimators, which are unavailable for standard finite differences or simplex gradients [CDV08, CuV07] in the nonsmooth case.

Up till now, Gupal's estimator has only been used in stochastic approximation algorithms (see [ENW95, MaP84] and the references therein). It is an open question whether similar estimators could provide nonderivative versions of bundle methods (see the references in [BLO05, Kiw96]).

The paper is organized as follows. The NS algorithm is presented in section 2, and its convergence analysis in section 3. Various modifications are discussed in section 4.

**2. The NS algorithm.** We assume that the objective function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is locally Lipschitzian and continuously differentiable on an open dense subset  $D$  of  $\mathbb{R}^n$ . The Clarke *subdifferential* [Cla83] of  $f$  at any point  $x$  is given by

$$\bar{\partial}f(x) = \text{co}\{\lim_j \nabla f(y^j) : y^j \rightarrow x, y^j \in D\},$$

where  $\text{co}$  denotes the convex hull, and the Clarke  $\epsilon$ -*subdifferential* [Gol77] by

$$(2.1) \quad \bar{\partial}_\epsilon f(x) := \text{co} \bar{\partial}f(B(x, \epsilon)),$$

where  $B(x, \epsilon) := \{y : \|y - x\| \leq \epsilon\}$  is the ball centered at  $x$  with radius  $\epsilon \geq 0$  and  $\|\cdot\|$  is the 2-norm. The  $\epsilon$ -subdifferential  $\bar{\partial}_\epsilon f(x)$  is approximated by the set of [BLO05]

$$(2.2) \quad G_\epsilon(x) := \text{cl co} \nabla f(B(x, \epsilon) \cap D),$$

since  $G_\epsilon(x) \subset \bar{\partial}_\epsilon f(x)$ , and  $\bar{\partial}_{\epsilon_1} f(x) \subset G_{\epsilon_2}(x)$  for  $0 \leq \epsilon_1 < \epsilon_2$ . We say that a point  $x$  is *stationary* for  $f$  if  $0 \in \bar{\partial}f(x)$ ;  $x$  is called  $\epsilon$ -*stationary* for  $f$  if  $0 \in \bar{\partial}_\epsilon f(x)$ .

For  $\alpha > 0$ , the *Steklov averaged* function  $f_\alpha$  (cf. [ENW95, Def. 3.1]) is defined by

$$(2.3) \quad f_\alpha(x) := \int_{\mathbb{R}^n} f(x - y) \psi_\alpha(y) dy,$$

where  $\psi_\alpha : \mathbb{R}^n \rightarrow \mathbb{R}_+$  is the Steklov mollifier defined by

$$\psi_\alpha(y) := \begin{cases} 1/\alpha^n & \text{if } y \in [-\alpha/2, \alpha/2]^n, \\ 0 & \text{otherwise;} \end{cases}$$

equivalently,

$$f_\alpha(x) = \frac{1}{\alpha^n} \int_{x_1 - \alpha/2}^{x_1 + \alpha/2} \cdots \int_{x_n - \alpha/2}^{x_n + \alpha/2} f(y) dy_1 \cdots dy_n.$$

The partial derivatives of  $f_\alpha$  are given by (cf. [ENW95, Prop. 3.11], [Gup77])

$$(2.4) \quad \frac{\partial f_\alpha}{\partial x_i}(x) = \int_{\mathbb{B}_\infty} \gamma_i(x, \alpha, \zeta) d\zeta$$

for  $i = 1, \dots, n$ , where  $\mathbb{B}_\infty := [-1/2, 1/2]^n$  is the unit cube centered at 0 and

$$(2.5) \quad \gamma_i(x, \alpha, \zeta) := \frac{1}{\alpha} [f(x_1 + \alpha\zeta_1, \dots, x_{i-1} + \alpha\zeta_{i-1}, x_i + \frac{1}{2}\alpha, x_{i+1} + \alpha\zeta_{i+1}, \dots, x_n + \alpha\zeta_n) - f(x_1 + \alpha\zeta_1, \dots, x_{i-1} + \alpha\zeta_{i-1}, x_i - \frac{1}{2}\alpha, x_{i+1} + \alpha\zeta_{i+1}, \dots, x_n + \alpha\zeta_n)].$$

Thus, for the unit cube  $Z := \prod_{i=1}^n \mathbb{B}_\infty$  in  $\mathbb{R}^{n \times n}$ , we may estimate  $\nabla f_\alpha(x)$  by

$$(2.6) \quad \gamma(x, \alpha, z) := (\gamma_1(x, \alpha, \zeta^1), \dots, \gamma_n(x, \alpha, \zeta^n)) \quad \text{for } z := (\zeta^1, \dots, \zeta^n) \in Z.$$

This needs just  $2n$  evaluations of  $f$ ;  $f_\alpha$  is convenient only for convergence analysis.

We now state our NS algorithm. It employs a sequence of mollifier parameters  $\alpha_k \downarrow 0$  in  $(0, 1]$ . For a closed convex set  $G$ ,  $\text{Proj}(0|G)$  is its minimum-norm element. ALGORITHM 2.1 (NS algorithm).

*Step 0* (initialization). Select an initial point  $x^1 \in \mathbb{R}^n$ , optimality tolerances  $\nu_{\text{opt}}, \epsilon_{\text{opt}} \geq 0$ , line search parameters  $\beta, \kappa, \underline{t}$  in  $(0, 1)$ , reduction factors  $\mu, \theta$  in  $(0, 1]$ , a sampling radius  $\epsilon_1 > 0$ , a stationarity target  $\nu_1 \geq 0$  and a sample size  $m \geq n + 1$ . Set  $k := 1$ .

*Step 1* (approximate the Clarke  $\epsilon$ -subdifferential by sampling estimates of mollifier gradients). Let  $\{x^{ki}\}_{i=1}^m$  and  $\{z^k\}_{i=1}^m$  be sampled independently and uniformly from  $B(x^k, \epsilon_k)$  and  $Z$ , respectively. Set

$$(2.7) \quad G_k := \text{co}\{\gamma(x^{k1}, \alpha_k, z^{k1}), \dots, \gamma(x^{km}, \alpha_k, z^{km})\}.$$

*Step 2* (direction finding). Set  $g^k := \text{Proj}(0|G_k)$ .

*Step 3* (stopping criterion). If  $|g^k| \leq \nu_{\text{opt}}$  and  $\epsilon_k \leq \epsilon_{\text{opt}}$ , terminate.

*Step 4* (sampling radius update). If  $|g^k| \leq \nu_k$ , set  $\nu_{k+1} := \theta\nu_k$ ,  $\epsilon_{k+1} := \mu\epsilon_k$ ,  $t_k := 0$ ,  $x^{k+1} := x^k$  and go to Step 7. Otherwise, set  $\nu_{k+1} := \nu_k$ ,  $\epsilon_{k+1} := \epsilon_k$  and

$$(2.8) \quad d^k := -g^k/|g^k|.$$

*Step 5* (limited Armijo line search). Find a step size  $t_k$  as follows:

(i) Choose an initial step size  $t = t_{\text{ini}}^k \geq t_{\text{min}}^k := \min\{\underline{t}, \kappa\epsilon_k/3\}$ .

(ii) If  $f(x^k + td^k) \leq f(x^k) - \beta t|g^k|$ , return  $t_k := t$ .

(iii) If  $\kappa t < t_{\text{min}}^k$ , return  $t_k := 0$ .

(iv) Set  $t := \kappa t$  and go to (ii).

*Step 6* (updating). Set  $x^{k+1} := x^k + t_k d^k$ .

*Step 7*. Increase  $k$  by 1 and go to Step 1.

Since  $|d^k| = 1$  by (2.8), Steps 5 and 6 ensure the usual Armijo condition

$$(2.9) \quad f(x^{k+1}) \leq f(x^k) - \beta t_k |g^k| = f(x^k) - \beta |x^{k+1} - x^k| |g^k|,$$

which also holds when  $x^{k+1} := x^k$  at Step 4.

**3. Convergence analysis.** We start with several technical lemmas. The first lemma on approximate least-norm elements extends [Kiw07, Lemma 3.1].

LEMMA 3.1. *Let  $\emptyset \neq C \subset \mathbb{R}^n$  be compact convex and  $\beta \in (0, 1)$ . If  $0 \notin C$ , there exists  $\delta > 0$  such that  $u \in \mathbb{R}^n$ ,  $\text{dist}(u|C) \leq \delta$ ,  $|u| \leq \text{dist}(0|C) + \delta$  and  $v \in C$  imply  $\langle v, u \rangle > \beta|u|^2$ .*

*Proof.* If the assertion were false, we could pick two sequences  $\{u^i\} \subset \mathbb{R}^n$ ,  $\{v^i\} \subset C$  satisfying  $\text{dist}(u^i|C) \leq 1/i$ ,  $|u^i| \leq \text{dist}(0|C) + 1/i$  and  $\langle v^i, u^i \rangle \leq \beta|u^i|^2$ . By

compactness, we may assume  $u^i \rightarrow \bar{u} \in C$ ,  $v^i \rightarrow \bar{v} \in C$ ; thus  $\langle \bar{v}, \bar{u} \rangle \leq \beta |\bar{u}|^2$ . However,  $\bar{u} = \text{Proj}(0|C) \neq 0$  satisfies  $\langle v, \bar{u} \rangle \geq |\bar{u}|^2$  for all  $v \in C$ , a contradiction.  $\square$

We now show that  $\gamma(\cdot, \alpha, z)$  approximates  $\nabla f$  on  $D$  when  $\alpha$  is small enough, for any  $z \in Z$ . To this end, for Lemmas 3.2 and 3.3 we could use other standard approximations (e.g., central differences with  $z = 0$ ). However, for asymptotic stationarity in Lemma 3.7, the random choice of  $z^{k_i}$  gives crucial connection between  $\gamma(x^{k_i}, \alpha_k, z^{k_i})$  and  $\nabla f_{\alpha_k}(x^{k_i})$  via Lemma 3.4, whereas Lemmas 3.5 and 3.6 relate  $\nabla f_\alpha$  with  $\tilde{\partial}_\epsilon f$ .

**LEMMA 3.2.** *Let  $\bar{x} \in D$  and  $\delta > 0$ . There exist  $\bar{\epsilon} > 0$  and  $\bar{\alpha} > 0$  such that  $|\nabla f(\bar{x}) - \gamma(x, \alpha, z)| < \delta$  for all  $x \in B(\bar{x}, \bar{\epsilon})$ ,  $\alpha \in (0, \bar{\alpha})$  and  $z \in Z$ .*

*Proof.* Since  $\nabla f$  is continuous on the open set  $D$ , there exist  $\bar{\epsilon} > 0$  and  $\bar{\alpha} > 0$  such that  $|\nabla f(\bar{x}) - \nabla f(x)| < \bar{\delta} := \delta/\sqrt{n}$  for all  $x \in B(\bar{x}, \bar{\epsilon}) + \bar{\alpha}\mathbb{B}_\infty \subset D$ . Let  $x \in B(\bar{x}, \bar{\epsilon})$ ,  $\alpha \in (0, \bar{\alpha})$ ,  $z := (\zeta^1, \dots, \zeta^n) \in Z$ . For each  $1 \leq i \leq n$ , by (2.5) and the mean value theorem, there is  $\bar{x}^i \in B(\bar{x}, \bar{\epsilon}) + \alpha\mathbb{B}_\infty$  with  $\gamma_i(x, \alpha, \zeta^i) = \frac{\partial f}{\partial x_i}(\bar{x}^i)$  and hence  $|\frac{\partial f}{\partial x_i}(\bar{x}) - \gamma_i(x, \alpha, \zeta^i)| < \bar{\delta}$ ; in effect,  $|\nabla f(\bar{x}) - \gamma(x, \alpha, z)| < \delta$  by (2.6).  $\square$

It is useful to note that  $B(x, \epsilon)$  in (2.2) may be replaced by its interior  $\mathring{B}(x, \epsilon)$ , since the set  $\nabla f(B(x, \epsilon) \cap D) = \text{cl} \nabla f(\mathring{B}(x, \epsilon) \cap D)$  is bounded by our assumption on  $f$ , whereas  $\text{cl} S = \text{cl co } S$  for any bounded set  $S \subset \mathbb{R}^n$ .

The next lemma states basic properties of the set of points close to a given point  $\bar{x}$  that can be used to provide a  $\delta$ -approximation to the least-norm element of  $G_\epsilon(\bar{x})$ ; it extends [BLO05, Lemma 3.2] and [Kiw07, Lemma 3.2] by replacing gradients  $\nabla f(y^i)$  with their estimates  $\gamma(y^i, \alpha, z^i)$  for points  $y^i \in D$  and  $z^i \in Z$ . For  $\epsilon, \delta, \bar{\alpha} > 0$  and  $\bar{x}, x \in \mathbb{R}^n$ , using the measure of proximity to  $\epsilon$ -stationarity

$$(3.1) \quad \rho_\epsilon(\bar{x}) := \text{dist}(0 | G_\epsilon(\bar{x})),$$

let

$$(3.2) \quad D_\epsilon^m(x) := \prod_1^m (\mathring{B}(x, \epsilon) \cap D) \subset \prod_1^m \mathbb{R}^n$$

and

$$(3.3) \quad \begin{aligned} V_{\epsilon, \bar{\alpha}}(\bar{x}, x, \delta) := \\ \{ (y^1, \dots, y^m) \in D_\epsilon^m(x) : \text{dist}(0 | \text{co}\{\gamma(y^i, \alpha, z^i)\}_{i=1}^m) \leq \rho_\epsilon(\bar{x}) + \delta, \\ \text{dist}(\{\gamma(y^i, \alpha, z^i)\}_{i=1}^m | G_\epsilon(\bar{x})) \leq \delta, \text{ for all } \{z^i\}_{i=1}^m \subset Z, \alpha \in (0, \bar{\alpha}) \}. \end{aligned}$$

**LEMMA 3.3.** *Let  $\epsilon > 0$  and  $\bar{x} \in \mathbb{R}^n$ .*

(i) *For any  $\delta > 0$ , there are  $\bar{\alpha} > 0$ ,  $\tau > 0$  and a nonempty open set  $\bar{V}$  satisfying  $\bar{V} \subset V_{\epsilon, \bar{\alpha}}(\bar{x}, x, \delta)$  for all  $x \in B(\bar{x}, \tau)$ .*

(ii) *Assuming  $0 \notin G_\epsilon(\bar{x})$ , pick  $\delta > 0$  as in Lemma 3.1 for  $C := G_\epsilon(\bar{x})$ , and then  $\bar{\alpha}$ ,  $\tau$  and  $\bar{V}$  as in statement (i). Suppose at iteration  $k$  of Algorithm 2.1, Step 5 is reached with  $x^k \in B(\bar{x}, \min\{\tau, \epsilon/3\})$ ,  $\epsilon_k = \epsilon$ ,  $\alpha_k \leq \bar{\alpha}$  and  $(x^{k_1}, \dots, x^{k_m}) \in \bar{V}$ . Then  $t_k \geq \min\{\epsilon, \kappa\epsilon/3\}$ .*

*Proof.* (i) Let  $u \in \text{co} \nabla f(\mathring{B}(\bar{x}, \epsilon) \cap D)$  be such that  $|u| < \rho_\epsilon(\bar{x}) + \delta$ . Then Carathéodory's theorem [Roc70] implies the existence of  $(\bar{x}^1, \dots, \bar{x}^m) \in D_\epsilon^m(\bar{x})$  and  $\bar{\lambda} \in \mathbb{R}_+^m$  with  $\sum_{i=1}^m \bar{\lambda}_i = 1$  such that  $u = \sum_{i=1}^m \bar{\lambda}_i \nabla f(\bar{x}^i)$ . By Lemma 3.2, there are  $\bar{\epsilon} \in (0, \epsilon)$  and  $\bar{\alpha} \in (0, 1)$  such that the set  $\bar{V} := \prod_{i=1}^m \mathring{B}(\bar{x}^i, \bar{\epsilon})$  lies in  $D_{\epsilon-\bar{\epsilon}}^m(\bar{x})$ ,  $|\sum_{i=1}^m \bar{\lambda}_i \gamma(y^i, \alpha, z^i)| \leq \rho_\epsilon(\bar{x}) + \delta$  and  $\text{dist}(\{\gamma(y^i, \alpha, z^i)\}_{i=1}^m | G_\epsilon(\bar{x})) \leq \delta$  for all points  $(y^1, \dots, y^m) \in \bar{V}$ ,  $\{z^i\}_{i=1}^m \subset Z$ ,  $\alpha \in (0, \bar{\alpha}]$ . Hence for all  $x \in B(\bar{x}, \tau)$  with  $\tau := \bar{\epsilon}$ , the fact that  $B(\bar{x}, \epsilon - \bar{\epsilon}) \subset B(x, \epsilon)$  yields  $\bar{V} \subset V_{\epsilon, \bar{\alpha}}(\bar{x}, x, \delta)$  by the definitions (3.2)–(3.3).

(ii) Since  $(x^{k_1}, \dots, x^{k_n}) \in \bar{V} \subset V_{\epsilon, \bar{\alpha}}(\bar{x}, \bar{x}, \delta)$  in statement (i), we get  $\text{dist}(0 | G_k) \leq \rho_\epsilon(\bar{x}) + \delta$  and  $\text{dist}(G_k | C) \leq \delta$  from (3.3) and (2.7). Thus, by the construction of  $g^k$  at Step 2,  $|g^k| \leq \rho_\epsilon(\bar{x}) + \delta$  and  $\text{dist}(g^k | C) \leq \delta$ . Hence by (3.1) and the choice of  $\delta$  in Lemma 3.1,

$$(3.4) \quad \langle v, g^k \rangle > \beta |g^k|^2 \quad \text{for all } v \in G_\epsilon(\bar{x}).$$

Let  $t \in (0, \epsilon/3]$ . By Lebourg's mean value theorem (cf. [Cla83, Theorem 2.3.7]),  $f(x^k + td^k) - f(x^k) = t \langle v, d^k \rangle$  for some  $v \in \bar{\partial}f(x)$  with  $x \in [x^k + td^k, x^k]$ . Then, using  $d^k := -g^k/|g^k|$ ,  $t|d^k| \leq \epsilon/3$  and  $|x^k - \bar{x}| \leq \epsilon/3$  imply  $x \in B(\bar{x}, 2\epsilon/3)$  and hence  $v \in G_\epsilon(\bar{x})$ , and so  $\langle v, d^k \rangle < -\beta|g^k|$  by (3.4). Therefore,  $f(x^k + td^k) < f(x^k) - \beta t|g^k|$  for all  $t \in (0, \epsilon/3]$ , and the conclusion follows from the rules of Step 5.  $\square$

The following result implies that  $\gamma(x, \alpha, z)$  provides a  $\rho$ -approximation to  $\nabla f_\alpha(x)$  when  $z$  happens to lie in a cube of side  $s := \min\{\rho/2Ln, 1/2\sqrt{n}\}$  contained in  $Z$ , where  $L$  is a Lipschitz constant for  $f$  on  $x + \mathbb{B}_\infty$  and  $\alpha \in (0, 1]$ ; this occurs with probability at least  $s^{n^2}$  when  $z$  is sampled from a uniform distribution on  $Z$ .

LEMMA 3.4. *Let  $\alpha \in (0, 1]$ ,  $\rho > 0$  and  $x \in X$ , where  $X \subset \mathbb{R}^n$  is bounded, and let  $L$  be a Lipschitz constant for  $f$  on  $X + \alpha\mathbb{B}_\infty$ .*

(i) *For each  $1 \leq i \leq n$ , there exists  $\zeta_i \in \mathbb{B}_\infty$  such that  $|\frac{\partial f_\alpha}{\partial x_i}(x) - \gamma_i(x, \alpha, \zeta)| \leq \rho$  for all  $\zeta$  in the set  $\mathbb{B}_\infty \cap B(\bar{\zeta}, \rho/2L)$ . Moreover, this set contains a cube of side  $s := \min\{\rho/2L\sqrt{n}, 1/2\}$ .*

(ii)  *$|\nabla f_\alpha(x) - \gamma(x, \alpha, z)| \leq \rho$  for all  $z$  in a cube of side  $s := \min\{\rho/2Ln, 1/2\sqrt{n}\}$  contained in  $Z$ .*

*Proof.* (i) Let  $\phi(\zeta) := \gamma_i(x, \alpha, \zeta) - \frac{\partial f_\alpha}{\partial x_i}(x)$ . By (2.4)–(2.5),  $\int_{\mathbb{B}_\infty} \phi(\zeta) d\zeta = 0$  and  $2L$  is a Lipschitz constant for  $\phi$  on  $\mathbb{B}_\infty$ . Hence there is  $\bar{\zeta} \in \mathbb{B}_\infty$  with  $\phi(\bar{\zeta}) = 0$ . Indeed, if  $\phi(\zeta') > 0$  for some  $\zeta' \in \mathbb{B}_\infty$ , then  $\phi(\zeta'') < 0$  for some  $\zeta'' \in \mathbb{B}_\infty$  (otherwise the continuity of  $\phi$  on  $\mathbb{B}_\infty$  would give  $\int_{\mathbb{B}_\infty} \phi(\zeta) d\zeta > 0$ , a contradiction), so  $\phi(\bar{\zeta}) = 0$  for some  $\bar{\zeta} \in [\zeta', \zeta'']$ ; similarly for  $\phi(\zeta') < 0$ . Since  $|\phi(\zeta) - \phi(\bar{\zeta})| \leq 2L|\zeta - \bar{\zeta}|$  for all  $\zeta \in \mathbb{B}_\infty$ , the first assertion follows. For the second assertion, using  $B(\bar{\zeta}, \rho/2L) \supset \bar{\zeta} + 2s\mathbb{B}_\infty$ , take the cube  $\bar{\zeta} + s\mathbb{B}_\infty$  with  $\hat{\zeta}_i := \bar{\zeta}_i - \text{sign}(\bar{\zeta}_i)s/2$  for  $i = 1, \dots, n$ .

(ii) This follows from (2.6) and statement (i) with  $\rho$  replaced by  $\rho/\sqrt{n}$ .  $\square$

For asymptotic stationarity, we need the following result of [MaP84, Prop. 2.2].

LEMMA 3.5. *Let  $x \in \mathbb{R}^n$ ,  $\alpha > 0$ . Then  $\nabla f_\alpha(x) \in \text{co } \bar{\partial}f(x + 2\alpha\mathbb{B}_\infty) \subset \bar{\partial}_{\sqrt{n}\alpha}f(x)$ .*

*Proof.* The derivative of  $f_\alpha$  at  $x$  in any direction  $d \in \mathbb{R}^n$ ,  $|d| = 1$ , is given by

$$\langle \nabla f_\alpha(x), d \rangle = \lim_{t \downarrow 0} \frac{1}{\alpha^n} \int_{\alpha\mathbb{B}_\infty} \{[f(x+y+td) - f(x+y)]/t\} dy = \lim_{t \downarrow 0} \frac{1}{\alpha^n} \int_{\alpha\mathbb{B}_\infty} \langle v(y, t), d \rangle dy,$$

where, by Lebourg's mean value theorem,  $v(y, t) \in \bar{\partial}f(x+y+atd)$  for some  $a \in [0, 1]$ . Hence,  $v(y, t) \in \text{co } \bar{\partial}f(x + 2\alpha\mathbb{B}_\infty)$  for all  $y \in \alpha\mathbb{B}_\infty$  and  $t \in [0, \alpha/2]$ , so that  $\nabla f_\alpha(x) \in \text{co } \bar{\partial}f(x + 2\alpha\mathbb{B}_\infty)$ . Since  $x + 2\alpha\mathbb{B}_\infty \subset B(x, \sqrt{n}\alpha)$ , (2.1) yields the conclusion.  $\square$

Actually, we mostly need only the following simple consequence of Lemma 3.5.

LEMMA 3.6. *Let  $\bar{x} \in \mathbb{R}^n$ ,  $\epsilon \geq 0$ ,  $\rho > 0$ . There exist  $\bar{\epsilon} > \epsilon$  and  $\hat{\alpha} \in (0, 1]$  such that  $\text{dist}(\nabla f_\alpha(B(\bar{x}, \bar{\epsilon})) | \bar{\partial}_\epsilon f(\bar{x})) \leq \rho$  for all  $\alpha \in (0, \hat{\alpha}]$ .*

*Proof.* Since  $\bar{\partial}_\epsilon f(\bar{x})$  is closed, there is  $\epsilon' > \epsilon$  with  $\bar{\partial}_{\epsilon'} f(\bar{x}) \subset \bar{\partial}_\epsilon f(\bar{x}) + B(0, \rho)$ . Pick  $\bar{\epsilon} > \epsilon$  and  $\hat{\alpha} \in (0, 1]$  such that  $\bar{\epsilon} + \sqrt{n}\hat{\alpha} \leq \epsilon'$ . By Lemma 3.5,  $\nabla f_\alpha(B(\bar{x}, \bar{\epsilon})) \subset \text{co } \bar{\partial}f(B(\bar{x}, \bar{\epsilon} + \sqrt{n}\alpha)) \subset \bar{\partial}_{\epsilon'} f(\bar{x})$  for all  $\alpha \in (0, \hat{\alpha}]$ , and the conclusion follows.  $\square$

In the GS algorithm, one has  $g^k \in \bar{\partial}_{\epsilon_k} f(x^k)$ , so when  $g^k$  vanishes around a cluster point  $\bar{x}$ , then  $0 \in \bar{\partial}_\epsilon f(\bar{x})$  for  $\epsilon_k \downarrow \epsilon$  because  $\bar{\partial}_\epsilon f(\cdot)$  is closed. Here we only have  $g^k \in G_k$

in (2.7), but we may relate  $\gamma(x^{kl}, \alpha_k, z^{kl})$  with  $\nabla f_{\alpha_k}(x^{kl})$  via Lemma 3.4, and then  $\nabla f_{\alpha_k}(x^{kl})$  with  $\bar{\partial}_\epsilon f(\bar{x})$  via Lemma 3.6 to get the following.

LEMMA 3.7. *Suppose  $\alpha_k \downarrow 0$ ,  $\epsilon_k \downarrow \epsilon \geq 0$ ,  $x^k \xrightarrow{K} \bar{x}$  and  $g^k \xrightarrow{K} 0$  for a subsequence  $K \subset \{1, 2, \dots\}$ . Then  $0 \in \bar{\partial}_\epsilon f(\bar{x})$  with probability 1.*

*Proof.* Let  $C := \bar{\partial}_\epsilon f(\bar{x})$ . Suppose  $0 \notin C$ , i.e.,  $\bar{\rho} := \text{dist}(0|C) > 0$ . Let  $\rho := \bar{\rho}/4$ . By Lemma 3.6, since  $\{x^{kl}\}_{l=1}^m \subset B(x^k, \epsilon_k)$ , there are  $\bar{\epsilon} > \epsilon$ ,  $\bar{\alpha} \in (0, 1]$  and  $\bar{k}$  such that for all  $k$  in  $\bar{K} := \{k \in K : k \geq \bar{k}\}$ , we have  $\alpha_k \leq \bar{\alpha}$ ,  $|g^k| \leq \rho$ ,  $\{x^{kl}\}_{l=1}^m \subset X := B(\bar{x}, \bar{\epsilon})$  and  $\text{dist}(\{\nabla f_{\alpha_k}(x^{kl})\}_{l=1}^m | C) \leq \rho$ . Hence, if we had for some  $k \in \bar{K}$ ,

$$(3.5) \quad |\nabla f_{\alpha_k}(x^{kl}) - \gamma(x^{kl}, \alpha_k, z^{kl})| \leq \rho \quad \text{for } l = 1, \dots, m,$$

then with  $g^k \in G_k$  in (2.7) we would get  $\text{dist}(g^k | C) \leq 2\rho$  and  $\text{dist}(0|C) \leq |g^k| + 2\rho \leq 3\rho$ , i.e.,  $\bar{\rho} \leq 3\bar{\rho}/4$ , a contradiction. Therefore, (3.5) must fail for all  $k \in \bar{K}$ . This event has probability 0, since for each  $k \in \bar{K}$  and  $l = 1, \dots, m$ ,  $z^{kl}$  is sampled independently and uniformly from  $Z$ , which by Lemma 3.4 contains a cube  $Z^{kl}$  of side  $s := \min\{\rho/2Lm, 1/2\sqrt{n}\}$  (with  $L$  a Lipschitz constant for  $f$  on  $X + \bar{\alpha}\mathbb{B}_\infty$ ) such that  $|\nabla f_{\alpha_k}(x^{kl}) - \gamma(x^{kl}, \alpha_k, z)| \leq \rho$  for all  $z \in Z^{kl}$ . The conclusion follows.  $\square$

Our convergence results parallel those in [Kiw07] for the GS algorithm. We start with the case where  $\epsilon_k$  and  $\nu_k$  are allowed to decrease.

THEOREM 3.8. *Let  $\{x^k\}$  be a sequence generated by Algorithm 2.1 with  $\nu_1 > \nu_{\text{opt}} = \epsilon_{\text{opt}} = 0$  and  $\mu, \theta < 1$ . With probability 1 either  $f(x^k) \downarrow -\infty$ , or  $\nu_k \downarrow 0$ ,  $\epsilon_k \downarrow 0$  and every cluster point of  $\{x^k\}$  is stationary for  $f$ .*

*Proof.* (i) If  $f(x^k) \downarrow -\infty$ , there is nothing to prove, so assume  $\inf_k f(x^k) > -\infty$ . Then summing  $\beta t_k |g^k| \leq f(x^k) - f(x^{k+1})$  in (2.9) gives

$$(3.6) \quad \sum_{k=1}^{\infty} t_k |g^k| < \infty,$$

$$(3.7) \quad \sum_{k=1}^{\infty} |x^{k+1} - x^k| |g^k| < \infty.$$

(ii) Suppose there is  $k_1$ ,  $\bar{\nu} > 0$  and  $\bar{\epsilon} > 0$  such that  $\nu_k = \bar{\nu}$  and  $\epsilon_k = \bar{\epsilon}$  for all  $k \geq k_1$ . Using  $|g^k| \geq \bar{\nu}$  in (3.6)–(3.7) yields  $t_k \rightarrow 0$ ,  $\sum_k |x^{k+1} - x^k| < \infty$ , and hence the existence of a point  $\bar{x}$  such that  $x^k \rightarrow \bar{x}$ . Let  $\epsilon := \bar{\epsilon}$ . First, suppose  $0 \notin G_\epsilon(\bar{x})$ . For  $\delta, \bar{\alpha}, \tau$  and  $\bar{V}$  chosen as in Lemma 3.3(ii), we can pick  $k_2 \geq k_1$  such that  $x^k \in B(\bar{x}, \min\{\tau, \epsilon/3\})$ ,  $\alpha_k \leq \bar{\alpha}$  and  $t_k < \min\{\frac{1}{2}, \kappa\epsilon/3\}$  yield  $(x^{k_1}, \dots, x^{k_m}) \notin \bar{V}$  for all  $k \geq k_2$ . This event has probability 0, since for each  $k \geq k_2$ ,  $(x^{k_1}, \dots, x^{k_m})$  is sampled independently and uniformly from  $\text{cl } D_\epsilon^m(x^k)$ , which contains the open set  $\bar{V} \neq \emptyset$ . Second, suppose  $0 \in G_\epsilon(\bar{x})$ . For  $\delta := \bar{\nu}/2$  and  $\bar{\alpha}, \tau, \bar{V}$  chosen as in Lemma 3.3(i), we can pick  $k_3 \geq k_1$  such that  $x^k \in B(\bar{x}, \tau)$ ,  $\alpha_k \leq \bar{\alpha}$ ,  $\bar{\nu} \leq |g^k| = \text{dist}(0|G_k)$  in (2.7) and  $\rho_\epsilon(\bar{x}) = 0$  imply  $(x^{k_1}, \dots, x^{k_m}) \notin \bar{V}$  for all  $k \geq k_3$ . This event has probability 0 as well.

(iii) Consider the event where  $\nu_k \downarrow 0$ ,  $\epsilon_k \downarrow 0$  and  $\{x^k\}$  has a cluster point  $\bar{x}$ . If  $x^k \not\rightarrow \bar{x}$ , we claim that  $\liminf_k \max\{|x^k - \bar{x}|, |g^k|\} = 0$ . Otherwise, there exist  $\bar{\nu} > 0$ ,  $\bar{k}$  and an infinite set  $K := \{k : k \geq \bar{k}, |x^k - \bar{x}| \leq \bar{\nu}\}$  such that  $|g^k| > \bar{\nu}$  for all  $k \in K$ , so (3.7) gives  $\sum_{k \in K} |x^{k+1} - x^k| < \infty$ . Since  $x^k \not\rightarrow \bar{x}$ , there is  $\hat{\nu} > 0$  such that for each  $k \in K$  with  $|x^k - \bar{x}| \leq \bar{\nu}/2$  there exists  $k' > k$  satisfying  $|x^{k'} - x^k| > \hat{\nu}$  and  $|x^i - \bar{x}| \leq \bar{\nu}$  for all  $k \leq i < k'$ . Therefore, by the triangle inequality, we have  $\hat{\nu} < |x^{k'} - x^k| \leq \sum_{i=k}^{k'-1} |x^{i+1} - x^i|$  with the right side being less than  $\hat{\nu}$  for large



$k \in K$  from  $\sum_{k \in K} |x^{k+1} - x^k| < \infty$ , a contradiction. If  $x^k \rightarrow \bar{x}$ , then  $\nu_k \downarrow 0$  at Step 4 also gives  $\lim_k \max\{|x^k - \bar{x}|, |g^k|\} = 0$ . Using this relation to pick a suitable  $K$  from Lemma 3.7 yields the conclusion.  $\square$

**THEOREM 3.9.** *Let  $\{x^k\}$  be a sequence generated by Algorithm 2.1 with  $\nu_1 > \nu_{\text{opt}} = \epsilon_{\text{opt}} = 0$  and  $\mu, \theta < 1$ . Suppose the level set  $\{x : f(x) \leq f(x^1)\}$  is bounded. Then with probability 1,  $\nu_k \downarrow 0$ ,  $\epsilon_k \downarrow 0$ , every cluster point of  $\{x^k\}$  is stationary for  $f$  and  $g^k \xrightarrow{R} 0$  for  $K := \{k : \nu_{k+1} < \nu_k\}$ .*

*Proof.* Since  $\{x^k\}$  lies in the bounded set  $\{x : f(x) \leq f(x^1)\}$ ,  $\inf_k f(x^k) > -\infty$  and the conclusion follows from Theorem 3.8.  $\square$

Our convergence results for fixed sampling radius follow.

**THEOREM 3.10.** *Let  $\{x^k\}$  be a sequence generated by Algorithm 2.1 with  $\nu_1 = \nu_{\text{opt}} = 0$ ,  $\epsilon_1 = \epsilon_{\text{opt}} = \epsilon > 0$  and  $\mu = 1$ . With probability 1 either the algorithm terminates at some iteration  $k$  with  $g^k = 0$ , or  $f(x^k) \downarrow -\infty$ , or there is a subsequence  $K \subset \{1, 2, \dots\}$  such that  $g^k \xrightarrow{R} 0$  and every cluster point  $\bar{x}$  of  $\{x^k\}_{k \in K}$  satisfies  $0 \in \partial_\epsilon f(\bar{x})$ .*

*Proof.* We may assume that no termination occurs and  $\inf_k f(x^k) > -\infty$ .

By part (ii) of the proof of Theorem 3.8, for  $\bar{\nu} := \lim_k |g^k|/2$ , the event  $\bar{\nu} > 0$  has probability 0. In the remaining case of  $\bar{\nu} = 0$ ,  $\lim_k |g^k| = 0$  and the conclusion follows from Lemma 3.7.  $\square$

**THEOREM 3.11.** *Let  $\{x^k\}$  be a sequence generated by Algorithm 2.1 with  $\nu_1 = \nu_{\text{opt}} = 0$ ,  $\epsilon_1 = \epsilon_{\text{opt}} = \epsilon > 0$  and  $\mu = 1$ . Suppose the set  $\{x : f(x) \leq f(x^1)\}$  is bounded. With probability 1 either the algorithm terminates at some iteration  $k$  with  $g^k = 0$ , or  $g^k \rightarrow 0$  and every cluster point  $\bar{x}$  of  $\{x^k\}$  satisfies  $0 \in \partial_\epsilon f(\bar{x})$ .*

*Proof.* Arguing by contradiction, it suffices to consider the case where there are a set  $J \subset \{1, 2, \dots\}$  and  $\bar{\nu} > 0$  such that  $\lim_{k \in J} |g^k| \geq 2\bar{\nu}$ . Since  $\{x^k\}$  lies in the bounded set  $\{x : f(x) \leq f(x^1)\}$ ,  $\inf_k f(x^k) > -\infty$  and we may assume with no loss of generality that there is a point  $\bar{x}$  such that  $x^k \xrightarrow{J} \bar{x}$ . Since (3.6) gives  $t_k \xrightarrow{J} 0$ , arguing as in part (ii) of the proof of Theorem 3.8 we deduce the existence of  $k_4$  and an open set  $\bar{V} \neq \emptyset$  such that  $(x^{k_4}, \dots, x^{k_m}) \notin \bar{V} \subset D_\epsilon^n(x^k)$  for all  $k \geq k_4$ ,  $k \in J$ . This event has probability 0, since for each  $k$ ,  $(x^{k_4}, \dots, x^{k_m})$  is sampled independently and uniformly from  $\text{cl } D_\epsilon^n(x^k)$ .  $\square$

**Remark 3.12.** If Step 3 is omitted, then Theorems 3.10 and 3.11 hold with the statements about termination omitted (by their proofs). In particular, if  $g^{k'} = 0$  for some  $k'$ , we may have  $0 \notin \partial_\epsilon f(x^{k'})$ , but if  $x^k = x^{k'}$  for all  $k \geq k'$ , then  $0 \in \partial_\epsilon f(x^{k'})$  with probability 1.

**4. Modifications.** In this section we propose several themes, supported by theory, that might prove useful in improving the practical performance of the method.

**4.1. Stopping criteria.** Recalling Remark 3.12, consider the following result.

**LEMMA 4.1.** *Let  $\epsilon'_k := \epsilon_k + \sqrt{n}\alpha_k$ . If (3.5) holds for some  $\rho > 0$ , then*

$$(4.1) \quad \text{dist}(0 | \partial_{\epsilon'_k} f(x^k)) \leq |g^k| + \rho.$$

Thus, for any  $\rho > 0$ , (3.5) and (4.1) hold with probability at least  $s^{mn^2}$ , where  $s := \min\{\rho/2Ln, 1/2\sqrt{n}\}$  with  $L$  a Lipschitz constant for  $f$  on  $B(x^k, \epsilon_k) \oplus \mathbb{B}_\infty$ .

*Proof.* Since  $\{x^k\}_{k=1}^m \subset B(x^k, \epsilon_k)$ , Lemma 3.5 gives  $G'_k := \text{co}\{\nabla f_{\alpha_k}(x^{k'})\}_{k=1}^m \subset \partial_{\epsilon'_k} f(x^k)$ . Thus  $\text{dist}(0 | \partial_{\epsilon'_k} f(x^k)) \leq \text{dist}(0 | G'_k)$ , where  $\text{dist}(0 | G'_k) \leq \text{dist}(0 | G_k) + \rho$  by (2.7) if (3.5) holds; then (4.1) follows with  $|g^k| = \text{dist}(0 | G_k)$ . The final assertion about probabilities stems from Lemma 3.4 as in the proof of Lemma 3.7.  $\square$

Remark 3.12 and Lemma 4.1 suggest that in practice, for  $\nu_{\text{opt}}, \epsilon_{\text{opt}} > 0$ , the stopping criterion of Step 3 should be augmented with the condition that, for a given integer  $k_{\text{last}}$ ,  $x^k$  has not changed during the last  $k_{\text{last}}$  iterations.

Alternatively, for a given integer  $\sigma_{\text{max}} \geq 1$  (playing the role of  $k_{\text{last}}$ ), we may consider a *resampling variant* where Step 0 sets the sampling counter  $\sigma_1 := 1$ , Step 7 sets  $\sigma_{k+1} := 1$ , and Step 3 is replaced by the following.

*Step 3'* (stopping criterion). If  $|g^k| > \nu_{\text{opt}}$  or  $\epsilon_k > \epsilon_{\text{opt}}$ , go to Step 4. If  $\sigma_k = \sigma_{\text{max}}$ , terminate. Otherwise, set  $\sigma_k := \sigma_k + 1$  and return to Step 1.

Upon termination in Step 3', by Lemma 4.1, for any  $\rho > 0$ , we have

$$(4.2) \quad \text{dist}(0 | \bar{\partial}_{\epsilon_k} f(x^k)) \leq \nu_{\text{opt}} + \rho$$

with probability  $p \geq 1 - (1 - s^{mn^2})^{\sigma_{\text{max}}}$ , where  $s$  is given in Lemma 4.1. In particular,  $p \geq \bar{p}$  for a given  $\bar{p} \in (0, 1)$  if  $\sigma_{\text{max}} \geq \log(1 - \bar{p}) / \log(1 - s^{mn^2})$ .

**4.2. Initial step sizes.** Since Algorithm 2.1 employs search directions  $d^k := -g^k/|g^k|$  of unit norm, the choice of an initial step size  $t_{\text{ini}}^k$  at Step 5(i) may be crucial in practice. For instance, if  $t_{\text{ini}}^k \equiv 1$ , then the number of  $f$ -evaluations per line search grows to infinity when  $t_k = |x^{k+1} - x^k| \rightarrow 0$  (e.g.,  $\{x^k\}$  converges). To provide more freedom for implementations, at Step 5(i) we may replace  $t_{\text{min}}^k$  by

$$(4.3) \quad t_{\text{min}}^k := \min\{\underline{t}, \kappa\epsilon_k/3, |g^k|\}.$$

Then, e.g.,  $t_{\text{ini}}^k = |g^k|$  corresponds to using a unit initial step size for the nonnormalized search direction  $-g^k$  as in [Kiw07, §4.1], whereas  $t_{\text{ini}}^k = \epsilon_k$  corresponds to searching within the sampled trust region  $B(x^k, \epsilon_k)$  as in [Kiw07, §4.2].

For (4.3), we need only replace  $t_k \geq \min\{\underline{t}, \kappa\epsilon/3\}$  by  $t_k \geq \min\{\underline{t}, \kappa\epsilon/3, |g^k|\}$  in Lemma 3.3(ii), and  $t_k < \min\{\underline{t}, \kappa\epsilon/3\}$  by  $t_k < \min\{\underline{t}, \kappa\epsilon/3, \bar{\nu}\}$  in part (ii) of the proof of Theorem 3.8 (where  $|g^k| \geq \bar{\nu}$ ). In effect, the preceding convergence results hold for this modification.

**4.3. Using the current gradient estimate.** Since the GS algorithm augments its bundle with the current gradient  $\nabla f(x^k)$ , we now consider a similar extension. At Step 1, let  $z^{k0}$  be sampled independently and uniformly from  $Z$ , set  $x^{k0} := x^k$  and

$$(4.4) \quad \hat{G}_k := \text{co}\{\gamma(x^{k0}, \alpha_k, z^{k0}), \gamma(x^{k1}, \alpha_k, z^{k1}), \dots, \gamma(x^{km}, \alpha_k, z^{km})\}.$$

To extend our preceding results, we replace Lemma 3.3(ii) by the following.

**LEMMA 4.2.** *Let  $\epsilon > 0$  and  $\bar{x} \in \mathbb{R}^n$ . Assuming  $0 \notin G_\epsilon(\bar{x})$ , pick  $\delta > 0$  as in Lemma 3.1 for  $C := G_\epsilon(\bar{x})$ , and then  $\bar{\alpha}$ ,  $\tau$  and  $\bar{V}$  as in Lemma 3.3(i). Let  $\hat{\alpha} \in (0, 1]$  be such that  $\text{dist}(\nabla f_\alpha(B(\bar{x}, \epsilon/2)) | \bar{\partial}_{\epsilon/2} f(\bar{x})) \leq \delta/2$  for all  $\alpha \in (0, \hat{\alpha}]$  (cf. Lemma 3.6), and let  $L$  be the Lipschitz constant of  $f$  on  $B(\bar{x}, \tau) + \mathbb{B}_\infty$ . Suppose at iteration  $k$  of Algorithm 2.1, Step 5 is reached with  $x^k \in B(\bar{x}, \min\{\tau, \epsilon/3\})$ ,  $\epsilon_k = \epsilon$ ,  $\alpha_k \leq \min\{\bar{\alpha}, \hat{\alpha}\}$  and  $(x^{k1}, \dots, x^{km}) \in \bar{V}$ . Using Lemma 3.4 with  $\rho := \delta/2$ , let  $Z^{k0}$  be a cube of side  $s := \min\{\rho/2L\tau, 1/2\sqrt{n}\}$  contained in  $Z$  such that  $|\nabla f_{\alpha_k}(x^k) - \gamma(x^k, \alpha_k, z)| \leq \rho$  for all  $z$  in this cube, and suppose  $z^{k0} \in Z^{k0}$ . Then  $t_k \geq \min\{\underline{t}, \kappa\epsilon/3\}$ .*

*Proof.* Since  $\alpha_k \leq \hat{\alpha}$ ,  $x^k \in B(\bar{x}, \epsilon/2)$ ,  $\bar{\partial}_{\epsilon/2} f(\bar{x}) \subset C$  and  $z^{k0} \in Z^{k0}$ , we have

$$\text{dist}(\gamma(x^{k0}, \alpha_k, z^{k0}) | C) \leq \text{dist}(\nabla f_{\alpha_k}(x^k) | C) + |\gamma(x^{k0}, \alpha_k, z^{k0}) - \nabla f_{\alpha_k}(x^k)| \leq \delta.$$

Let  $\hat{G}_k := \text{co}\{\gamma(x^{k1}, \alpha_k, z^{k1})\}_{l=1}^m$ . Since  $(x^{k1}, \dots, x^{km}) \in \bar{V} \subset V_{\epsilon, \hat{\alpha}}(\bar{x}, \bar{\delta})$  in Lemma 3.3(i), we get  $\text{dist}(0 | \hat{G}_k) \leq \rho_\epsilon(\bar{x}) + \bar{\delta}$  and  $\text{dist}(\hat{G}_k | C) \leq \bar{\delta}$  from (3.3). Thus, by

(4.4) and the construction of  $g^k$  at Step 2,  $\text{dist}(G_k | C) \leq \delta$ ,  $|g^k| \leq \rho_\epsilon(\bar{x}) + \delta$  and  $\text{dist}(g^k | C) \leq \delta$ . The conclusion follows as in the proof of Lemma 3.3(ii).  $\square$

The proof of Lemma 3.7 goes through (with  $l = 1$  replaced by  $l = 0$  and (2.7) by (4.4)). In part (ii) of the proof of Theorem 3.8 for  $0 \notin G_\epsilon(\bar{x})$ , replacing Lemma 3.3(ii) by Lemma 4.2, we can pick  $k_2$  such that  $((x^{k_1}, \dots, x^{k_m}), z^{k_0}) \notin \bar{V} \times Z^{k_0}$  for all  $k \geq k_2$ , again concluding that this event has probability 0 by the uniform and independent sampling of  $(x^{k_1}, \dots, x^{k_m})$  in  $\text{cl } D_\epsilon^m(x^k)$  and  $z^{k_0}$  in  $Z$ , since  $\text{cl } D_\epsilon^m(x^k)$  contains the open set  $\bar{V} \neq \emptyset$  and  $Z$  contains the cube  $Z^{k_0}$  of side  $s := \min\{\delta/4Ln, 1/2\sqrt{n}\}$ . The proof of Theorem 3.11 is modified accordingly. In effect, the preceding convergence results hold for this modification.

**4.4. Larger samples for Monte Carlo estimates.** At Step 1, the estimates  $\gamma(x^{kl}, \alpha_k, z^{kl})$  stem from single samples  $z^{kl} \in Z$ . In the Monte Carlo spirit, we may consider estimates that are averages over several samples. Thus, for  $\sigma \geq 1$  denoting the  $z$ -sample size, let  $Z := \prod_{j=1}^\sigma Z$  be the sample set, and let

$$(4.5) \quad \gamma(x, \alpha, z) := \frac{1}{\sigma} \sum_{j=1}^\sigma \gamma(x, \alpha, z^j) \quad \text{for } z := (z^1, \dots, z^\sigma) \in Z.$$

Then at Step 1,  $\{z^{kl}\}_{l=1}^m$  and  $Z$  are replaced by  $\{z^{kl}\}_{l=1}^m$  and  $Z$ , respectively.

The preceding convergence results extend easily to this modification. Indeed, it suffices to notice that if  $|g - \gamma(x, \alpha, z^j)| \leq \rho$  for  $j = 1, \dots, \sigma$ , then (4.5) gives  $|g - \gamma(x, \alpha, z)| \leq \rho$ , where  $g = \nabla f(\bar{x})$  and  $\rho = \delta$  for Lemma 3.2, and  $g = \nabla f_\alpha(x)$  with  $\alpha = \alpha_k$  and  $x = x^{kl}$  in the proof of Lemma 3.7.

By similar arguments, we could use a variable sample size  $\sigma_k$  with  $\sup_k \sigma_k < \infty$ .

**4.5. Sampling in cubes instead of balls.** Replacing the ball  $B(x, \epsilon)$  in (2.1)–(2.2) by the cube  $B_\infty(x, \epsilon) := \{y : |y - x|_\infty \leq \epsilon\} = x + 2\epsilon\mathbb{B}_\infty$  centered at  $x$  of side  $2\epsilon$ , where  $|\cdot|_\infty$  is the  $\infty$ -norm, we may replace  $B(x^k, \epsilon_k)$  by  $B_\infty(x^k, \epsilon_k)$  at Step 1.

The preceding results extend easily with  $B_\infty(\cdot, \cdot)$  replacing  $B(\cdot, \cdot)$  in Lemma 3.2, (3.2), Lemma 3.3 and its proof (also using  $|d^k|_\infty \leq 1$  there),  $\sqrt{n}$  omitted in Lemma 3.5 and the proof of Lemma 3.6, and  $\epsilon'_k := \epsilon_k + \alpha_k$  in Lemma 4.1.

**4.6. Incremental sampling.** At Step 1, the whole bundle of gradient estimates  $G_k$  of size  $m \geq n + 1$  is generated in one stage. Instead, we may build the bundle

$$(4.6) \quad G_k := \text{co}\{\gamma(x^{kl}, \alpha_k, z^{kl})\}_{l=1}^{m_k}$$

incrementally by increasing its size  $m_k$  until either descent occurs or  $m_k = m$ . This may save the oracle work when  $m_k < m$  suffices for descent or reducing  $|g^k|$ .

To this end, setting  $m_0 := 0$  at Step 0, replace Step 1 by the following.

*Step 1* (approximate the Clarke  $\epsilon$ -subdifferential by sampling estimates of mollifier gradients). Pick the current sample size  $m_k \in [m_{k-1} + 1, m]$ . Let  $\{x^{kl}\}_{l=m_{k-1}+1}^{m_k}$  and  $\{z^{kl}\}_{l=m_{k-1}+1}^{m_k}$  be sampled independently and uniformly from  $B(x^k, \epsilon_k)$  and  $Z$ , respectively. Compute  $\{\gamma(x^{kl}, \alpha_k, z^{kl})\}_{l=m_{k-1}+1}^{m_k}$  and set  $G_k$  by (4.6).

At Step 4, if  $|g^k| \leq \nu_k$ , set  $m_k := 0$  to start new sampling. At Step 6, if  $t_k = 0$  and  $m_k < m$ , set  $m_{k-1} := m_k$  and go back to Step 1; otherwise, set  $m_k := 0$ .

Thus Steps 4 and 6 restart sampling by setting  $m_k := 0$  if there is progress in stationarity ( $|g^k| \leq \nu_k$ ), descent ( $t_k > 0$ ) or the sample is full ( $m_k = m$ ).

The preceding convergence results extend easily to this modification. Indeed, in view of (4.6), we may replace  $m$  by  $m_k$  in the proof of Lemma 3.7. In part (ii) of the

proof of Theorem 3.8, we have  $t_k = 0$  for  $k \geq k_2$  in general (cf.  $t_{\min}^k$  at Step 5), and hence  $m_k = m$  by the modified Step 6 above.

*Remark 4.3.*

(i) If Step 5 chooses an initial  $t_{\min}^k = t_{\min}^k$ , it gives either  $t_k = t_{\min}^k$  or  $t_k = 0$ ; thus, the additional worst-case cost of the incremental version is relatively small: at most  $m - 1$   $f$ -evaluations relative to  $2mn$  for computing the full  $G_k$  with  $m_k = m$ .

(ii) For  $m_k = m_{k-1} + 1$  and  $t_{\min}^k = t_{\min}^k$ , Steps 1 and 5 need  $2n + 1$   $f$ -evaluations.

**4.7. Bundling past information.** The incremental version of Sect. 4.6 can be equipped with additional bundle memory, using the following notation.

The current *sampled bundle*  $\hat{G}_k := \text{co}\{\gamma(x^{kl}, \alpha_k, z^{kl})\}_{l=1}^{m_k}$  is managed as in Sect. 4.6. Some gradient estimates obtained earlier are stored in the *past bundle*

$$(4.7) \quad \check{G}_k := \text{co}\{\gamma(x^{jl}, \alpha_j, z^{jl})\}_{(j,l) \in I_k} \quad \text{with} \quad I_k \subset \{1, \dots, k-1\} \times \{1, \dots, m\},$$

where  $I_k$  is chosen so that for a fixed *past bundle size*  $\check{m} \geq 1$ ,

$$(4.8) \quad |I_k| \leq \check{m} \quad \text{and} \quad \{x^{jl}\}_{(j,l) \in I_k} \subset B(x^k, 0.99\epsilon_k),$$

$$(4.9) \quad \check{\alpha}_{\max}^k := \max_{(j,l) \in I_k} \alpha_j \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty.$$

To check  $|x^{jl} - x^k| \leq 0.99\epsilon_k$  in (4.8) without storing  $x^{jl}$ ,  $|x^{jl} - x^k|$  may be replaced by its overestimate  $|x^{jl} - x^j| + \sum_{i=j}^{k-1} |x^i - x^{i+1}|$ . As for (4.9), since  $\alpha_k \downarrow 0$ , we may require that  $\check{\alpha}_{\max}^k \leq \kappa_\alpha \alpha_k$  for a constant  $\kappa_\alpha > 1$ . Finally, the total bundle is

$$(4.10) \quad G_k := \text{co}\{\hat{G}_k \cup \check{G}_k\}.$$

Formally, this version employs the modifications of Sect. 4.6 for managing  $m_k$ . Additionally, Step 0 chooses  $\check{m} \geq 0$  and sets  $I_1 := \emptyset$ , whereas Step 1 sets  $\hat{G}_k$  as in (4.6), chooses  $\check{G}_k$  via (4.7)–(4.9) and sets  $G_k$  by (4.10). Of course, there is still room for implementation choices. For instance, when Step 6 returns to Step 1, we may keep the same  $\hat{G}_k$ . At Step 7, we may obtain  $\check{G}_{k+1}$  from  $G_k$  by dropping points with “too large” values of  $|x^{jl} - x^{k+1}|$  or  $\alpha_j$ .

To extend our preceding results, we replace Lemma 3.3(ii) by the following.

**LEMMA 4.4.** *Let  $\epsilon > 0$  and  $\bar{x} \in \mathbb{R}^n$ . Assuming  $0 \notin G_\epsilon(\bar{x})$ , pick  $\delta > 0$  as in Lemma 3.1 for  $C := G_\epsilon(\bar{x})$ , and then  $\bar{\alpha}$ ,  $\tau$  and  $\bar{V}$  as in Lemma 3.3(i). For  $\rho := \delta/2$ , let  $\hat{\alpha} \in (0, 1)$  be such that  $\text{dist}(\nabla f_\alpha(B(\bar{x}, 0.999\epsilon)) | \partial_{0.999\epsilon} f(\bar{x})) \leq \rho$  for all  $\alpha \in (0, \hat{\alpha})$  (cf. Lemma 3.6), and let  $L$  be the Lipschitz constant of  $f$  on  $B(\bar{x}, \epsilon) + \mathbb{B}_\infty$ . Suppose at iteration  $k$  of Algorithm 2.1, Step 5 is reached with  $x^k \in B(\bar{x}, \min\{\tau, \epsilon/1000\})$ ,  $\epsilon_k = \epsilon$ ,  $\max\{\alpha_k, \check{\alpha}_{\max}^k\} \leq \min\{\bar{\alpha}, \hat{\alpha}\}$ ,  $m_k = m$  and  $(x^{k1}, \dots, x^{km}) \in \bar{V}$ . Using Lemma 3.4, for each  $(j, l) \in I_k$ , let  $Z^{jl}$  be a cube of side  $s := \min\{\rho/2Ln, 1/2\sqrt{m}\}$  contained in  $Z$  such that  $|\nabla f_{\alpha_j}(x^{jl}) - \gamma(x^{jl}, \alpha_j, z)| \leq \rho$  for all  $z$  in this cube, and suppose  $z^{jl} \in Z^{jl}$ . Then  $t_k \geq \min\{\underline{t}, \kappa\epsilon/3\}$ .*

*Proof.* For each  $(j, l) \in I_k$ , since  $\alpha_j \leq \hat{\alpha}$  by (4.9),  $x^{jl} \in B(\bar{x}, 0.999\epsilon)$  by (4.8) with  $|x^k - \bar{x}| \leq \epsilon/1000$ ,  $\partial_{0.999\epsilon} f(\bar{x}) \subset C$  and  $z^{jl} \in Z^{jl}$ , we have

$$\text{dist}(\gamma(x^{jl}, \alpha_j, z^{jl}) | C) \leq \text{dist}(\nabla f_{\alpha_j}(x^{jl}) | C) + |\gamma(x^{jl}, \alpha_j, z^{jl}) - \nabla f_{\alpha_j}(x^{jl})| \leq \delta.$$

For  $\hat{G}_k := \text{co}\{\gamma(x^{kl}, \alpha_k, z^{kl})\}_{l=1}^{m_k}$ , since  $(x^{k1}, \dots, x^{km}) \in \bar{V} \subset V_{\epsilon, \bar{\alpha}}(\bar{x}, \bar{x}, \delta)$  in Lemma 3.3(i), we get  $\text{dist}(0 | \hat{G}_k) \leq \rho_\epsilon(\bar{x}) + \delta$  and  $\text{dist}(\hat{G}_k | C) \leq \delta$  from (3.3). Thus, by

(4.10) and the construction of  $g^k$  at Step 2,  $\text{dist}(G_k | C) \leq \delta$ ,  $|g^k| \leq \rho_\epsilon(\bar{x}) + \delta$  and  $\text{dist}(g^k | C) \leq \delta$ . The conclusion follows as in the proof of Lemma 3.3(ii).  $\square$

The proof of Lemma 3.7 goes through, with the additional conditions that for  $k$  in  $\bar{K}$ ,  $\hat{\alpha}_{\max}^k \leq \hat{\alpha}$  (cf. (4.9)),  $\{x^{j_l}\}_{(j,l) \in I_k} \subset X$  (cf. (4.8)),  $\text{dist}(\{\nabla f_{\alpha_j}(x^{j_l})\}_{(j,l) \in I_k} | C) \leq \rho$ , (3.5) is augmented with  $|\nabla f_{\alpha_j}(x^{j_l}) - \gamma(x^{j_l}, \alpha_j, z^{j_l})| \leq \rho$  for all  $(j, l) \in I_k$ , and (4.10) replaces (2.7). In part (ii) of the proof of Theorem 3.8 for  $0 \notin G_\epsilon(\bar{x})$ , replacing Lemma 3.3(ii) by Lemma 4.4, we may argue as in Sect. 4.3, with  $z^{k_0}$  and  $Z^{k_0}$  replaced by  $z^{j_l}$  and  $Z^{j_l}$ . The proof of Theorem 3.11 is modified accordingly. In effect, the preceding convergence results hold for this modification.

*Remark 4.5.*

(i) By modifying Lemma 4.4 and its proof, we may replace the factor 0.99 in (4.8) by any number in  $(0, 1)$ ; e.g., 0.999999.

(ii) The sampling region may change relatively slowly to keep most of past estimates; e.g., we have  $|x^{k+1} - x^k| \leq \epsilon_k/6$  for  $\kappa = 1/2$  and  $\epsilon_{\text{ini}}^k = \epsilon_{\text{min}}^k = \epsilon_k/6$ .

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the 1990s, the number of people in the world who are undernourished has increased from 600 million to 800 million (FAO 2001).

There are a number of reasons for this increase. One of the main reasons is the increase in the world population. The world population has increased from 5 billion in 1987 to 6 billion in 2000, and is projected to reach 9 billion by 2050 (FAO 2001). This increase in population has led to an increase in the demand for food, and this has led to an increase in the number of people who are undernourished.

Another reason for the increase in the number of people who are undernourished is the increase in the number of people who are living in poverty. The number of people who are living in poverty has increased from 1 billion in 1987 to 2 billion in 2000, and is projected to reach 3 billion by 2050 (FAO 2001). This increase in poverty has led to an increase in the number of people who are unable to afford the food that they need.

A third reason for the increase in the number of people who are undernourished is the increase in the number of people who are living in rural areas. The number of people who are living in rural areas has increased from 2 billion in 1987 to 3 billion in 2000, and is projected to reach 4 billion by 2050 (FAO 2001). This increase in rural population has led to an increase in the number of people who are unable to access the food that they need.

There are a number of ways in which the number of people who are undernourished can be reduced. One of the main ways is to increase the production of food. This can be done by increasing the number of people who are working in agriculture, and by increasing the amount of land that is used for agriculture. Another way is to reduce the number of people who are living in poverty. This can be done by increasing the number of people who are employed, and by increasing the wages of those who are employed.

A third way is to increase the number of people who are living in urban areas. This can be done by increasing the number of people who are working in industry, and by increasing the number of people who are working in services. A fourth way is to increase the number of people who are living in rural areas. This can be done by increasing the number of people who are working in agriculture, and by increasing the amount of land that is used for agriculture.

There are a number of challenges that must be overcome in order to reduce the number of people who are undernourished. One of the main challenges is to increase the production of food. This can be done by increasing the number of people who are working in agriculture, and by increasing the amount of land that is used for agriculture. Another challenge is to reduce the number of people who are living in poverty. This can be done by increasing the number of people who are employed, and by increasing the wages of those who are employed.

