Raport Badawczy
Research Report

RB/8/2008

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Warszawa 2008

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Abstract. We present a concept of altruistic equilibrium in non-cooperative non-zero sum games based on the assumption that players take care of other player's interests. This concept is quantified by introducing altruistic coefficients which determine intensity of player altruism. We also introduce altruistic trade-off coefficients to characterize a strategy profile in terms of lower bounds on altruistic coefficients, for which the strategy profile is an altruistic equilibrium. Our concept allows to obtain equilibrium in any game for any Pareto optimal strategy profile, even if a Nash equilibrium does not exist.

Keywords: game theory, altruistic equilibrium, altruistic trade-off

1. An approach to resolving the Prisoner's Dilemma

Prisoner's Dilemma is an example of two person non-zero sum game showing that Nash equilibria are not necessary Pareto optimal. We propose a conceptual framework, still on the ground of game theory, which enables resolving this dilemma.

The game formulation of Prisoner's Dilemma is following.

There are two players, Player A and Player B. Each player has two strategies: "*stay silent*" and "*betray*". The payoff matrix is

	Player A stays silent	**Player A betrays**
Player B stays silent	(-0.5, -0.5)	(0, -10)
Player B betrays	(-10, 0)	(-2, -2)

Here two numbers in parentheses mean Player A and Player B payoffs, respectively.

The classic interpretation of the game is that both players are suspects in a crime committed together. They are separated from each other and interrogated simultaneously. Each of them have to decide either to betray the partner or to stay silent. The absolute values of payoffs indicate how many years of imprisonment will a player get depending on its own and its partner's decision.

Whichever strategy a player chooses, the other player is better off by betraying. Under the assumption that each player considers rational to pursue his own interest, both players will decide to betray. In other words, (*betray*, *betray*) is the Nash equilibrium of this game. The paradox is that there exists a better solution where players both stay silent and get 0.5 year of imprisonment each. Solution (*stay silent*, *stay silent*) dominates solution (*betray*, *betray*) in the Pareto sense, but is not attainable under the above assumption of rational behavior. Even being agreed beforehand to stay silent, each of players will prefer to betray. Indeed, betraying increases the payoff from -0.5 to 0 if the partner stays silent. Thus rational behavior of two actors leads to an irrational (Pareto dominated) result.

Formulated originally as Prisoner's Dilemma, the described game model gain in importance as many its applications are found in social sciences such as economy, politics, sociology, environmental studies, and also in biology and some other sciences.

The solution (*stay silent*, *stay silent*) is usually referred to as *cooperation*, because achieving this solution means coordinated rejection of own interests by each player. Attempts to establish conditions for mutual cooperation in Prisoner's Dilemma are generally based on studying repeatedly played games. Axelrod (1980, 1984) conducted game tournaments of two players and found out long-term incentive for cooperation in their behavior. In the framework of evolutionary approach, Robson (1990) proposed a model where Prisoner's Dilemma is repeatedly played in a population of players and there are "mutants" who cooperate playing with other "mutants" and betray playing with the rest of individuals. Invasion of "mutants" displays advantage of cooperation strategy. The evolutionary approach to Prisoner's Dilemma was further developed in later works (see for example Nowak and Sigmund (2004), Szabó and Fáth (2007)).

In contrast to evolutionary approach, we study games played only once. Our approach is based on idea to endow players altruistic trait. While in non-cooperative games it is assumed that each player pursues only his own interest, we challenge this principle by admitting the following assumption:

each player evaluating one strategy versus another,
prefers not to gain in his payoff if this leads to
disproportionally large loss in payoffs of other players.

To quantify this assumption, for each pair of players we introduce *altruistic coefficient*. Denote two players by k and l. The altruistic coefficient of player k with respect to player l applies in the following situation. Knowing the strategies of the other players, player k evaluates one of his strategies, say i, over another his strategy, say j. At that strategy i in comparison to j

gives player k a greater payoff, but if player k chooses i over j, then player l loses in his payoff. In this situation, our assumption is formulated as follows:

*player **k** does not prefer strategy **i** to strategy **j**,*
*if the payoff loss of player **l** multiplied by the altruistic coefficient*
*is greater or equal to the payoff gain of player **k**.*

Let us illustrate how this assumption affects analysis of Prisoner's Dilemma.

Suppose that altruistic coefficient equal to 0.1 applies to Player A with respect to Player B and also to Player B with respect to Player A.

Suppose now that Player A stays silent. If Player B had betrayed instead of staying silent, he would condemn Player A to additional 9.5 years of imprisonment while avoiding only 0.5 year imprisonment by himself. The payoff loss of Player A multiplied to the altruistic coefficient is greater than the payoff gain of Player B ($9.5 \cdot 0.1 > 0.5$). According to our assumption, Player B prefers to stay silent. Analogously, if Player B stays silent, then Player A prefers to stay silent too. Thus solution (*stay silent*, *stay silent*) is an equilibrium in the sense that no one player deviates from his strategy if the partner does not.

Let us formalize the concept of equilibrium with account of player altruism.

2. Altruism and equilibria in the multi-player non-zero sum game

Consider p person, $p>1$, non-cooperative non-zero sum game $(\mathbf{S},\mathbf{a})$, where

$\mathbf{S}=S_1\times S_2\times\ldots\times S_p$ is the set of *strategy profiles*, $S_k:=\{1,2,\ldots,m_k\}$, $m_k\in\mathbf{N}$, $m_k>1$, is the strategy set of k-th player, $k\in N_p:=\{1,2,\ldots,p\}$;

$\mathbf{a}=(a^1,a^2,\ldots,a^p)$: $\mathbf{S}\to\mathbf{R}^p$ is the vector of payoff functions, where a^k: $\mathbf{S}\to\mathbf{R}$ is the *payoff function* of k-th player yielding payoff $a^k(I)$ for each strategy profile $I\in\mathbf{S}$.

For any strategy profile $I=(i_1,i_2,\ldots,i_p)$ and any player $k\in N_p$, define another strategy profile which differs from I by strategy of player k only:

$$I_{\langle k,j\rangle}=(i'_1,\, i'_2,\, \ldots,\, i'_p), \text{ where } i'_l=i_l \text{ for any } l\neq k \text{ and } i'_k=j,\, j\in S_k,\, j\neq i_k.$$

Definition 1. *Strategy profile I is a* ***Nash equilibrium*** *in game* $(\mathbf{S},\mathbf{a})$, *if*

$$a^k(I)\geq a^k(I_{\langle k,j\rangle}) \text{ for any } k\in N_p \text{ and any } j\in S_k.$$

Definition 2. *Strategy profile I is* ***Pareto optimal*** *in game* $(\mathbf{S},\mathbf{a})$, *if there does not exist another strategy profile* I' *such that*

$$\mathbf{a}(I')\geq\mathbf{a}(I),\quad \mathbf{a}(I')\neq\mathbf{a}(I).$$

We denote the altruistic coefficient of player k with respect to player l, $k\neq l$, by α_{kl}, $\alpha_{kl}\geq 0$. Let us introduce the matrix of altruistic coefficients $A=(\alpha_{kl})_{p\times p}$ where $\alpha_{kk}:=1$, $k\in N_p$ for definiteness.

Definition 3. *Strategy profile I is called* ***altruistic equilibrium*** *or* ***A-equilibrium****, if for any player $k \in N_p$ and any strategy from his strategy set $j \in S_k$, $j \neq i_k$ the following implication holds:*

if $a^k(I_{\langle k,j\rangle}) > a^k(I)$, then for some player $l \in N_p$, $l \neq k$, we have

$$\alpha_{kl}(a^l(I) - a^l(I_{\langle k,j\rangle})) \geq a^k(I_{\langle k,j\rangle}) - a^k(I) \qquad (1)$$

Literally, A-equilibrium is a strategy profile such that no one player wants to change his strategy by the following reason: if the player can gain in payoff by changing his strategy, this leads to payoff loss of another player such that the absolute value of the payoff loss multiplied to the corresponding altruistic coefficient is greater or equal to the payoff gain.

Player's altruistic behavior restricts domains in which players act exclusively in their own interests. The greater is a player's altruistic coefficient, the more severe this restriction is. If $\alpha_{kl}=0$, then player k does not feel any altruism with respect to player l and acts as in "ordinary" game. Indeed, for $\alpha_{kl}=0$ the implication in Definition 3 takes the form

$$\text{if } a^k(I_{\langle k,j\rangle}) > a^k(I) \text{ then } a^k(I_{\langle k,j\rangle}) - a^k(I) \leq 0$$

which holds true if and only if $a^k(I_{\langle k,j\rangle}) \leq a^k(I)$. It follows that the definition of A-equilibrium is equivalent to the definition of Nash equilibrium in the case, where all the altruistic coefficients are equal to zero.

One important consequence of altruistic behavior is that it may lead to cooperation among players. We will prove that introducing big enough altruistic coefficients guarantees existence of an altruistic equilibrium which is Pareto optimal.

Definition 4. *We call strategy profile I* ***locally efficient****, if there does not exist $k \in p$, $j \in S_k \backslash \{i_k\}$ such that*

$$a^k(I_{\langle k,j\rangle}) > a^k(I) \text{ and } a^l(I_{\langle k,j\rangle}) \geq a^l(I) \text{ for all } l \in N_p \backslash \{k\}.$$

In other words, I is locally efficient if it is not "dominated" by any "neighbor" strategy profile $I_{\langle k,j\rangle}$, where "domination" differs from the Pareto domination by the requirement of $a^k(I_{\langle k,j\rangle}) > a^k(I)$ and "neighborhood" means difference in one player's strategy only.

Theorem 1. *Let $I \in S$. There exists $A=(\alpha_{kl}) \in \mathbf{R}_{>}^{p\times p}$ such that I is an A-equilibrium if and only if I is locally efficient.*

Proof. Suppose that I is locally efficient. Then for any player k such that

$$a^k(I_{\langle k,j\rangle}) > a^k(I) \text{ for some strategy } j \in S_k,\ j \neq i_k$$

there exists another player l such that

$$a^l(I_{\langle k,j\rangle}) < a^l(I).$$

For altruistic coefficient α_{lk} satisfying

$$\alpha_{lk} \geq \frac{a^k\left(I_{\langle k,j\rangle}\right)-a^k(I)}{a^l(I)-a^l\left(I_{\langle k,j\rangle}\right)}$$

we have (1). It follows that I is an A-equilibrium if altruistic coefficients are big enough.

If I is not locally efficient, then for some $k \in p$ and some $j \in S_k \setminus \{i_k\}$ we have

$$a^k(I_{\langle k,j\rangle}) > a^k(I) \text{ and } a^l(I_{\langle k,j\rangle}) \geq a^l(I) \text{ for all } l \in N_p \setminus \{k\}.$$

It follows that there does not exist $l \in N_p$, $l \neq k$, and positive α_{lk} satisfying (1). Therefore I is not an A-equilibrium for any altruistic coefficients. □

It is evident that any Pareto optimal strategy profile is locally efficient. Therefore Theorem 1 implies

Corollary 1. *For any Pareto optimal strategy profile $I \in \mathbf{S}$ in game ($\mathbf{S}$, $\mathbf{a}$), there exists a positive matrix of altruistic coefficients A such that I is an A-equilibrium.*

Actually Corollary 1 is a stronger proposition than existence of an altruistic equilibrium being Pareto optimal. We have proved that **any** Pareto optimal strategy profile can be an altruistic equilibrium, if the altruistic coefficients are big enough. Observe that existence of a Nash equilibrium in the game is not required.

3. Altruistic trade-offs

Altruistic coefficients can be considered as parameters determining whether a given locally efficient strategy profile is an altruistic equilibrium or not. The following question arises: "*for which values of altruistic coefficients a given strategy profile is an altruistic equilibrium?*". Answering this question means characterizing a strategy profile by lower bounds of altruistic coefficients such that if altruistic coefficients satisfy these lower bounds, then this strategy profile is an altruistic equilibrium. We build such a characterization with the help of trade-off concept. Trade-off coefficients are widely used in multiple criteria decision making for characterizing solutions in terms of relative preference expressing (see Kaliszewski I. (2006)). We define the trade-off coefficient in a game as ratio between improvement of player's payoff and worsening of another player's payoff caused by player's strategy change.

Definition 5. *For any strategy profile I, any pair of players k, $p \in N_p$, $k \neq l$, and any k-th player's strategy $j \in S_k$ such that $a^k(I_{\langle k,j\rangle}) > a^k(I)$ and $a^l(I_{\langle k,j\rangle}) < a^l(I)$, the number*

$$T_{kl}(I,j) = \frac{a^k\left(I_{\langle k,j\rangle}\right)-a^k(I)}{a^l(I)-a^l\left(I_{\langle k,j\rangle}\right)}$$

*is called **altruistic trade-off coefficient** of player k with respect to player l for strategy profile I and strategy j.*

In the following evident proposition we reformulate the definition of A-equilibrium in terms of altruistic trade-off coefficients.

Proposition 1. *A locally efficient strategy profile I is an A-equilibrium if and only if for any player $k \in N_p$ and any strategy from his strategy set $j \in S_k$, $j \neq i_k$ the following implication holds:*

$$\text{if } a^k(I_{(k,j)}) > a^k(I), \text{ then}$$

$$\text{there exists another player } l \in N_p \text{ such that } a^l(I_{(k,j)}) < a^l(I) \text{ and } T_{kl}(I,j) \leq \alpha_{kl}.$$

Example 1. *Consider the Prisoner's Dilemma game described in Section 0, where the players and the strategies are numbered by the following way: Player A = 1, Player B = 2, "stay silent" = 1 and "betray" = 2. Consider the strategy profile I:=(1,1). It is locally efficient. Let us calculate altruistic trade-off coefficients for I:*

$$T_{12}(I,2) = T_{21}(I,2) = 0.5/9.5 = 1/19.$$

According to Proposition 1, strategy profile I is an A-equilibrium if and only if $\alpha_{12} \geq 1/19$ and $\alpha_{21} \geq 1/19$. So it suffices that each player takes care of the other player's interests 19 times less intense than of his own interests, for the cooperation to emerge.

It is easy to characterize a strategy profile in a game with two players with the help of the following evident corollary from Proposition 1.

Corollary 2. *Let p = 2. A locally efficient strategy profile $I=(i_1,i_2)$ is an A-equilibrium if and only if $\alpha_{12} \geq \tau_{12}$ and $\alpha_{21} \geq \tau_{21}$, where*

$$\tau_{kl} = \max\{T_{kl}(I,j) : j \in S_k,\ j \neq i_k,\ a^k(I_{(k,j)}) > a^k(I),\ a^l(I_{(k,j)}) < a^l(I)\},\ (k,l) \in \{(1,2),(2,1)\}\ \}$$

and maximum over empty set is assumed to be zero.

Unfortunately, in a game with more than two players it is impossible to characterize a strategy profile by lower bounds of altruistic coefficients, where bounds are independent on strategy changes. In other words, it is impossible to represent the characterization in the following form: *the strategy profile is A-equilibrium if and only if $\alpha_{kl} \geq \tau_{kl}$ for any $k,l \in N_p$, $k \neq l$, where τ_{kl} is the lower bound for altruistic coefficient*. This difficulty is illustrated by the following example.

Example 2. *Consider the game with 3 players each having 2 strategies and following payoff functions:*

Player 1 payoff function

	$i_2=1$		$i_2=2$	
	$i_3=1$	$i_3=2$	$i_3=1$	$i_3=2$
$i_1=1$	4	3	0	1
$i_1=2$	5	1	1	3

Player 2 payoff function

	$i_1=1$		$i_1=2$	
	$i_3=1$	$i_3=2$	$i_3=1$	$i_3=2$
$i_2=1$	3	3	5	1
$i_2=2$	4	1	1	4

Player 3 payoff function

	$i_1=1$		$i_1=2$	
	$i_2=1$	$i_2=2$	$i_2=1$	$i_2=2$
$i_3=1$	10	2	0	1
$i_3=2$	8	1	1	6

It is easy to check that strategy profile I:=(1,1,1) is locally efficient. Let us characterize I with the help of altruistic trade-off coefficients.

Altruistic trade-off coefficients of Player 1: $T_{13}(I, 2) = 0.1$; $T_{12}(I,2)$ *is undefined because when Player 1 changes his strategy from 1 to 2, the Player's 3 payoff is not decreased.*

Altruistic trade-off coefficients of Player 2: $T_{21}(I,2) = 0.5$; $T_{23}(I,2) = 0.25$.

Altruistic trade-off coefficients of Player 3 are undefined and are not taken into account when answering question whether I is A-equilibrium. Indeed, $a^3(1,1,2)<a^3(1,1,1)$ *which means that Player 3 is not interested in changing his strategy anyway.*

Applying Proposition 1, we obtain that I is A-equilibrium if and only if

$$\alpha_{13}\geq 0.1 \text{ and } (\alpha_{21}\geq 0.5 \text{ or } \alpha_{23}\geq 0.25). \tag{2}$$

The characterization of a strategy profile with the help of altruistic trade-off coefficients based on Proposition 1 can be written out by using logical operators:

locally efficient strategy profile I is A-equilibrium if and only if

$$\bigwedge_{k\in\hat{N}(I)} \bigwedge_{j\in\hat{S}_k(I)} \bigvee_{l\in\tilde{N}(I,k,j)} \left(\alpha_{kl} \geq T_{kl}(I,j)\right), \tag{3}$$

where

$\hat{N}(I)=\{k\in N_p:\ \hat{S}_k(I)\neq\emptyset\}$ is the subset of players who can improve their payoffs by changing their strategies;

$\hat{S}_k(I)=\{j\in S_k:\ j\neq i_k,\ a^k(I_{\langle k,j\rangle})>a^k(I)\}$ is the subset of player k's strategies, for which his payoff is greater than the payoff for the initial strategy;

$\tilde{N}(I,k,j)=\{l\in N_p: a^l(I_{\langle k,j\rangle})<a^l(I)\}$ is the set of players who suffer from player k changing his strategy to j.

4. The linear altruistic equilibrium

Let us present one more variant of modeling altruism in player behavior. This variant differs from the approach presented in Section 2 by the method of how a player evaluates losses he inflicts on the other players when changing his strategy. In Section 2 we assume that player k checks for each other player, if the other player loss multiplied to the corresponding altruistic coefficient is greater or equal to player k's gain. If yes, then player k refuses to change his strategy. At that player k checks payoff losses independently on each other.

Now we admit some other assumptions about player behavior which allow to formulate the linear model of altruistic equilibrium.

1) We assume that the evaluation of the other player losses is additive. Namely player k sums up the other player losses multiplied to the corresponding altruistic coefficients.

2) We assume that player k considers payoff gains obtained by some players when player k changes his strategy, as a compensation of losses caused to other players at the same time. Namely the other player payoff gains multiplied to altruistic coefficients are subtracted from the sum of other player payoff losses. Thus player k evaluates not only loss, but the cumulative effect of his strategy change on the "income" of the "player community", through the prism of his feeling of altruism expressed by his altruistic coefficients.

3) We assume that altruism may compel player k not only to refuse improving his payoff by changing his strategy, but also to sacrifice some of his payoff in other player favor. Player k agrees to change his strategy to decrease his own payoff, if his loss will be less than the sum of gains of the other players multiplied to the corresponding altruistic coefficients.

According to these assumptions we define the notion of linear altruistic equilibrium.

Definition 6. *Strategy profile I is called* ***linear altruistic equilibrium*** *or* ***linear A-equilibrium****, if for any player* $k\in N_p$ *and any strategy from his strategy set* $j\in S_k$, $j\neq i_k$, *the following inequality holds:*

$$a^k\left(I_{\langle k,j\rangle}\right)-a^k(I)\leq \sum_{\substack{l\in N_p\\ l\neq k}}\alpha_{kl}\left(a^l(I)-a^l\left(I_{\langle k,j\rangle}\right)\right). \qquad (4)$$

If passing from i_k to j improves the payoff of player k, inequality (4) means that in k-th player belief, this improvement does not overweighs the cumulative worsening of the other player payoffs, so player k refuses to change his strategy to j.

If passing from i_k to j worsens the payoff of player k, inequality (4) is equivalent to

$$a^k(I)-a^k\left(I_{\langle k,j\rangle}\right)\geq \sum_{\substack{l\in N_p\\ l\neq k}}\alpha_{kl}\left(a^l\left(I_{\langle k,j\rangle}\right)-a^l(I)\right)$$

meaning that this worsening is considered by player k as too large to be a reasonable price for improvement of the situation with payoffs of the rest of "player community".

Let us make use of linearity of the later model to connect the linear altruistic equilibrium with Nash equilibrium. We define the following transformation of payoff functions using altruistic coefficients:

$$\tilde{a}^k(I)=a^k(I)+\sum_{\substack{l\in N_p\\ l\neq k}}\alpha_{kl}\left(a^l(I)\right),\ k\in N_p, I\in \mathbf{S},$$

$$\tilde{\mathbf{a}}=\left(\tilde{a}^1,\tilde{a}^2,\ldots\tilde{a}^p\right).$$

The next theorem follows directly from Definition 6.

Theorem 2. *Strategy profile I is a linear A-equilibrium in game (S,a) if and only if it is a Nash equilibrium in game* $(\mathbf{S},\tilde{\mathbf{a}})$.

The following example illustrates the fact that the concepts of altruistic equilibrium and linear altruistic equilibrium are principally different in the sense that being altruistic equilibrium does not implies being linear altruistic equilibrium and *vice versa*.

Example 3. *Consider the game defined in Example 2.*

Put $\alpha_{kl}=0.2$ *for each* $k,l\in N_3$, $k\neq l$. *Then (2) does not hold which means that* $I=(1,1,1)$ *is not an altruistic equilibrium. But checking up inequalities (4) we make sure that I is a linear altruistic equilibrium.*

Now suppose that $\alpha_{23}=0.25$, $\alpha_{12}>1$ *and the rest of altruistic coefficients are as before. Then (2) is satisfied which means that I is an altruistic equilibrium. But I is not a linear altruistic equilibrium, because one of inequalities (4) is violated:*

$$a^1\left(I_{\langle 1,2\rangle}\right)-a^1(I)-\sum_{l\in\{2,3\}}\alpha_{kl}\left(a^l(I)-a^l\left(I_{\langle 1,2\rangle}\right)\right)=5-4-\alpha_{12}(3-4)-0.2(10-0)=-1+\alpha_{12}>0.$$

The later inequality has following interpretation in terms of the above assumptions about player behavior: for Player 1, the payoff of Player 2 is more important than his own payoff

(α_{12}>1) which compels Player 1 to change his strategy from 1 to 2 improving the payoff of Player 2. In contrast to this, the first altruistic equilibrium concept does not suppose Player 1 take into account improvements of payoffs of other players which does not make him to prefer strategy 2 to strategy 1.

Conclusion

We presented two concepts of equilibrium in multi-player non-zero sum games, which allow to take into account altruistic behavior of players. The intensity of altruism of each player with respect to each other player is defined by a nonnegative number called altruistic coefficient. In particular case, where altruistic coefficients are zero, both concepts are reduced to Nash equilibrium.

The two concepts of altruistic equilibrium are principally different from each other in the sense that no one of them implies the other. The advantage of the first concept is simplicity of interpretation, the advantage of the second (linear) concept is simplicity of mathematical definition via the Nash equilibrium of the game with linearly transformed vector of payoff functions.

The main application of the concepts of altruistic equilibrium is to model player behavior which can lead to cooperation among players. The altruistic coefficients play role of parameters of such a model. By controlling these parameters one can vary the intensity of altruism continuously from zero to big enough values providing that cooperation emerges. As stated by Corollary 1, any Pareto optimal strategy profile can be an altruistic equilibrium if the altruistic coefficients are big enough. The question "*how big?*" is answered by altruistic trade-off characterization (3). The priority direction of further research is to obtain analogous results for the linear variant of altruistic equilibrium concept.

Let us list some other perspective directions of further research:

- to study how altruistic behavior affects solutions in mixed strategies;
- to build a theoretical model of altruistic behavior for a cooperative game, combining the approach of altruistic coefficients (between commitments) with traditional bounds on trade-offs (between players inside each commitment);
- to develop approach how to evaluating effect of altruistic behavior in evolutional games.

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