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Abstract

We consider so-called *generic combinatorial optimization problem*, where the set of feasible solutions is some family of nonempty subsets of a finite ground set with specified positive initial weights of elements, and the objective function represents the total weight of elements of the feasible solution. We assume that the set of feasible solutions is fixed, but the weights of elements may be perturbed or are given with errors. All possible realizations of weights form the set of *scenarios*. A feasible solution, which for a given set of scenarios guarantees the minimum value of the worst-case relative regret among all the feasible solutions, is called a *robust solution*. The maximum percentage perturbation of a single weight, which does not destroy the robustness of a given solution, is called the *robustness tolerance* of this weight with respect to the solution considered. In this paper we present formulae which allow calculating the robustness tolerances with respect to an optimal solution obtained for some initial weights.

Keywords: Combinatorial optimization; Robustness and sensitivity analysis; Robustness tolerances

1 Optimality and robustness

We consider a combinatorial optimization problem in the following generic form:

$$v(c) = \min\{w(c, F) : F \in \mathcal{F}\}, \tag{1}$$

where the set of feasible solutions $\mathcal{F}$ is a family of nonempty subsets of a given ground set $E = \{e_1, \ldots, e_n\}$ and $c = (c(e_1), \ldots, c(e_n))^{\mathsf{T}} \in \mathbb{R}^n$ denotes the vector of weights of the elements of E. For $c \in \mathbb{R}^n$ and $F \in \mathcal{F}$, the objective function in (1) represents the total weight of this solution, i.e.,

$$w(c, F) = \sum_{e \in F} c(e).$$

Numerous discrete optimization problems, like e.g. the traveling salesman problem, the minimum spanning tree problem, the shortest path problem, the linear 0–1 programming problem, can be stated in this general form. In the following we will use as an example of problem (1) an instance of the minimum spanning tree problem.

Example 1

Consider an undirected graph $G = (V, E)$, where $V = \{1, 2, 3, 4, 5\}$ and $E = \{e_1, \ldots, e_7\} = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,4\}, \{3,4\}, \{3,5\}, \{4,5\}\}$. Let $\mathcal{F}$ be a family of subsets of E corresponding to all spanning trees in G, and let $c^o = (14, 11, 14, 15, 13, 18, 17)^{\mathsf{T}}$ be a vector of the initial weights of edges in G. Then the combinatorial optimization problem (1) for $c = c^o$ is the minimum spanning tree problem in the weighted graph G.

The graph G with indicated weights of its edges is shown in Figure 1. In Figure 2 all of the spanning trees in G with corresponding weights for $c = c^o$ are presented. It is easy to check that the subset of edges $T_9 = \{\{1,2\}, \{1,3\}, \{3,4\}, \{4,5\}\}$ is the unique optimal solution for this instance of problem (1) and $v(c^o) = 55$.

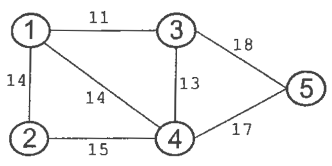

Figure 1: Graph G with indicated weights of edges.

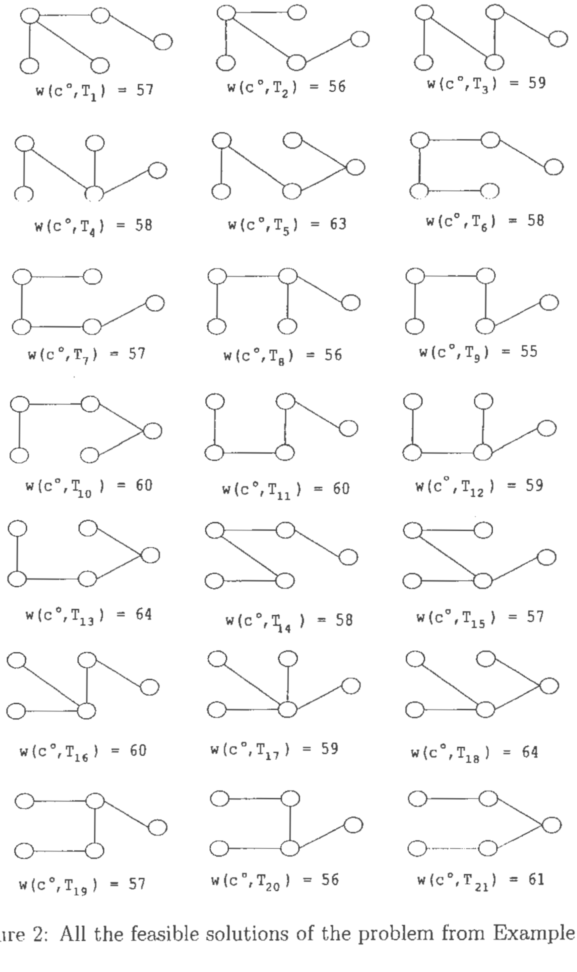

Figure 2: All the feasible solutions of the problem from Example 1.

In the robustness analysis (see e.g. [3, 9, 15]) it is usually assumed that the set of feasible solutions $\mathcal{F}$ in problem (1) is fixed but the vector of weights can change or it is given with errors. Let $C \subseteq \mathbb{R}^n$ denotes a set of all possible realizations of the vector c, called the *scenarios*. We will assume that no additional information concerning possible realizations of weights, like e.g. a probability distribution of the vector c on C, is available.

There are various concepts of robustness of solutions in optimization (see e.g. [3, 4, 5, 9, 17, 18] and the bibliography therein). In this paper we will use as a robustness measure the maximum relative error (worst case relative regret) of the solution considered, over the set of all scenarios. Namely, assume that for any $F \in \mathcal{F}$ and $c \in C$, we have $w(c, F) > 0$. Let $Z(F, C)$ denotes the *worst-case relative regret* of the solution F on the set C, i.e.,

$$Z(F, C) = \max_{c \in C} \max_{Y \in \mathcal{F}} \frac{w(c, F) - w(c, Y)}{w(c, Y)}. \tag{2}$$

A feasible solution $X \in \mathcal{F}$ will be called a *robust* solution for the set of scenarios $C \subseteq \mathbb{R}^n$ if the following inequalities hold:

$$Z(X, C) \leq Z(F, C) \qquad \text{for any} \quad F \in \mathcal{F}. \tag{3}$$

Thus, a feasible solution is robust if it guarantees the minimum value of the worst-case relative regret on the set C among all the feasible solutions.

Usually, in so-called *robust optimization* (see e.g. [1, 2, 3, 8, 9, 16, 26]) the set C of possible scenarios is given and one seeks for a feasible solution which is robust for C. In this paper we are interested in the opposite problem: Namely, we assume that some feasible solution X^o is given an we try to find a set of scenarios, for which this solution is robust. Thus, this approach is similar to standard sensitivity analysis (see e.g., [7, 10, 11, 23, 19, 21, 25]), where one wants to find a subset of the problem data, for which some given solution is optimal.

To state the problem formally, assume that the set $\mathcal{F}$ of feasible solutions in (1) is fixed and that some initial vector of weights $c^o > 0$ is specified. Consider the solution $X^o \in \Omega(c^o)$, where $\Omega(c^o) \subseteq \mathcal{F}$ denotes the set of optimal solutions in problem (1) for $c = c^o$.

In sensitivity analysis one seeks for the maximal under inclusion subset $S(X^o) \subseteq \mathbb{R}^n$ of vectors of the weights, for which the solution X^o remains optimal. Such a set is called the *optimality* (or – *stability*) *region* of the solution X^o. It is well known (see e.g. [7, 11, 12]) that the optimality region is a polyhedral convex cone in $\mathbb{R}^n$. It is also obvious that an optimal solution $X^o \in \Omega(c^o)$ is robust for arbitrary scenario $c \in S(X^o)$.

Given $X \in \mathcal{F}$, let $R(X)$ denotes the maximal subset of scenarios in $\mathbb{R}^n$ for which X is a robust solution. We will call this set the *robustness region* of the feasible solution X.

It is rather difficult to find the robustness region of a given feasible solution of the combinatorial optimization problem (1); some attempts to obtain a subset of $R(X^o)$ for $X^o \in \Omega(c^o)$ are made in [15], where the maximal ball with a center in c^o, belonging to the robustness region of X^o is investigated.

Also in standard sensitivity analysis one usually seeks for some subsets of the optimality region. In particular, instead of changes of all the weights, one may be interested in changes of the weight of some single element only. This leads to the analysis of so-called *tolerances* of weights. The problem of finding the tolerances as well as exploiting them in optimization algorithms received significant attention in combinatorial optimization (see e.g. [6, 7, 10, 23, 19, 21, 22, 24, 25]). In the following we consider analogous problems in the robustness analysis framework.

Let X^o be an optimal solution in problem (1) for $c = c^o$. Assume that only the weight of a single element $e \in E$ can be perturbed, i.e., $c(e_i) = c^o(e_i)$ for $e_i \neq e$. It is known (see e.g. [11]) that then X^o remains optimal if and only if the following inequalities holds:

$$c^o(e) - t^-(e) \leq c(e) \leq c^o(e) + t^+(e), \tag{4}$$

where $t^+(e), t^-(e) \in \mathbb{R} \cup \{\infty\}$ denote, respectively, so-called *upper* and *lower tolerance* of the weight $c(e)$.

Let

$$\mathcal{F}^e = \{F \in \mathcal{F} : \; e \in F\},$$
$$\mathcal{F}_e = \{F \in \mathcal{F} : \; e \notin F\},$$

and denote

$$v^e(c) = \min_{F \in \mathcal{F}^e} w(c, F), \tag{5}$$

$$v_e(c) = \min_{F \in \mathcal{F}_e} w(c, F). \tag{6}$$

According to standard conventions, we take $v^e(c) = \infty$ or $v_e(c) = \infty$ if $\mathcal{F}^e = \emptyset$ or $\mathcal{F}_e = \emptyset$, respectively.

Observe that given an algorithm for solving problem (1) for arbitrary $c \in \mathbb{R}^n$ and $\mathcal{F} \subseteq 2^E$, we may use them also for solving the optimization problems (5), (6).

It is well known (see e.g. [10, 11, 21]), that the following facts hold:

Proposition 1 *If* $e \in X^o$, *then* $t^-(e) = \infty$ *and*

$$t^+(e) = v_e(c^o) - v(c^o). \tag{7}$$

If $e \notin X^o$, *then* $t^+(e) = \infty$ *and*

$$t^-(e) = v^e(c^o) - v(c^o). \tag{8}$$

From Proposition 1 it follows that if the optimization problem (1) is polynomially solvable, then also the tolerances $t^+(e)$, $t^-(e)$, $e \in E$, can be computed in polynomial time. Moreover, the opposite implication also holds under some mild assumptions (see [6, 23, 19]).

In the next section we will introduce an analogue of the tolerances $t^+(e)$, $t^-(e)$ in the robustness analysis context. Our approach is similar to the Wendell's tolerance approach in linear programming (see [24, 25]), which is actually more general, since it allows simultaneous changes of all weights in the objective function or right-hand-side vector of linear program).

2 Robustness tolerances

Consider the following model of perturbations of the weights of elements in problem (1):

Assume that some initial vector of weights $c^o > 0$ is given as well as a subset $Q \subseteq E$ is specified. The set Q represents all of the elements, for which the weights may be perturbed simultaneously and independently, Moreover, assume that the maximum percentage perturbation of any weight does not exceed $\delta \cdot 100\%$ of its initial value for some $\delta \in [0, 1)$. This means that for a given value of the parameter δ we are faced with the set of scenarios $\mathcal{C}(c^o, Q, \delta)$, where

$$\mathcal{C}(c^o, Q, \delta) = \{(c(e_1), \ldots, c(e_n))^{\mathrm{T}} \in \mathbb{R}^n : \ |c(e_i) - c^o(e_i)| \leq \delta \cdot c^o(e_i), \text{ if } e_i \in Q; \\ c(e_i) = c^o(e_i), \quad \text{if } \ e_i \notin Q\}.$$

Consider an optimal solution $X^o \in \Omega(c^o)$ and let $Q = \{e\}$, where $e \in E$. Obviously, X^o is a robust solution for the set of scenarios $\mathcal{C}(c^o, \{e\}, 0)$. The maximum value $t^r(e)$ of the parameter δ, such that X^o remains robust for the set of scenarios $\mathcal{C}(c^o, \{e\}, \delta)$, will be called the *robustness tolerance* of the weight $c(e)$. Formally,

$$t^r(e) = \sup\{\delta \in [0,1) : \ Z(X^o, \mathcal{C}(c^o, \{e\}, \delta)) \leq Z(X, \mathcal{C}(c^o, \{e\}, \delta)), \ X \in \mathcal{F}\}.$$

In order to find the exact values of the robustness tolerances for $e \in E$ we will exploit some properties of the so-called *accuracy function* of a feasible solution of problem (1) introduced in [14].

Let X be an arbitrary feasible solution of problem (1). Given $Q \subseteq E$ and $\delta \in [0,1)$, the value $a(X,Q,\delta)$ of the accuracy function $a(X,Q,\cdot) : [0,1) \rightarrow \mathbb{R}$ is equal to the maximum relative error (relative regret) of the solution X on the set of scenarios $\mathcal{C}(c^o,Q,\delta)$, i.e.,

$$a(X,Q,\delta) = \max_{c \in \mathcal{C}(c^o,Q,\delta)} \max_{Y \in \mathcal{F}} \frac{w(c^o,X) - w(c^o,Y)}{w(c^o,Y)}. \tag{9}$$

Observe that this means, that $a(X,Q,\delta) = Z(X,\mathcal{C})$ for $\mathcal{C} = \mathcal{C}(c^o,Q,\delta)$. The properties of the accuracy function can be therefore directly used in the robustness analysis for the set of scenarios $\mathcal{C}(c^o,Q,\delta)$. In particular, it is shown in [14] that the following fact holds:

Lemma 1 *For $X \in \mathcal{F}$, $Q \subseteq E$ and $\delta \in [0,1)$,*

$$a(X,Q,\delta) = \max_{Y \in \mathcal{F}} \frac{w(c^o,X) - w(c^o,Y) + \delta \cdot w(c^o,(X \otimes Y) \cap Q)}{w(c^o,Y) - \delta \cdot w(c^o, Y \cap Q)}, \tag{10}$$

where $X \otimes Y = (X \setminus Y) \cup (Y \setminus X)$.

Formula (10) can be easily specified for the case $Q = \{e\}$. From Lemma 1 we obtain the following corollary:

Corollary 1 *For $X \in \mathcal{F}$, $e \in E$, and $\delta \in [0,1)$,*

$$Z\left(X, \mathcal{C}(c^o,\{e\},\delta)\right) = a(X,\{e\},\delta) = \max\left\{a', a''\right\}, \tag{11}$$

where

$$\begin{aligned} a' &= \frac{w(c^o,X) - v_e(c^o) + \delta \cdot w(c^o, X \cap \{e\})}{v_e(c^o)}, \\ a'' &= \frac{w(c^o,X) - v^e(c^o) + \delta \cdot [c^o(e) - w(c^o, X \cap \{e\})]}{v^e(c^o) - \delta \cdot c^o(e)}. \end{aligned}$$

Proof From (10) it follows that

$$Z\left(X, \mathcal{C}(c^o,\{e\},\delta)\right) = a(X,\{e\},\delta) = \max\left\{z', z''\right\},$$

where

$$\begin{aligned}
z' &= \max_{Y \in \mathcal{F}_e} \frac{w(c^o, X) - w(c^o, Y) + \delta \cdot w(c^o, (X \otimes Y) \cap \{e\})}{w(c^o, Y) - \delta \cdot w(c^o, Y \cap \{e\})}, \\
z'' &= \max_{Y \in \mathcal{F}^e} \frac{w(c^o, X) - w(c^o, Y) + \delta \cdot w(c^o, (X \otimes Y) \cap \{e\})}{w(c^o, Y) - \delta \cdot w(c^o, Y \cap \{e\})}.
\end{aligned}$$

We will show, that $z' = a'$ and $z'' = a''$. Indeed, for $Y \in \mathcal{F}_e$ we have $Y \cap \{e\} = \emptyset$ and $(X \otimes Y) \cap \{e\} = X \cap \{e\}$, which implies

$$\begin{aligned}
z' &= \max_{Y \in \mathcal{F}_e} \frac{w(c^o, X) - w(c^o, Y) + \delta \cdot w(c^o, (X \otimes Y) \cap \{e\})}{w(c^o, Y) - \delta \cdot w(c^o, Y \cap \{e\})} \\
&= \max_{Y \in \mathcal{F}_e} \frac{w(c^o, X) - w(c^o, Y) + \delta \cdot w(c^o, X \cap \{e\})}{w(c^o, Y)} \\
&= \frac{w(c^o, X) - \min_{Y \in \mathcal{F}_e} w(c^o, Y) + \delta \cdot w(c^o, X \cap \{e\})}{\min_{Y \in \mathcal{F}_e} w(c^o, Y)} \\
&= \frac{w(c^o, X) - v_e(c^o) + \delta \cdot w(c^o, X \cap \{e\})}{v_e(c^o)} = a'.
\end{aligned}$$

Similarly, for $Y \in \mathcal{F}^e$ we have, $Y \cap \{e\} = \{e\}$ and $w(c^o, (X \otimes Y) \cap \{e\}) = w(c^o, X \cap \{e\}) + w(c^o, Y \cap \{e\}) - 2w(c^o, X \cap Y \cap \{e\}) = c^o(e) - w(c^o, X \cap \{e\})$, which implies

$$\begin{aligned}
z'' &= \max_{Y \in \mathcal{F}^e} \frac{w(c^o, X) - w(c^o, Y) + \delta \cdot w(c^o, (X \otimes Y) \cap \{e\})}{w(c^o, Y) - \delta \cdot w(c^o, Y \cap \{e\})} \\
&= \max_{Y \in \mathcal{F}^e} \frac{w(c^o, X) - w(c^o, Y) + \delta \cdot [c^o(e) - w(c^o, X \cap \{e\})}{w(c^o, Y) - \delta \cdot c^o(e)} \\
&= \frac{w(c^o, X) - \min_{Y \in \mathcal{F}^e} w(c^o, Y) + \delta \cdot [c^o(e) - w(c^o, X \cap \{e\})}{\min_{Y \in \mathcal{F}^e} w(c^o, Y) - \delta \cdot c^o(e)} \\
&= \frac{w(c^o, X) - v^e(c^o) + \delta \cdot [c^o(e) - w(c^o, X \cap \{e\})]}{v^e(c^o) - \delta \cdot c^o(e)} = a''.
\end{aligned}$$

■

To simplify the notation let for $X \in \mathcal{F}$, $e \in X$, and $\delta \in [0, 1)$,

$$Z_e(X, \delta) = Z\left(X, C(c^o, \{e\}, \delta)\right).$$

It will be also convenient to state formulae for calculating $Z_e(X, \delta)$ separately in both cases: $e \in X$ and $e \notin X$. From Corollary 1 we have the following facts:

If $X \in \mathcal{F}^e$ and $\delta \in [0,1)$, then

$$Z_e(X,\delta) = \max\left\{\frac{w(c^o,X) - v_e(c^o) + \delta \cdot c^o(e)}{v_e(c^o)},\ \frac{w(c^o,X) - v^e(c^o)}{v^e(c^o) - \delta \cdot c^o(e)}\right\}. \quad (12)$$

If $X \in \mathcal{F}_e$ and $\delta \in [0,1)$, then

$$Z_e(X,\delta) = \max\left\{\frac{w(c^o,X) - v_e(c^o)}{v_e(c^o)},\ \frac{w(c^o,X) - v^e(c^o) + \delta \cdot c^o(e)}{v^e(c^o) - \delta \cdot c^o(e)}\right\}. \quad (13)$$

In the following example we will apply the above formulae to illustrate the behavior of the worst-case regret functions for two feasible solutions of the minimum spanning tree problem considered in Example 1.

Example 2

Consider again the minimum spanning tree problem from Example 1. Assume that only a weight of the edge $e = \{3,4\}$ can be perturbed and that all the remaining weights are given by the initial vector c^o as in Example 1. We are therefore faced with the set of scenarios $\mathcal{C}(c^o, \{\{3,4\}\}, \delta)$. Moreover, we have $v(c^o) = v^e(c^o) = 55$, $v_e(c^o) = 56$. Consider the following two feasible solutions: T_1 and T_9, where

$T_1 = \{\{1,2\},\{1,3\},\{1,4\},\{3,5\}\}$,

$T_9 = \{\{1,2\},\{1,3\},\{3,4\},\{4,5\}\}$.

The solution T_9 is an optimal solution in problem (1) for $c = c^o$ and $e \in T_9$, therefore the formula (13) is appropriate to calculate $Z_e(T_9,\delta)$. To find $Z_e(T_1,\delta)$ one can use (12), because $e \notin T_1$.

In Figure 3 the worst-case regret functions for the feasible solutions T_1 and T_2 are shown. □

The following theorem gives simple formulae for calculating the robustness tolerances $t^r(e)$, $e \in E$, for an initially optimal solution $X^o \in \Omega(c^o)$.

Theorem 1 *For $X^o \in \Omega(c^o)$,*

$$t^r(e) = \begin{cases} 1 & \text{if } e \in X^o, \\ \min\left\{1, [v^e(c^o)^2 - v(c^o)^2]^{\frac{1}{2}} \cdot c^o(e)^{-1}\right\} & \text{if } e \notin X^o. \end{cases} \quad (14)$$

Proof From the definition of the robustness tolerances we have for $e \in E$, $X^o \in \Omega(c^o)$,

$$t^r(e) = \sup\{\delta \in [0,1):\ Z_e(X^o,\delta) \le Z_e(X,\delta),\ X \in \mathcal{F}\}.$$

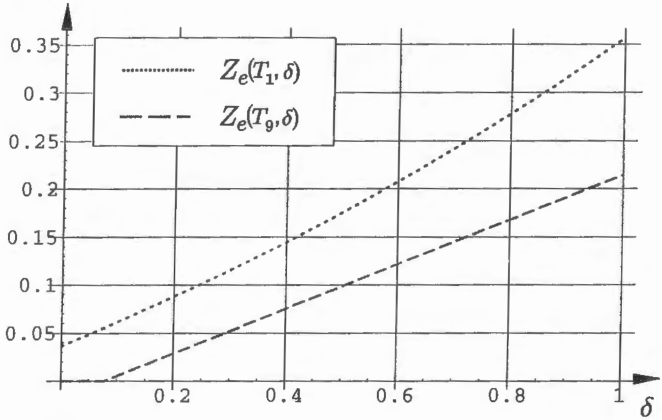

Figure 3: Worst-case regret functions of the solutions T_1 and T_9.

(i) Consider first the case when $e \in X^o$, which implies $v(c^o) = v^e(c^o)$. It is easy to see that then $Z_e(X^o, \delta) \leq Z_e(X, \delta)$ for arbitrary $\delta \in [0, 1)$, $X \in \mathcal{F}^e$. Indeed, from (12) we have for $X = X^o$,

$$Z_e(X^o, \delta) = \max\left\{\frac{v(c^o) - v_e(c^o) + \delta \cdot c^o(e)}{v_e(c^o)},\ 0\right\},$$

and for arbitrary $X \in \mathcal{F}^e$,

$$\begin{aligned} Z_e(X, \delta) &= \max\left\{\frac{w(c^o, X) - v_e(c^o) + \delta \cdot c^o(e)}{v_e(c^o)},\ \frac{w(c^o, X) - v^e(c^o)}{v^e(c^o) - \delta \cdot c^o(e)}\right\} \\ &\geq \max\left\{\frac{v(c^o) - v_e(c^o) + \delta \cdot c^o(e)}{v_e(c^o)},\ 0\right\}. \end{aligned}$$

Thus, given $X^o \in \mathcal{F}^e \cap \Omega(c^o)$ for any $e \in X^o$,

$$t^r(e) = \sup\left\{\delta \in [0, 1) :\ Z_e(X^o, \delta) \leq Z_e(X, \delta),\ X \in \mathcal{F}_e\right\}.$$

But according to (13) for $X \in \mathcal{F}_e$ and $\delta \in [0,1)$,

$$\begin{aligned}
Z_e(X,\delta) &= \max\left\{\frac{w(c^o,X)-v_e(c^o)}{v_e(c^o)},\ \frac{w(c^o,X)-v^e(c^o)+\delta\cdot c^o(e)}{v^e(c^o)-\delta\cdot c^o(e)}\right\} \\
&\geq \max\left\{0,\ \frac{w(c^o,X)-v^e(c^o)+\delta\cdot c^o(e)}{v^e(c^o)-\delta\cdot c^o(e)}\right\} \\
&= \frac{w(c^o,X)-v^e(c^o)+\delta\cdot c^o(e)}{v^e(c^o)-\delta\cdot c^o(e)} \\
&\geq \frac{v(c^o)-v^e(c^o)+\delta\cdot c^o(e)}{v^e(c^o)-\delta\cdot c^o(e)} = Z_e(X^o,\delta).
\end{aligned}$$

This means that X^o remains robust for arbitrary $\delta \in [0,1)$, which implies that $t^r(e) = 1$ when $e \in X^o$.

(ii) Consider now the case when $e \notin X^o$. From (13) for arbitrary $\delta \in [0,1)$,

$$Z_e(X^o,\delta) = \max\left\{0,\ \frac{v(c^o)-v^e(c^o)+\delta\cdot c^o(e)}{v^e(c^o)-\delta\cdot c^o(e)}\right\},$$

and for $X \in \mathcal{F}_e$,

$$\begin{aligned}
Z_e(X,\delta) &= \max\left\{\frac{w(c^o,X)-v_e(c^o)}{v_e(c^o)},\ \frac{w(c^o,X)-v^e(c^o)+\delta\cdot c^o(e)}{v^e(c^o)-\delta\cdot c^o(e)}\right\} \\
&\geq \max\left\{0,\ \frac{v(c^o)-v^e(c^o)+\delta\cdot c^o(e)}{v^e(c^o)-\delta\cdot c^o(e)}\right\}.
\end{aligned}$$

This implies $Z_e(X^o,\delta) \leq Z_e(X,\delta)$ for any $X \in \mathcal{F}_e$, and consequently,

$$t^r(e) = \sup\left\{\delta \in [0,1):\ Z_e(X^o,\delta) \leq Z_e(X,\delta),\ X \in \mathcal{F}^e\right\}.$$

Substituting $v(c^o) = v_e(c^o)$ in (13) we obtain for $X \in \mathcal{F}^e$, $\delta \in [0,1)$,

$$Z_e(X,\delta) = \max\left\{\frac{w(c^o,X)-v(c^o)+\delta\cdot c^o(e)}{v(c^o)},\ \frac{w(c^o,X)-v^e(c^o)}{v^e(c^o)-\delta\cdot c^o(e)}\right\}. \quad (15)$$

From (15) it follows that for arbitrary $X^e \in \arg\min_{F\in\mathcal{F}^e} w(c^o,F)$ and for any $\delta \in [0,1)$,

$$Z_e(X,\delta) \geq Z_e(X^e,\delta) = \max\left\{\frac{v^e(c^o)-v(c^o)+\delta\cdot c^o(e)}{v(c^o)},\ 0\right\}.$$

This implies, that when $e \notin X^o$, we have

$$Z_e(X^o,\delta) \leq Z_e(X,\delta), \quad \text{for any } X \in \mathcal{F}^e,$$

if and only if

$$\max\left\{0,\ \frac{v(c^o) - v^e(c^o) + \delta \cdot c^o(e)}{v^e(c^o) - \delta \cdot c^o(e)}\right\} \leq \max\left\{\frac{v^e(c^o) - v(c^o) + \delta \cdot c^o(e)}{v(c^o)},\ 0\right\}.$$

But this inequality holds for

$$\delta \leq \min\left\{1,\ \frac{[v^e(c^o)^2 - v(c^o)^2]^{\frac{1}{2}}}{c^o(e)}\right\},$$

which finally proves (14). ∎

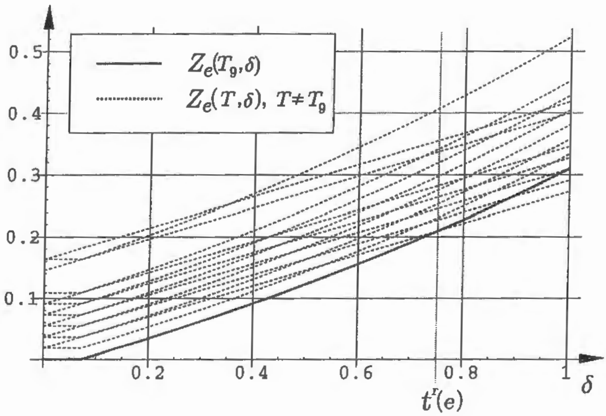

Figure 4: Worst-case regret functions of feasible solutions in problem (1).

Example 3

Consider again the minimum spanning tree problem from Example 1 and its optimal solution $T_9 = \{\{1,2\},\{1,3\},\{3,4\},\{4,5\}\}$. From Theorem 1 it follows that the robustness tolerances of all the edges belonging to T_9 are equal to 1. Consider therefore some edge from the set $E \setminus T_9$, e.g. the edge $e = \{1,4\}$, and the corresponding set of scenarios $C = C(c^o, \{\{1,4\}\}, \delta)$. We have $c^o(e) = 14$, $v(c^o) = w(c^o, T_9) = 55$, $v^e(c^o) = 56$. Calculating $t^r(e)$ from (14) we obtain:

$$t^r(e) = \frac{(56^2 - 55^2)^{\frac{1}{2}}}{14} \approx \frac{10.54}{14} \approx 0.75.$$

Thus, the spanning tree T_9 guarantees the minimum value of the worst-case relative regret if the weight of the edge $e = \{1, 4\}$ is perturbed by no more than approximately 75%, and all the remaining weights are unchanged.

In Figure 4 the worst case regret functions of all the feasible solutions in problem (1), are shown; bold line indicates the worst-case regret function of the spanning tree T_9. Observe that the solution T_9 guarantees, indeed, the minimum value of the worst-case regret among all the feasible solutions provided $\delta \leq t^r(e) \approx 0.75$. For larger values of δ a feasible solution T_2 appears to be better from the robustness point of view. The worst-case regret functions for both the solutions T_2 and T_9 are shown in Figure 5.

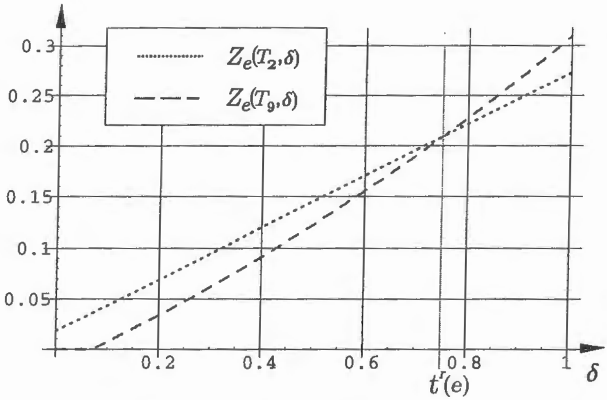

Figure 5: Worst-case regret functions of the solutions T_2 and T_9.

□

3 Conclusions

In this paper we consider an influence of perturbations of single weights on the robustness of an initially optimal solution for the generic combinatorial optimization problem. Maximum percentage perturbation of the weight, which do not destroy the robustness of the solution considered, is called the robustness tolerance of this weight. It is shown, that the robustness tolerances of the weights for all elements belonging to the optimal solution are equal to 1, and the tolerances of weights for all remaining elements can be computed easily if the optimal value of an auxiliary optimization problem is known. This auxiliary problem consists in forcing an additional requirement, that the element considered does not belong to any feasible solution. Observe that this leads to polynomial solvability of the robustness tolerance problem provided that the original optimization problem is polynomially solvable.

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