ANALYTICAL SOLUTION METHOD FOR RHEOLOGICAL PROBLEMS OF SOLIDS

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A large variety of solid materials – like plastics, rocks, asphalt, biomaterials etc. – possess viscoelastic/rheological characteristics. Correspondingly, one can observe some kind of delayed and damped elastic behaviour. In a linear regime, rheology of solids can be expressed as a generalization of Hooke's law where elasticity coefficients are replaced by polynomials of the time derivative operator. Then time dependent boundary conditions, like those when gradually creating a hole, tunnel etc., induce time dependent processes, which also influence the elasticity originated space dependence.

Volterra's principle [1,2] is a long-known method to treat such problems. Here, we introduce and present another approach that is simpler to apply (no operator inverse is required to compute but only linear ordinary differential equations to solve). Our method starts with the elastic solution, replaces the elasticity coefficients with time dependent functions, derives differential equations on them, and determines the solution corresponding to the initial conditions.

More concretely, Hooke's law, written in the deviatoric–spherical decomposition

$$\boldsymbol{\sigma}^{\mathrm{dev}} = E^{\mathrm{dev}} \boldsymbol{\varepsilon}^{\mathrm{dev}}, \qquad \boldsymbol{\sigma}^{\mathrm{sph}} = E^{\mathrm{sph}} \boldsymbol{\varepsilon}^{\mathrm{sph}}, \qquad E^{\mathrm{dev}} = 2G, \quad E^{\mathrm{sph}} = 3K \tag{1}$$

with the spherical and deviatoric parts defined according to

$$\varepsilon^{\mathrm{sph}} = \frac{1}{3}(\operatorname{tr} \boldsymbol{\varepsilon}), \qquad \boldsymbol{\varepsilon}^{\mathrm{sph}} = \varepsilon^{\mathrm{sph}} \mathbf{1}, \qquad \boldsymbol{\varepsilon}^{\mathrm{dev}} = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{sph}}, \tag{2}$$

is replaced by

$$\mathcal{S}^{\mathrm{dev}} \boldsymbol{\sigma}^{\mathrm{dev}} = \mathcal{E}^{\mathrm{dev}} \boldsymbol{\varepsilon}^{\mathrm{dev}}, \qquad \mathcal{S}^{\mathrm{sph}} \boldsymbol{\sigma}^{\mathrm{sph}} = \mathcal{E}^{\mathrm{sph}} \boldsymbol{\varepsilon}^{\mathrm{sph}} \tag{3}$$

with the operator polynomials

$$\mathcal{S}^{\mathrm{dev}} = 1 + \tau_1^{\mathrm{dev}} \frac{\partial}{\partial t} + \tau_2^{\mathrm{dev}} \frac{\partial^2}{\partial t^2} + \ldots, \qquad \mathcal{E}^{\mathrm{dev}} = E_0^{\mathrm{dev}} + E_1^{\mathrm{dev}} \frac{\partial}{\partial t} + E_2^{\mathrm{dev}} \frac{\partial^2}{\partial t^2} + \ldots, \tag{4}$$

$$\mathcal{S}^{\mathrm{sph}} = 1 + \tau_1^{\mathrm{sph}} \frac{\partial}{\partial t} + \tau_2^{\mathrm{sph}} \frac{\partial^2}{\partial t^2} + \ldots, \qquad \mathcal{E}^{\mathrm{sph}} = E_0^{\mathrm{sph}} + E_1^{\mathrm{sph}} \frac{\partial}{\partial t} + E_2^{\mathrm{sph}} \frac{\partial^2}{\partial t^2} + \ldots. \tag{5}$$

In our applications, we concentrate on the Kluitenberg–Verhás model family [3]

$$\boldsymbol{\sigma}^{\mathrm{dev}} + \tau^{\mathrm{dev}} \dot{\boldsymbol{\sigma}}^{\mathrm{dev}} = E_0^{\mathrm{dev}} \boldsymbol{\varepsilon}^{\mathrm{dev}} + E_1^{\mathrm{dev}} \dot{\boldsymbol{\varepsilon}}^{\mathrm{dev}} + E_2^{\mathrm{dev}} \ddot{\boldsymbol{\varepsilon}}^{\mathrm{dev}}, \tag{6}$$

$$\boldsymbol{\sigma}^{\mathrm{sph}} + \tau^{\mathrm{sph}} \dot{\boldsymbol{\sigma}}^{\mathrm{sph}} = E_0^{\mathrm{sph}} \boldsymbol{\varepsilon}^{\mathrm{sph}} + E_1^{\mathrm{sph}} \dot{\boldsymbol{\varepsilon}}^{\mathrm{sph}} + E_2^{\mathrm{sph}} \ddot{\boldsymbol{\varepsilon}}^{\mathrm{sph}}, \tag{7}$$

which is important from both theoretical [3] and experimental [4] aspects (overdot denoting time derivative).

We work in the small-strain region with the geometric compatibility equation

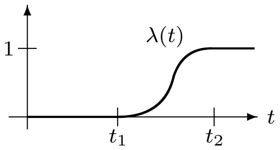
$$\overset{\rightarrow}{\nabla} \times \boldsymbol{\varepsilon} \times \overset{\leftarrow}{\nabla} = \mathbf{0} \tag{8}$$

for the strain tensor ε, and take the mechanical equation of motion in the force equilibrial approximation (i.e., acceleration neglected)

$$\boldsymbol{\sigma} \cdot \overset{\leftarrow}{\nabla} = -\varrho \mathbf{g}, \tag{9}$$

where $\boldsymbol{\sigma}$ denotes the Cauchy stress tensor and $\varrho\mathbf{g}$ is the volumetric force density.

In the examples treated by us, a hole/tunnel in a pre-stressed medium is established progressively via switching on a stress-free boundary condition gradually, described by a smooth multiplying function



$$\lambda(t) = \begin{cases} 0 & \text{if} \quad t \le t_1, \\ 1 & \text{if} \quad t \ge t_2, \\ \text{smooth in between.} & \end{cases} \tag{10}$$

The solution of the corresponding elastic problem (typically already known from sources like [5–7]) is assumed in finite sum form, then we replace the elasticity coefficients with time dependent functions, on which a set of ordinary differential equations is derived from (3), and for suitable switch-on functions $\lambda(t)$ the solution can be obtained analytically.

We present several examples solved via this new method, like tunnels and spherical hollows opened in various initial stress states, and pressurizing of thick-walled tubes and spherical tanks. These examples are useful for applications and, in parallel, are suitable for testing and validating numerical methods of various kinds.

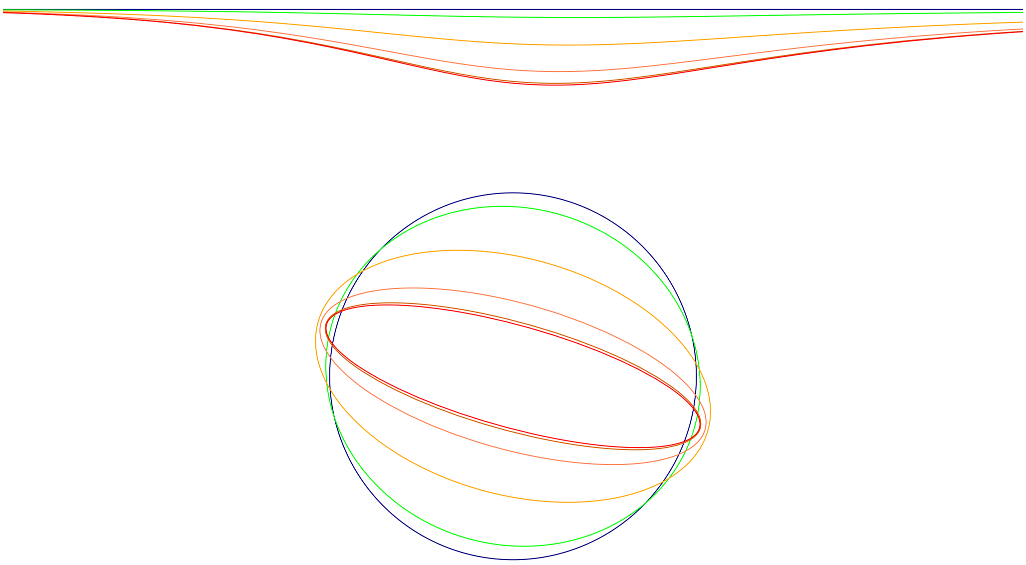


Figure 1: Time evolution of the displacement field when a cylindrical hole is opened in an anisotropically pre-stressed medium: change of the shape of the hole and of a nearby plane surface.

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