891.

ON THE BINODAL QUARTIC AND THE GRAPHICAL REPRESENTATION OF THE ELLIPTIC FUNCTIONS.

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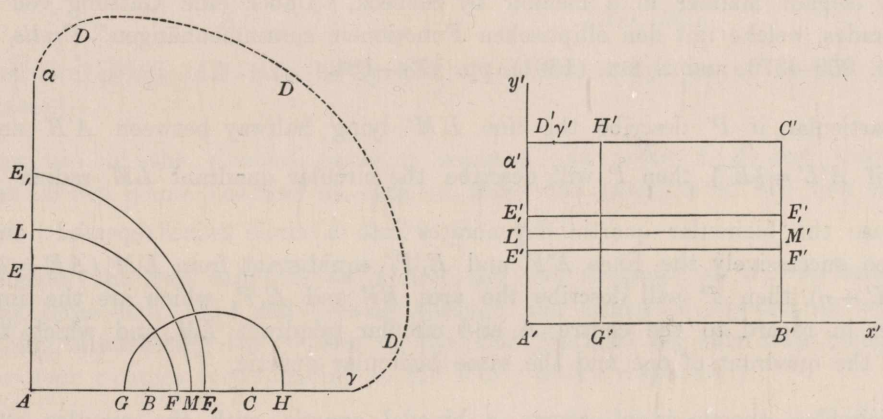
I APPROACH the subject from the question of the graphical representation of the elliptic functions: assuming as usual that the modulus is real, positive, and less than unity, and to fix the ideas considering the function sn (but the like considerations are applicable to the functions cn and dn), then the equation $x+iy=\text{sn}(x'+iy')$ establishes a (1, 1) correspondence between the xy infinite quarter plane, and the $x'y'$ rectangle (sides K and K'): viz. to any given point $x+iy$, x and y each positive, there corresponds a single point $x'+iy'$, x', y' each positive and less than K, K' respectively: and conversely to any such point $x'+iy'$, there corresponds a single point $x+iy$, x and y each positive.

I draw in the $x'y'$-figure the rectangle $A'B'C'D'$ (sides K and K'), and in the xy-figure, I take on the axis of x, the points B, C where $AB=1$, $AC=\frac{1}{k}$: and the



point D at infinity. We have thus in the $x'y'$-figure the closed curve or contour

$A'B'C'D'A'$: and corresponding hereto we have in the xy-figure the closed curve or contour $ABCDA$, viz. here D is the point at infinity considered as a line always at infinity, extending from the point at infinity on the positive part of the axis of x, to the point at infinity on the positive part of the axis of y, the contour being thus AB, BC, CD (D at infinity on the axis of x); and then D (at infinity on the axis of y) A. And thus to a point P' describing successively the lines $A'B'$, $B'C'$, $C'D'$, $D'A'$ there corresponds a point P describing successively the lines AB, BC, CD, DA: to P' at D' there corresponds P at D, viz. this is any point at infinity from D on the axis of x to D on the axis of y. There is no real breach of continuity: in further illustration, suppose that P', instead of actually coming to D', just cuts off the corner, viz. that it passes from a point γ' on $C'D'$ to a point α' on $D'A'$ (γ', α' each of them very near to D'): then P passes from a point γ very near D on the axis of x (that is, at a great distance from A) to a point α very near D on the axis of y (that is, at a great distance from A): and to the indefinitely small arc $\gamma'\alpha'$ described by P' there corresponds the indefinitely large arc $\gamma\alpha$ described by P.

We thus see that, if P' describe any arc $E'F'$ passing from a point E' of $A'D'$ to a point F' of $B'C'$, then P will describe an arc EF passing from a point E of AD to a point F of BC: and similarly, if P' describe any arc $G'H'$ passing from a point G' of $A'B'$ to a point H' of $C'D'$, then P will describe an arc GH passing from a point G of AB to a point H of CD.

Supposing $E'F'$ is a straight line parallel to $A'x'$, that is, cutting $A'D'$ and $B'C'$ each at right angles, then EF will be an arc cutting AD and BC each at right angles: and so if $G'H'$ is a straight line parallel to $A'y'$, that is, cutting $A'B'$ and $C'D'$ each at right angles, then GH will be an arc cutting AB and CD each at right angles: and moreover, since $E'F'$ and $G'H'$ cut each other at right angles, then also EF and GH cut each other at right angles.

Supposing, as above, that $E'F'$ and $G'H'$ are straight lines, we have EF and GH each of them the arc of a special bicircular quartic: the theory was in fact established in a very elegant manner in a memoir by Siebeck, "Ueber eine Gattung von Curven vierten Grades, welche mit den elliptischen Functionen zusammenhängen," *Crelle*, t. LVII. (1860), pp. 359—370, and t. LIX. (1861), pp. 173—184.

In particular, if P' describe the line $L'M'$ lying halfway between $A'B'$ and $D'C'$ (that is, if $A'L' = \frac{1}{2}K'$), then P will describe the circular quadrant LM, radius $\frac{1}{\sqrt{k}}$, viz. in this case the bicircular quartic degenerates into a circle twice repeated: and so if P' describe successively the lines $E'F'$ and $E_1'F_1'$ equidistant from $L'M'$ ($AE' = \frac{1}{2}K' - \eta$, $AE_1' = \frac{1}{2}K' + \eta$), then P will describe the arcs EF and E_1F_1 which are the images of each other in regard to the centre A and circular quadrant LM, and which together constitute the quadrant of one and the same bicircular quartic.

A bicircular quartic is of course a binodal quartic with the circular points at infinity for the two nodes: there is no real gain of generality in considering the binodal quartic rather than the bicircular quartic, but I have preferred to do so, and I have accordingly introduced the term Binodal Quartic into the title of the present Memoir. I present in a compendious form the properties of the general curve, and I show how the curve is to be particularised so as to obtain from it the special bicircular quartics which present themselves as above in the graphical representation of the elliptic functions.

A binodal quartic has the Plückerian numbers

$$\begin{array}{cccccc} m & n & \delta & \kappa & \tau & \iota \\ =4 & 8 & 2 & 0 & 8 & 12. \end{array}$$

The number of tangents to the curve which can be drawn from either of the nodes is $n-4$, $=4$; and the pencil of tangents from the one node is homographic with the pencil of tangents from the other node. Call the nodes I and J: and let the tangents from I be called (a, b, c, d) and those from J be called (a', b', c', d'), then if the tangents which correspond to (a', b', c', d') respectively are (a, b, c, d), they may also be taken to be (b, a, d, c), (c, d, a, b) or (d, c, b, a): and considering the intersections of corresponding tangents, we have thus four tetrads of points, say the f-points, such that the points of each tetrad lie in a conic through the two nodes: and we have consequently four conics each passing through the two nodes, say these are the f-conics.

Starting as above with the correspondence (a, b, c, d), (a', b', c', d'), if the intersections of a and a', b and b', c and c', d and d' are called A, B, C, D respectively, then we have A, B, C, D for a tetrad of f-points, lying on the f-conic $(ABCD)$.

Writing AB for the two points, the intersections of IA, JB and of IB, JA respectively, and so in other cases, then the remaining three tetrads of f-points are

AB, CD lying on the f-conic (AB, CD),
AC, BD „ „ (AC, BD),
AD, BC „ „ (AD, BC).

The two points AB may be spoken of as the antipoints of A, B: and so in other cases.

Any two of the f-conics have in common the nodes I, J, and they therefore intersect in two points besides: at each of these the tangents to the two conics, and the lines to I, J respectively, form a harmonic pencil.

Consider the two tangents at I and the two tangents at J: we have a conic touching these four lines and passing through the tetrad of f-points, or what is the same thing, intersecting the f-conic in the four f-points: say this is a c-conic. There are thus four c-conics corresponding to the four f-conics respectively.

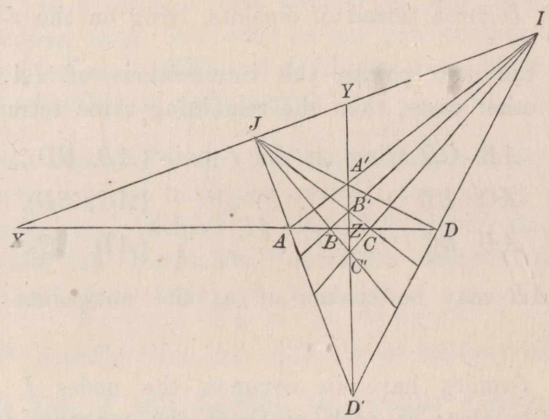
We may consider a variable conic, passing through the points I and J; and such that the tangents thereto at these points respectively meet on a point of a

c-conic: the variable conic and the corresponding f-conic each pass through the points I, J and they meet in two points besides: and the variable conic may be such that at each of these the tangents to the variable conic and the f-conic form with the lines drawn to the points I and J a harmonic pencil. The variable conic, as thus defined, contains a single variable parameter; and it has for its envelope the binodal quartic: the binodal quartic is thus in four different ways the envelope of a variable conic. This is of course Casey's Theorem for the fourfold generation of the bicircular quartic as the envelope of a variable circle.

One of the f-conics, say $(ABCD)$, may break up into the line IJ and a line, say the axis $(ABCD)$: we have then the four f-points A, B, C, D on this line. The other f-conics say (AB, CD), (AC, BD), (AD, BC) are proper conics as before: any one of these meets the line $(ABCD)$ in two points: and at each of these the line and the tangent to the conic form with the lines drawn to the points I and J a harmonic pencil. The binodal quartic is in this case said to have an axis.

But a second f-conic, say (AD, BC), may break up into the line IJ and a line, say the axis (AD, BC): we have then the four f-points AD, BC on this line. Writing moreover A', D' for the two points AD, or say $(AI, DJ) = A'$, $(AJ, DI) = D'$; and similarly B', C' for the two points BC, or say $(BI, CJ) = B'$, $(BJ, CI) = C'$; then the four f-points are A', B', C', D', and the axis (AD, BC) may be called $(A'B'C'D')$.

The relation of the f-points A, B, C, D and A', B', C', D' and of the two axes is as shown in the figure: taking X for the intersection of IJ with $(ABCD)$ and Y for



that of IJ with $(A'B'C'D')$, also Z for the intersection of the two axes, then X, Z are the sibiconjugate points of the involution AD, BC and Y, Z the sibiconjugate points of the involution $A'D'$, $B'C'$. The two axes intersect in Z, and form with the lines ZI, ZJ a pencil in involution.

The two remaining f-conics (AB, CD) and (AC, BD), or as they might also be called $(A'B', C'D')$ and $(A'C', B'D')$, are proper conics as before: they touch each other at the points I, J, and have for their common tangents at these points the lines ZI,

ZJ respectively. Each conic meets each axis in two points; and at each of these points the axis and the tangent to the conic form with the lines to I, J a harmonic pencil. The binodal quartic is in this case said to be biaxial. The point Z, which is the intersection of the two axes, may be called the centre.

In the case where an f-conic breaks up into the line IJ, and a line containing four f-points, say an f-line, the corresponding c-conic coincides with the f-conic, viz. it also breaks up into the line IJ and the f-line: the variable conic is a conic through the points I, J such that the tangents thereto at these points respectively meet on the f-line. Moreover, the variable conic must be such that at each of its intersections with the f-conic, that is, the f-line, the tangent to the variable conic and the f-line must be harmonics in regard to the lines drawn from the point to the points I, J respectively: but this condition is satisfied *ipso facto* for each of the intersections of the variable conic and the f-line. This depends on the theorem that, taking on a conic any three points P, I, J, then the tangent at P and the line drawn from P to the pole of IJ are harmonics in regard to the lines PI, PJ. Thus we have only three conditions for the variable conic, or, as above defined, it would in the case in question (of four f-points in a line) depend upon two variable parameters. There is really another condition—but what this in general is I have not ascertained: and this being so the variable conic in the case in question (of the four f-points in a line) depends upon a single variable parameter, and we have as before the bicircular quartic as the envelope. The foregoing is the axial case; in the biaxial case, the same thing happens in regard to the variable conics belonging to the two axes respectively. Thus in every case we have the fourfold generation of the curve as the envelope of a variable conic: only in the axial case, the variable conics belonging to the axis, and in the biaxial case the variable conics belonging to the two axes respectively, are not by the foregoing definitions completely defined. It will be seen further on how, in the case of the biaxial bicircular quartic, we complete the definition of the variable circles belonging to the two axes respectively.

Taking the points I, J to be the circular points at infinity, we have a bicircular quartic. The f-points are the foci, and the f-conics are circles, viz. we have 16 foci situate in fours upon four focal circles. The harmonic relation of two lines to the lines through I, J means of course that the lines cut at right angles; hence the focal circles cut each other at right angles: this must certainly be a known property, but it is not mentioned in Salmon's *Higher Plane Curves*, Ed. 3, Dublin, 1879, and I cannot find it in Darboux or Casey: it is given No. 81 in Lachlan's Memoir "On Systems of Circles and Spheres," *Phil. Trans.*, vol. CLXXVII. (1886), and I find it as a question in the *Educational Times*, March 1, 1889, 10034 (Prof. Morley). "Prove that, of the four focal circles of a circular cubic or bicircular quartic, any two are orthogonal, and the radii are connected by the relation $\Sigma(\mu^{-2}) = 0$." The theorem is not as well known as it should be.

The c-conics are confocal conics having for their real foci the so-called double-foci of the quartic (more accurately, the common foci are the four quadruple foci of the quartic); we have thus four conics corresponding to the four focal circles respectively, each conic intersecting the corresponding circle in the four foci upon this circle. And we have then the quartic as the envelope of a variable circle having its centre upon one of these conics and cutting at right angles the corresponding focal circle: the bicircular quartic is thus generated in four different ways.

Instead of one of the focal circles, we may have a line or axis, and the quartic is then said to be axial: the foci on the axis may be any four points; and for a real curve they may be all real, or two real and two imaginary, or all four imaginary. The remaining focal circles are real or imaginary circles, cut by the axis at right angles, that is, having their centres on the axis, and cutting each other at right angles.

But instead of another of the focal circles, we may have a line or axis, and the quartic is then said to be biaxial: the two axes cut at right angles at a point which may be called the centre of the curve. The foci cn each axis form pairs of points situate symmetrically in regard to the centre. If on one of the axes the foci are real, then on the other axis they form two imaginary conjugate pairs; and conversely: but if on one of the axes the foci are two of them real and the other two conjugate imaginaries, then this is so for the other axis also. There are thus only the two cases: 1°, foci on the one axis real, and on the other conjugate imaginaries; 2°, foci on each axis two of them real and the other two conjugate imaginaries: there is however a limiting case where on each axis two foci are united at the centre, the other two foci being real on the one axis and conjugate imaginaries on the other. The remaining two focal circles are real or imaginary circles, cutting each axis at right angles, that is, having their centres at the centre; and cutting each other at right angles, that is, having the sum of the squares of their radii $=0$.

The biaxial form of bicircular quartic is, in fact, that which presents itself in the theory of the representation of the elliptic functions.

I consider for a moment the case of a variable circle having its centre upon a given line, and cutting at right angles a given circle. The variable circles pass all of them through two fixed points, the antipoints of the intersections of the given line and circle, and which are thus real or imaginary according as the intersections of the given line and circle are imaginary or real. Hence, considering any one variable circle and the consecutive variable circle, these intersect in two real points, when the given line does not meet the given circle (meets it in two imaginary points); but when the given line meets the given circle in two real points, then the two variable circles intersect in two imaginary points: if however the given line touches the given circle, then the two variable circles touch each other. Taking now the curve of centres to be any given curve whatever, and considering one of the variable circles, and the consecutive variable circle, it at once appears that, if the tangent to the curve of centres at the centre of the variable circle does not meet the given circle, then the two variable circles intersect in two real points (which, if the tangent touch the given circle, unite in a single real point): but if the tangent to the curve of centres meets the given circle, then the two variable circles do not intersect. It hence appears that the real portions of the envelope arise exclusively from those portions of the curve of centres which are such that at any point thereof the tangent to the curve of centres does not meet the given circle. In particular, if the given circle be real, and the curve of centres is a real ellipse enclosing the given circle, then the real portion of the envelope arises from the whole ellipse: but if the curve of centres be a real ellipse cutting the given circle in four real points, then drawing the four common tangents of the ellipse and circle, it is at once seen that there are on the ellipse two detached portions such that, at any point of either portion, the tangent to the ellipse does not meet the circle: and the real portions of the envelope arise exclusively from these portions of the ellipse.

In the case just referred to, there are on the ellipse four portions each lying outside the circle and terminating in the four intersections respectively of the ellipse and circle, such that at a point of any one of these portions the tangent to the ellipse meets the circle in two real points. Starting from the extremity of one of these portions of the ellipse and proceeding to the other extremity on the circle, the corresponding variable circles do not intersect each other, but each of them is a circle lying wholly inside that which immediately precedes it; and the variable circle becomes ultimately a point, viz. this point is a focus of the curve: this agrees with the foregoing statement that the f-conic intersects the circle in the four foci upon this circle. For the two portions of the ellipse which lie inside the circle, the variable circle is of course always imaginary. The like considerations apply to the case where the locus of the centre of the variable circle is a hyperbola or parabola. The foregoing remarks illustrate the actual generation of a bicircular quartic as the envelope of the variable circle.

Starting now from the equation

$$x + iy = \operatorname{sn}(x' + iy'),$$

we have

$$x = \frac{\operatorname{sn} x' \operatorname{cn} iy' \operatorname{dn} iy'}{1 - k^2 \operatorname{sn}^2 x' \operatorname{sn}^2 iy'}, \quad iy = \frac{\operatorname{sn} iy' \operatorname{cn} x' \operatorname{dn} x'}{1 - k^2 \operatorname{sn}^2 x' \operatorname{sn}^2 iy'},$$

or putting

$$\operatorname{sn} x' = p, \quad \operatorname{sn} iy' = iq;$$

these equations are

$$x = \frac{p\sqrt{1+q^2 \,.\, 1+k^2q^2}}{1+k^2p^2q^2}, \quad y = \frac{q\sqrt{1-p^2 \,.\, 1-k^2p^2}}{1+k^2p^2q^2};$$

whence also

$$x^2 + y^2 = \frac{p^2+q^2}{1+k^2p^2q^2} = r^2, \quad (\text{if } x^2+y^2 \text{ be put } = r^2).$$

These equations, considering therein q as a given constant, and p as a variable parameter, determine the curve EF: and considering p as a given constant, and q as a variable parameter, they determine the curve GH. But the eliminations are easily effected; we have

$$p^2(1 - k^2q^2r^2) = r^2 - q^2, \quad q^2(1 - k^2p^2r^2) = r^2 - p^2.$$

Hence, for EF,

$$p^2 = \frac{r^2 - q^2}{1 - k^2r^2q^2}, \qquad 1 - p^2 = \frac{1 + q^2 - (1 + k^2q^2) r^2}{1 - k^2r^2q^2},$$

$$1 - k^2p^2 = \frac{1 + k^2q^2 - k^2(1 + q^2) r^2}{1 - k^2r^2q^2}, \qquad 1 + k^2p^2q^2 = \frac{1 - k^2q^4}{1 - k^2r^2q^2},$$

and consequently

$$x = \frac{\sqrt{1 + q^2 . 1 + k^2q^2}}{1 - k^2q^4} \sqrt{r^2 - q^2 . 1 - k^2r^2q^2},$$

$$y = \frac{q \sqrt{1 + q^2 - (1 + k^2q^2) r^2} \sqrt{1 + k^2q^2 - k^2(1 + q^2) r^2}}{1 - k^2q^4},$$

giving x and y each of them in terms of r. And from the first of these we at once derive

$$(x^2 + y^2)^2 - 2Ax^2 - 2By^2 + \frac{1}{k^2} = 0,$$

where

$$2A = \frac{1 + q^2}{1 + k^2q^2} + \frac{1}{k^2} \frac{1 + k^2q^2}{1 + q^2}, \qquad 2B = q^2 + \frac{1}{k^2q^2}.$$

Similarly, for GH,

$$q^2 = \frac{r^2 - p^2}{1 - k^2r^2p^2}, \qquad 1 + q^2 = \frac{(1 - p^2) + r^2(1 - k^2p^2)}{1 - k^2r^2p^2},$$

$$1 + k^2q^2 = \frac{1 - k^2p^2 + k^2(1 - p^2) r^2}{1 - k^2r^2p^2}, \qquad 1 + k^2p^2q^2 = \frac{1 - k^2p^4}{1 - k^2r^2p^2};$$

and consequently

$$x = \frac{p \sqrt{(1 - p^2) + (1 - k^2p^2) r^2} \sqrt{1 - k^2p^2 + k^2(1 - p^2) r^2}}{1 - k^2p^4},$$

$$y = \frac{\sqrt{1 - p^2 . 1 - k^2p^2}}{1 - k^2p^4} \sqrt{r^2 - p^2 . 1 - k^2r^2p^2},$$

giving x, y each of them in terms of r. And from the second of them we at once derive

$$(x^2 + y^2)^2 - 2Ax^2 - 2By^2 + \frac{1}{k^2} = 0,$$

where

$$2A = p^2 + \frac{1}{k^2p^2}, \qquad 2B = -\frac{1 - p^2}{1 - k^2p^2} - \frac{1}{k^2} \frac{1 - k^2p^2}{1 - p^2}.$$

Consider, in particular, the case where the line EF is the midway line LM: here $y' = \tfrac{1}{2}K'$, and thence $iq = \text{sn}\, iy' = \text{sn}\, \tfrac{1}{2}iK' = \frac{i}{\sqrt{k}}$, that is, $q = \frac{1}{\sqrt{k}}$: and we thence obtain $A = B = \frac{1}{k}$; the equation of the bicircular quartic is

$$(x^2 + y^2)^2 - \frac{2}{k}(x^2 + y^2) + \frac{1}{k^2} = 0,$$

viz. this is the circle $x^2+y^2-\frac{1}{k}=0$ twice repeated. As a direct verification, observe that we have here

$$x+iy=\text{sn}\,(x'+\tfrac{1}{2}iK')=\frac{\frac{1+k}{\sqrt{k}}\,\text{sn}\,x'+\frac{i}{\sqrt{k}}\,\text{cn}\,x'\,\text{dn}\,x'}{1+k^2\,.\,\frac{1}{k}\,\text{sn}^2\,x'}=\frac{(1+k)\,p+i\sqrt{1-p^2\,.\,1-k^2p^2}}{\sqrt{k}\,(1+kp^2)},$$

and hence

$$x^2+y^2=\frac{(1+k)^2p^2+1-(1+k^2)\,p^2+k^2p^4}{k\,(1+2kp^2+k^2p^4)}=\frac{1}{k},$$

as it should be.

Reverting to the equation

$$x+iy=\text{sn}\,(x'+iy'),$$

I write successively

$$y'=\tfrac{1}{2}K'-z',\quad \text{sn}\,iy'=iq_1=\text{sn}\,i\,(\tfrac{1}{2}K'-z'),$$

and

$$y'=\tfrac{1}{2}K'+z',\quad \text{sn}\,iy'=iq_2=\text{sn}\,i\,(\tfrac{1}{2}K'+z');$$

we then have

$$iq_1\,.\,iq_2=\text{sn}\,i\,(\tfrac{1}{2}K'-z')\,\text{sn}\,i\,(\tfrac{1}{2}K'+z'),\quad =-\frac{1}{k},$$

that is, $q_1q_2=\frac{1}{k}$; hence for q writing q_1 or q_2, we have in each case the same values of A and B; that is, we have the same bicircular quartic corresponding to the lines $E'F'$ and $E_1'F_1'$, equidistant from the line $L'M'$: but to one of these lines there corresponds the quadrant lying inside, to the other that lying outside, the circular quadrant

$$x^2+y^2-\frac{1}{k}=0.$$

The curve

$$(x^2+y^2)^2-2Ax^2-2By^2+\frac{1}{k^2}=0$$

is in four different ways the envelope of a variable circle: viz. the circle may have its centre on a conic $\alpha x^2+\beta y^2-1=0$, and cut at right angles one of the circles

$$x^2+y^2\pm\frac{1}{k}=0\,;$$

or it may have its centre on the axis of x, or on the axis of y. The circle, having its centre on either axis, cuts this axis at right angles; but this condition being *ipso facto* satisfied, we do not thereby determine the radius of the circle having for its centre a given point on the axis: the expression for the radius must be sought for independently.

Write for shortness

$$\lambda^2=\frac{B-\frac{1}{k^2}}{A-B},$$

and consider the circle

$$x^2 + y^2 - 2\alpha x = \lambda \sqrt{A - B - 2\alpha^2} + B,$$

where α is a variable parameter. Differentiating, we have

$$x = \frac{\lambda\alpha}{\sqrt{A - B - 2\alpha^2}}, \text{ giving } \alpha = \frac{x\sqrt{A-B}}{\sqrt{2x^2 + \lambda^2}}:$$

the equation of the circle then gives

$$x^2 + y^2 - B = \frac{\alpha}{x}(2x^2 + \lambda^2), \ = \sqrt{A - B}\,\sqrt{2x^2 + \lambda^2};$$

that is,

$$(x^2 + y^2)^2 - 2Ax^2 - 2By^2 = B^2 + \lambda^2 (A - B), \quad = -\frac{1}{k^2};$$

or we have

$$(x^2 + y^2)^2 - 2Ax^2 - 2By^2 + \frac{1}{k^2} = 0$$

as the envelope of the variable circle

$$x^2 + y^2 - 2\alpha x = \frac{\sqrt{B - \frac{1}{k^2}}}{\sqrt{A - B}} \sqrt{A - B - 2\alpha^2} + B,$$

having its centre on the axis of x.

And similarly, the same quartic curve is the envelope of the variable circle

$$x^2 + y^2 - 2\beta y = \frac{\sqrt{A - \frac{1}{k^2}}}{\sqrt{B - A}} \sqrt{B - A - 2\beta^2} + A,$$

having its centre on the axis of y.

If we have in like manner the equation $x + iy = \text{cn}\,(x' + iy')$, then writing, as before,

$$\text{sn}\, x' = p, \quad \text{sn}\, iy' = iq,$$

we find

$$x^2 = \frac{(1 - p^2)(1 + q^2)}{(1 + k^2p^2q^2)^2}, \quad y^2 = \frac{p^2q^2(1 - k^2p^2)(1 + k^2q^2)}{(1 + k^2p^2q^2)^2},$$

and hence

$$r^2 = \frac{1 - p^2 + q^2 - k^2p^2q^2}{1 + k^2p^2q^2}, \text{ where } r^2 = x^2 + y^2.$$

For the curve EF corresponding to a line $E'F'$ parallel to the axis of x', we have to eliminate p from these equations; the expression for r^2 gives

$$p^2 = \frac{1 + q^2 - r^2}{1 + k^2q^2 + k^2q^2r^2}, \qquad 1 - p^2 = \frac{-(1 - k^2)q^2 + r^2(1 + k^2q^2)}{1 + k^2q^2 + k^2q^2r^2},$$

$$1 - k^2p^2 = \frac{(1 - k^2) + k^2r^2(1 + q^2)}{1 + k^2q^2 + k^2q^2r^2}, \qquad 1 + k^2p^2q^2 = \frac{1 + 2k^2q^2 + k^2q^4}{1 + k^2q^2 + k^2q^2r^2},$$

and thence

$$x = \frac{\sqrt{1+q^2}\sqrt{-(1-k^2)q^2 + r^2(1+k^2q^2)}\sqrt{1+k^2q^2+k^2q^2r^2}}{1+2k^2q^2+k^2q^4},$$

$$y = \frac{q\sqrt{1+q^2-r^2}\sqrt{1+k^2q^2}\sqrt{1-k^2+k^2r^2(1+q^2)}}{1+2k^2q^2+k^2q^4},$$

from which we deduce

$$(x^2+y^2)^2 - 2Ax^2 - 2By^2 - \frac{k'^2}{k^2} = 0,$$

where

$$2A = \frac{-1+2k^2+2k^2q^2+k^2q^4}{k^2(1+q^2)}, \quad 2B = \frac{-1-2k^2q^2+k^2(1-2k^2)q^4}{k^2q^2(1+k^2q^2)},$$

which is a bicircular quartic of the foregoing form.

Similarly, for the curve GH corresponding to a line $G'H'$ parallel to the axis of y', we have to eliminate q: the expression for r^2 gives

$$q^2 = \frac{-1+p^2+r^2}{1-k^2p^2-k^2p^2r^2}, \qquad 1+q^2 = \frac{(1-k^2)p^2+r^2(1-k^2p^2)}{1-k^2p^2-k^2p^2r^2},$$

$$1+k^2q^2 = \frac{(1-k^2)+k^2r^2(1-p^2)}{1-k^2p^2-k^2p^2r^2}, \quad 1+k^2p^2q^2 = \frac{1-2k^2p^2+k^2p^4}{1-k^2p^2-k^2p^2r^2}.$$

Hence

$$x = \frac{\sqrt{1-p^2}\sqrt{(1-k^2)p^2+r^2(1-k^2p^2)}\sqrt{1-k^2p^2-k^2p^2r^2}}{1-2k^2p^2+k^2p^4},$$

$$y = \frac{p\sqrt{-1+p^2+r^2}\sqrt{1-k^2p^2}\sqrt{1-k^2+k^2r^2(1-p^2)}}{1-2k^2p^2+k^2p^4},$$

from which we deduce

$$(x^2+y^2)^2 - 2Ax^2 - 2By^2 - \frac{k'^2}{k^2} = 0,$$

where

$$2A = \frac{-1+2k^2-2k^2p^2+k^2p^4}{k^2(1-p^2)}, \quad 2B = \frac{1-2k^2p^2-k^2(1-2k^2)p^4}{k^2p^2(1-k^2p^2)},$$

which is again a bicircular quartic of the foregoing form. And we have the like results for the equation

$$x + iy = \text{dn}(x'+iy');$$

so that, for the sn, cn, and dn, the curve EF or GH is in each case a biaxial bicircular quartic of the form

$$(x^2+y^2)^2 - 2Ax^2 - 2By^2 + C = 0.$$