889.

ON A DIFFERENTIAL EQUATION AND THE CONSTRUCTION OF MILNER'S LAMP.

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WHAT sort of an equation is

$$b^3 \cos(\alpha+\theta) = a\cos\theta \int_\theta^\beta r^2 d\theta - \tfrac{2}{3}\left\{\cos\theta \int_\theta^\beta r^3 \cos\theta d\theta + \sin\theta \int_\theta^\beta r^3 \sin\theta d\theta\right\}\ ?\ \ldots\ldots(1).$$

Write

$$X = \int_\theta^\beta r^2 d\theta,\quad Y = \int_\theta^\beta r^3\cos\theta d\theta,\quad Z = \int_\theta^\beta r^3 \sin\theta d\theta \ldots\ldots(2),$$

and start with the equations

$$d\theta = \frac{dX}{-r^2} = \frac{dY}{-r^3\cos\theta} = \frac{dZ}{-r^3\sin\theta}\ \ldots\ldots(3),$$

$$\left(\frac{d^2}{d\theta^2}+1\right)\{a\cos\theta . X - \tfrac{2}{3}(Y\cos\theta + Z\sin\theta)\} = 0 \ldots\ldots(4).$$

This last gives

$$(r - a\cos\theta)\,dr + ar\sin\theta . d\theta = 0\ \ldots\ldots(5),$$

and the system thus is

$$d\theta = \frac{dX}{-r^2} = \frac{dY}{-r^3\cos\theta} = \frac{dZ}{-r^3\sin\theta} = \frac{(r-a\cos\theta)\,dr}{-ar\sin\theta}\ \ldots\ldots(6),$$

viz. this is a system of ordinary differential equations between the five variables θ, r, X, Y, Z: the system can therefore be integrated with four arbitrary constants, and these may be so determined that for the value β of θ, X, Y, Z shall be each $=0$; and r shall have the value r_0.

But this being so, from the assumed equations (3) and (4) we have

$$X=\int_\theta^\beta r^2 d\theta,\quad Y=\int_\theta^\beta r^3\cos\theta d\theta,\quad Z=\int_\theta^\beta r^3\sin\theta d\theta,$$

and further, by integration of (4),

$$L\cos\theta+M\sin\theta=a\cos\theta\,.\,X-\tfrac{2}{3}(Y\cos\theta+Z\sin\theta).$$

Here L and M denote properly determined constants: viz. the conclusion is that r, X, Y, Z admit of being determined as functions of θ and of an arbitrary constant r_0, in such wise that

$$a\cos\theta\,.\,X-\tfrac{2}{3}(Y\cos\theta+Z\sin\theta)$$

shall be a function of θ, of the proper form $L\cos\theta+M\sin\theta$, but not so that it shall be the precise function $b^3\cos(\alpha+\theta)$. To make it have this value, we must have $L=b^3\cos\alpha$, $M=-b^3\sin\alpha$ (where L, M are given functions of a, β, r_0), i.e. we must have *two* given relations between a, b, α, β, r_0: or treating r_0 as a disposable constant, we must have *one* given relation between a, b, α, β.

The equation $d\theta=\dfrac{r-a\cos\theta}{-ar\sin\theta}dr$ gives $r^2-2ar\cos\theta=C$, where $C=r_0^2-2ar_0\cos\beta$. There would be considerable difficulty in working the question out with r_0 arbitrary, but we may do it easily enough for the particular value $r_0=0$ or $r_0=2a\cos\beta$, giving $C=0$ and therefore $r=2a\cos\theta$: and we ought in this case to be able to satisfy the given equation not in general but with *two* determinate relations between the constants a, b, α, β.

We have

$$\int\cos^2\theta d\theta=\tfrac{1}{2}\theta+\tfrac{1}{4}\sin 2\theta,$$

$$\int\cos^4\theta d\theta=\tfrac{3}{8}\theta+\tfrac{1}{4}\sin 2\theta+\tfrac{1}{32}\sin 4\theta,$$

$$\int\cos^3\theta\sin\theta d\theta=-\tfrac{1}{4}\cos^4\theta.$$

And thence

$$\begin{aligned}
&a\cos\theta\,.\,X-\tfrac{2}{3}(Y\cos\theta+Z\sin\theta)\\
&=\quad 4a^3\cos\theta\,\{\tfrac{1}{2}(\beta-\theta)+\tfrac{1}{4}(\sin 2\beta-\sin 2\theta)\}\\
&\quad-\tfrac{16}{3}a^3\cos\theta\,\{\tfrac{3}{8}(\beta-\theta)+\tfrac{1}{4}(\sin 2\beta-\sin 2\theta)+\tfrac{1}{32}(\sin 4\beta-\sin 4\theta)\}\\
&\quad-\tfrac{16}{3}a^3\sin\theta\,\{\qquad\qquad\qquad\qquad\qquad\qquad-\tfrac{1}{4}(\cos^4\beta-\cos^4\theta)\}\\
&=-\ \tfrac{1}{3}a^3\cos\theta\,(\sin 2\beta-\sin 2\theta)\\
&\quad-\ \tfrac{1}{6}a^3\cos\theta\,(\sin 4\beta-\sin 4\theta)\\
&\quad+\ \tfrac{4}{3}a^3\sin\theta\,(\cos^4\beta-\cos^4\theta),
\end{aligned}$$

where the terms containing β are readily reduced to $\frac{4}{3}a^3\cos^3\beta\sin(\theta-\beta)$; hence also the terms without β disappear of themselves: and we have

$$a\cos\theta\,.\,X-\tfrac{2}{3}(Y\cos\theta+Z\sin\theta)=\tfrac{4}{3}a^3\cos^3\beta\,.\,\sin(\theta-\beta),$$

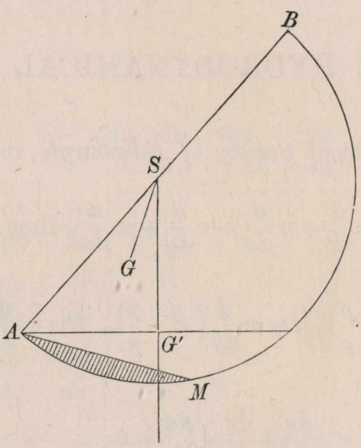
which may be put

$$=b^3\cos(\theta+\alpha):$$

viz. this will be so if we have the *two* relations

$$\alpha=\tfrac{1}{2}\pi-\beta;\quad\text{and}\quad b^3=-\tfrac{4}{3}a^3\cos^3\beta.$$

I make (see figure) Milner's lamp, with a circular section, β arbitrary, but a



segment AM ($\angle SAM=\beta$) made solid. G in the line SG at right angles to AM is the C. G. of the lamp, and G' the C. G. of the oil.

And this seems to be the *only* form—for the pole of r must, it seems to me, be *on* the bounding circle—viz. in the equation $r^2-2ar\cos\theta=C$, we must have $C=0$.