702.

ON THE TRIPLE ϑ-FUNCTIONS.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), t. LXXXVII. (1878), pp. 190—198.]

A QUARTIC curve has the deficiency 3, and depends therefore on the triple ϑ-functions: and these, as functions of 3 arguments, should be connected with functions of 3 points on the curve; but it is easy to understand that it is possible, and may be convenient, to introduce a fourth point, and so regard them as functions of 4 points on the curve: thus in the circle, the functions $\cos u$, $\sin u$ may be regarded as functions of one point $\cos u = x$, $\sin u = y$, or as functions of two points,

$$\cos u = xx_1 + yy_1, \quad \sin u = xy_1 - x_1 y.$$

And accordingly in Weber's memoir "Theorie der Abel'schen Functionen vom Geschlecht 3," (1876), see p. 156, the triple ϑ-functions are regarded as functions of 4 points on the curve: viz. it is in effect shown that (disregarding constant factors) each of the 64 functions is proportional to a determinant, the four lines of which are algebraical functions of the coordinates of the four points respectively: the form of this determinant being different according as the characteristic of the ϑ-function is odd or even, or say according as the ϑ-function is odd or even. But the geometrical signification of these formulæ requires to be developed.

A quartic curve may be touched in six points by a cubic curve: but (Hesse, 1855*) there are two kinds of such tangent cubics, according as the six points of contact are on a conic, or are not on a conic; say we have a conic hexad of points on the quartic, and a cubic hexad of points on the quartic. In either case, three points of the hexad may be assumed at pleasure; we can then in 28 different ways determine the remaining three points of the conic hexad, and in 36 different

* See the two memoirs "Ueber Determinanten und ihre Anwendung in der Geometrie" and "Ueber die Doppeltangenten der Curven vierter Ordnung," *Crelle*, t. XLIX. (1855).

ways the remaining three points of the cubic hexad: or what is the same thing, there are 28 systems of cubics touching in a conic hexad, and 36 systems of cubics touching in a cubic hexad. The condition in order that four points of the quartic curve may belong to a hexad (conic or cubic) is given by an equation $\Omega = 0$, where Ω is a determinant the four lines of which are algebraical functions of the coordinates of the four points respectively: but the form of such determinant is different according as the condition belongs to a conic hexad, or to a cubic hexad: we have thus 28 conic determinants and 36 cubic determinants, Ω; and the 64 ϑ-functions are proportional to constant multiples of these determinants; viz. the odd functions correspond to the conic determinants, and the even functions to the cubic determinants.

First, as to the conic hexads: the points of a conic hexad lie in a conic with the two points of contact of some one of the bitangents of the quartic curve: so that, given any three points of the hexad, these together with the two points of contact of the bitangent determine a conic which meets the quartic in the remaining three points of the hexad. Suppose that a, b, c, f, g, h are linear functions of the coordinates such that the equation of the quartic curve is

$$\sqrt{af} + \sqrt{bg} + \sqrt{ch} = 0;$$

then $a = 0$, $b = 0$, $c = 0$, $f = 0$, $g = 0$, $h = 0$ are six of the bitangents of the curve, and the bitangent $a = 0$ touches the curve at the two points of intersection of this line with the conic $bg - ch = 0$. The general equation of a conic through these two points $a = 0$, $bg - ch = 0$, may be written

$$bg - ch + a(Ax + By + Cz) = 0,$$

where for x, y, z we may if we please substitute any three of the six linear functions a, b, c, f, g, h, or any other linear functions of the coordinates (x, y, z): and the equation may also be written

$$af \pm (bg - ch) + a(Ax + By + Cz) = 0.$$

Adopting this latter form, and considering the intersections of the conic with the quartic, that is, considering the relation

$$\sqrt{af} + \sqrt{bg} + \sqrt{ch} = 0$$

as holding good, we have

$$af + bg - ch = -2\sqrt{afbg},$$

$$af - bg + ch = -2\sqrt{afch},$$

and we thus have at pleasure one or other of the two equations

$$-2\sqrt{afbg} + a(Ax + By + Cz) = 0,$$

$$-2\sqrt{afch} + a(Ax + By + Cz) = 0,$$

that is,

$$-2\sqrt{fbg} + \sqrt{a}(Ax + By + Cz) = 0,$$

$$-2\sqrt{fch} + \sqrt{a}(Ax + By + Cz) = 0.$$

Hence the condition in order that the four points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4), assumed to be points of the quartic, may belong to the conic hexad, may be written

$$\begin{vmatrix} \sqrt{f_1b_1g_1}, & x_1\sqrt{a_1}, & y_1\sqrt{a_1}, & z_1\sqrt{a_1} \\ \sqrt{f_2b_2g_2}, & x_2\sqrt{a_2}, & y_2\sqrt{a_2}, & z_2\sqrt{a_2} \\ \sqrt{f_3b_3g_3}, & x_3\sqrt{a_3}, & y_3\sqrt{a_3}, & z_3\sqrt{a_3} \\ \sqrt{f_4b_4g_4}, & x_4\sqrt{a_4}, & y_4\sqrt{a_4}, & z_4\sqrt{a_4} \end{vmatrix} = 0, \text{ or } \begin{vmatrix} \sqrt{f_1c_1h_1}, & x_1\sqrt{a_1}, & y_1\sqrt{a_1}, & z_1\sqrt{a_1} \\ \sqrt{f_2c_2h_2}, & x_2\sqrt{a_2}, & y_2\sqrt{a_2}, & z_2\sqrt{a_2} \\ \sqrt{f_3c_3h_3}, & x_3\sqrt{a_3}, & y_3\sqrt{a_3}, & z_3\sqrt{a_3} \\ \sqrt{f_4c_4h_4}, & x_4\sqrt{a_4}, & y_4\sqrt{a_4}, & z_4\sqrt{a_4} \end{vmatrix} = 0,$$

where, as before, the x, y, z may be replaced by any three of the letters a, b, c, f, g, h, or by any other linear functions of (x, y, z): and, moreover, although in obtaining the condition we have used for the quartic the equation

$$\sqrt{af} + \sqrt{bg} + \sqrt{ch} = 0,$$

depending upon six bitangents, yet from the process itself it is clear that the condition can only depend upon the particular bitangent $a = 0$: calling the condition $\Omega = 0$, all the forms of condition which belong to the same bitangent $a = 0$, will be essentially identical, that is, the several determinants Ω will differ only by constant factors; or disregarding these constant factors, we have for the bitangent $a = 0$, a single determinant Ω, which may be taken to be any one of the determinants in question. And we have thus 28 determinants Ω, corresponding to the 28 bitangents respectively.

Coming now to the cubic hexads, Hesse showed that the equation of a quartic curve could be (and that in 36 different ways) expressed in the form, symmetrical determinant $= 0$, or say

$$\begin{vmatrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & d \end{vmatrix} = 0,$$

where $(a, b, c, d, f, g, h, l, m, n)$ are linear functions of the coordinates; and from each of these forms he obtains the equation of a cubic

$$\begin{vmatrix} a, & h, & g, & l, & \alpha \\ h, & b, & f, & m, & \beta \\ g, & f, & c, & n, & \gamma \\ l, & m, & n, & d, & \delta \\ \alpha, & \beta, & \gamma, & \delta & \end{vmatrix} = 0,$$

containing the four constants α, β, γ, δ, or say the 3 ratios of these constants, touching the quartic in a cubic hexad of points: that the cubic does touch the quartic in six points appears, in fact, from Hesse's identity

$$\begin{vmatrix} a, & h, & g, & l, & \alpha \\ h, & b, & f, & m, & \beta \\ g, & f, & c, & n, & \gamma \\ l, & m, & n, & d, & \delta \\ \alpha, & \beta, & \gamma, & \delta & \end{vmatrix} \begin{vmatrix} a, & h, & g, & l, & \alpha' \\ h, & b, & f, & m, & \beta' \\ g, & f, & c, & n, & \gamma' \\ l, & m, & n, & d, & \delta' \\ \alpha', & \beta', & \gamma', & \delta' & \end{vmatrix} - \begin{vmatrix} a, & h, & g, & l, & \alpha \\ h, & b, & f, & m, & \beta \\ g, & f, & c, & n, & \gamma \\ l, & m, & n, & d, & \delta \\ \alpha', & \beta', & \gamma', & \delta' & \end{vmatrix}^2 = \begin{vmatrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & d \end{vmatrix} U,$$

where U is an easily calculated function of the second order in $a, b, c, d, f, g, h, l, m, n$, and also of the second order in the determinants $\alpha\beta' - \alpha'\beta$, etc.

We can obtain such a form of the equation of the quartic, from the beforementioned equation

$$\sqrt{af} + \sqrt{bg} + \sqrt{ch} = 0,$$

viz. this equation gives

$$\begin{vmatrix} *, & h, & g, & a \\ h, & *, & f, & b \\ g, & f, & *, & c \\ a, & b, & c, & * \end{vmatrix} = 0,$$

which is of the required form, symmetrical determinant $= 0$; the equation is, in fact,

$$a^2f^2 + b^2g^2 + c^2h^2 - 2bcgh - 2cahf - 2abfg = 0,$$

which is the rationalised form of

$$\sqrt{af} + \sqrt{bg} + \sqrt{ch} = 0,$$

and we hence have the cubic

$$\begin{vmatrix} *, & h, & g, & a, & \alpha \\ h, & *, & f, & b, & \beta \\ g, & f, & *, & c, & \gamma \\ a, & b, & c, & *, & \delta \\ \alpha, & \beta, & \gamma, & \delta, & * \end{vmatrix} = 0,$$

the developed form of which is

$$\begin{aligned} &\alpha^2bcf + \beta^2cag + \gamma^2abh + \delta^2fgh \\ &- (\alpha\beta\gamma + f\alpha\delta)(-af + bg + ch) \\ &- (b\gamma\alpha + g\beta\delta)(\ \ af - bg + ch) \\ &- (c\alpha\beta + h\gamma\delta)(\ \ af + bg - ch) = 0. \end{aligned}$$

Considering the intersections with the quartic

$$\sqrt{af} + \sqrt{bg} + \sqrt{ch} = 0,$$

we have

$$-af + bg + ch,\ \ af - bg + ch,\ \ af + bg - ch = -2\sqrt{bcgh},\ -2\sqrt{cahf},\ -2\sqrt{abfg},$$

and the equation thus becomes

$$(\alpha\sqrt{bcf} + \beta\sqrt{cag} + \gamma\sqrt{abh} + \delta\sqrt{fgh})^2 = 0\,;$$

viz. for the points of the cubic hexad we have

$$\alpha\sqrt{bcf}+\beta\sqrt{cag}+\gamma\sqrt{abh}+\delta\sqrt{fgh}=0,$$

and hence the condition in order that the four points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4) may belong to the cubic hexad is

$$\begin{vmatrix} \sqrt{b_1c_1f_1}, & \sqrt{c_1a_1g_1}, & \sqrt{a_1b_1h_1}, & \sqrt{f_1g_1h_1} \\ \sqrt{b_2c_2f_2}, & \sqrt{c_2a_2g_2}, & \sqrt{a_2b_2h_2}, & \sqrt{f_2g_2h_2} \\ \sqrt{b_3c_3f_3}, & \sqrt{c_3a_3g_3}, & \sqrt{a_3b_3h_3}, & \sqrt{f_3g_3h_3} \\ \sqrt{b_4c_4f_4}, & \sqrt{c_4a_4g_4}, & \sqrt{a_4b_4h_4}, & \sqrt{f_4g_4h_4} \end{vmatrix}=0,$$

viz. we have thus the form of the determinant Ω which belongs to a cubic hexad.

It is to be observed that the equation

$$\sqrt{af}+\sqrt{bg}+\sqrt{ch}=0$$

remains unaltered by any of the interchanges a and f, b and g, c and h; but we thus obtain only two cubic hexads; those answering to the equations

$$\alpha\sqrt{bcf}+\beta\sqrt{cag}+\gamma\sqrt{abh}+\delta\sqrt{fgh}=0,$$

and

$$\alpha\sqrt{agh}+\beta\sqrt{bhf}+\gamma\sqrt{cfg}+\delta\sqrt{abc}=0,$$

which give distinct hexads. The whole number of ways in which the equation of the quartic can be expressed in a form such as

$$\sqrt{af}+\sqrt{bg}+\sqrt{ch}=0,$$

attending only to the pairs of bitangents, and disregarding the interchanges of the two bitangents of a pair, is $=1260$, and hence the number of forms for the determinant Ω of a cubic hexad is the double of this, $=2520$, which is $=36\times 70$: but the number of distinct hexads is $=36$, and thus there must be for each hexad, 70 equivalent forms.

To explain this, observe that every even characteristic except $\begin{matrix}000\\000\end{matrix}$, and every odd characteristic, can be (and that in 6 ways) expressed as a sum of two different odd characteristics; we have thus (see Weber's Table I.) a system of $(35+28=)$ 63 hexpairs; and selecting at pleasure any three pairs out of the same hexpair, we have a system of $(63\times 20=)$ 1260 tripairs; giving the 1260 representations of the quartic in a form such as

$$\sqrt{af}+\sqrt{bg}+\sqrt{ch}=0.$$

Each even characteristic $\left(\text{not excluding } \begin{matrix}000\\000\end{matrix}\right)$ can be in 56 different ways (Weber, p. 23) expressed as a sum of three different odd characteristics, and these are such that no two of them belong to the same pair, in any tripair; or we may say that each even characteristic gives rise to 56 hemi-tripairs. But a hemi-tripair can be in 5 different ways completed into a tripair; and we have thus, belonging to the same even characteristic $(56 \times 5 =)$ 280 tripairs, which are however 70 tripairs each taken 4 times. A tripair contains in all $(2^3 =)$ 8 hemi-tripairs, but these divide themselves into two sets each of 4 hemi-tripairs such that for each hemi-tripair of the first set the three characteristics have a given sum, and for each hemi-tripair of the second set the three characteristics have a different given sum. Hence considering the 70 tripairs corresponding as above to a given even characteristic, in any one of the 70 tripairs, there is a set of 4 hemi-tripairs such that in each of them the sum of the three characteristics is equal to the given even characteristic; and taking the bitangents f, g, h to correspond to any one of these hemi-tripairs, the bitangents which correspond to the other three hemi-tripairs will be b, c, f; c, a, g and a, b, h respectively; and we thus obtain from any one of these one and the same representation

$$\alpha \sqrt{bcf} + \beta \sqrt{cag} + \gamma \sqrt{abh} + \delta \sqrt{fgh} = 0$$

of the cubic hexad. And the 70 tripairs give thus the 70 representations of the same cubic hexad.

The whole number of hemi-tripairs is $36 \times 56 = 2016$: it may be remarked that there exists a system of 288 heptads, each of 7 odd characteristics such that selecting at pleasure any 3 characteristics out of the heptad, we obtain always a hemi-tripair: we have thus in all $288 \times 35 = 10080$ hemi-tripairs: this is $= 2016 \times 5$, or we have the 2016 hemi-tripairs each taken 5 times. Weber's Table II. exhibits 36 out of the 288 heptads.

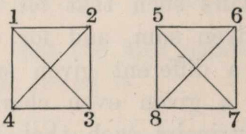
I recall that in the algorithm derived from Hesse's theory the bitangents are represented by the duads 12, 13, ..., 78 formed with the eight figures 1, 2, 3, 4, 5, 6, 7, 8; these duads correspond to the odd characteristics as shown in the Table, and the table shows also triads corresponding to all the even characteristics except
000
000.

Top line of characteristic.

Bottom line of characteristic.

	000	100	010	110	001	101	011	111
000		236	345	137	467	156	124	257
100	237	67	136	12	157	48	256	35
010	245	127	23	68	134	357	15	47
110	126	13	78	145	356	25	46	234
001	567	146	125	247	45	17	38	26
101	147	58	246	34	16	123	27	367
011	135	347	14	57	17	36	167	456
111	346	24	56	235	37	267	457	18

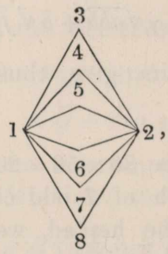
See my "Algorithm of the triple ϑ-functions," *Crelle*, t. LXXXVII. p. 165, [701]. The $(35+28=)$ 63 hexpairs then are

35 hexpairs such as 

, say this is 1234.5678 or for shortness 567 (the 8 going always with the expressed triad): that is, 567 denotes the hexpair

$$12.34;\ 13.24;\ 14.23;\ 56.78;\ 57.68;\ 58.67:$$

and

28 hexpairs such as 

, say this is 12; that is, 12 denotes the hexpair

$$13.32;\ 14.42;\ 15.52;\ 16.62;\ 17.72;\ 18.82.$$

It is to be noticed that the odd characteristics, as represented by their duad symbols, can be added by the formulæ

$$12+23=13, \text{ etc.},$$

or, what is the same thing,

$$12+13+23=0,\ =\frac{000}{000},\ \text{etc.},$$

and

$$12+34=13+24=14+23=56+78=57+68=58+67=567, \text{ etc.}$$

Thus, referring to the table,

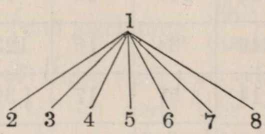
$$12+23=\ 13 \text{ means } \frac{110}{100}+\frac{010}{010}=\frac{100}{110},$$

and

$$12+34=567 \text{ means } \frac{110}{100}+\frac{110}{101}=\frac{000}{001},$$

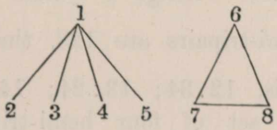
which are right.

The 288 heptads are

8 heptads such as 

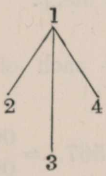
, say this is the heptad 1, denoting the seven duads 12, 13, 14, 15, 16, 17, 18:

and

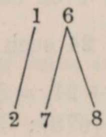
280 heptads such as 

, say this is the heptad 1 . 678, denoting the seven duads 12, 13, 14, 15, 67, 68, 78.

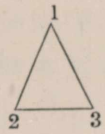
We hence see that the 2016 hemi-tripairs are:

280 hemi-tripairs 

(I.), say this is 1 . 234, denoting the three duads 12, 13, 14:

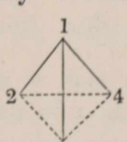
1680 hemi-tripairs 

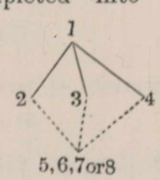
(II.), say this is 12 (6 . 78), denoting the three duads 12, 67, 68:

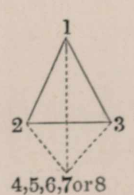
56 hemi-tripairs 

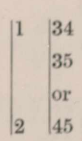
(III.), say this is 123, denoting the three duads 12, 13, 23:

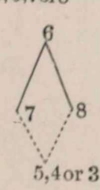
2016.

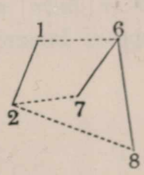
We further see how each hemi-tripair may be completed into a tripair in 5 different ways: thus (I.) gives the 5 tripairs 

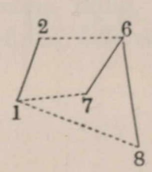


; (III.) gives the 5 tripairs 

; while (II.) gives the 3 tripairs 



and the 2 tripairs 

and 

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To each even characteristic there belongs a system of 56 hemi-tripairs; thus for the characteristic $\frac{000}{000}$, the 56 hemi-tripairs are 123, that is, 12, 13, 23, etc.: whence the 70 tripairs are 1234, that is, 12.34; 13.24; 14.23, etc.; and in any such tripair, say in 1234, we have the set of four hemi-tripairs 123, 124, 134, 234, for each of which the sum of the three characteristics is

$$=\frac{000}{000}\left(12+23+13=\frac{000}{000}, \text{ etc.}\right):$$

and the other set 1.234, 2.134, 3.124, 4.123, for each of which the sum of the three characteristics is

$$=567\left(12+13+14,\ =23+14,\ =567,\ =\frac{000}{001}\right).$$

To find the hemi-tripairs that belong to any other even characteristic; for instance, $\frac{000}{001}$, corresponds to 567: we have 4 such as 1.234; 24 such as (5.12)34; 4 such as 5.678; and 24 such as (1.56)78; in all $4+24+4+24, =56$. The tripairs are the 2, 1234, 5678; 16 such as 54(123); 16 such as 15(678); 36 such as (5162)34.78; in all $2+16+16+36, =70$; and in each of these it is easy to select the hemi-tripairs for which the sum of the 3 duads is $=567$.

Cambridge, 27 *December*, 1878.