699.

ON THE TRIPLE ϑ-FUNCTIONS.

[From the *Journal für die reine und angewandte Mathematik* (Crelle), t. LXXXVII. (1878), pp. 134—138.]

THERE should be in all 64 functions proportional to irrational algebraical functions of three independent variables x, y, z; there is no difficulty in obtaining the expression of these 64 functions in the case of the system of differential equations connected with the integral

$$\int dx : \sqrt{a-x\,.\,b-x\,.\,c-x\,.\,d-x\,.\,e-x\,.\,f-x\,.\,g-x\,.\,h-x}\,;$$

but this is *not the general form* of the system for the deficiency (Geschlecht) $p=3$; and I do not know how to deal with the general form: the present note relates therefore exclusively to the above-mentioned hyper-elliptic form.

I.

If in the Memoir, Weierstrass, "Theorie der Abel'schen Functionen," *Crelle*, t. LII. (1856), pp. 285—380, we take $\rho=3$, and write x, y, z; u, v, w; a, b, c, d, e, f, g instead of x_1, x_2, x_3; u_1, u_2, u_3; $a_1, a_2, a_3, a_4, a_5, a_6, a_7$; then, neglecting throughout mere constant factors, we have

$$X = a-x\,.\,b-x\,.\,c-x\,.\,d-x\,.\,e-x\,.\,f-x\,.\,g-x,$$

with the like values for Y and Z: the differential equations are

$$du = \frac{b-x\,.\,c-x\,.\,dx}{\sqrt{X}} + \frac{b-y\,.\,c-y\,.\,dy}{\sqrt{Y}} + \frac{b-z\,.\,c-z\,.\,dz}{\sqrt{Z}},$$

$$dv = \frac{c-x\,.\,a-x\,.\,dx}{\sqrt{X}} + \frac{c-y\,.\,a-y\,.\,dy}{\sqrt{Y}} + \frac{c-z\,.\,a-z\,.\,dz}{\sqrt{Z}},$$

$$dw = \frac{a-x\,.\,b-x\,.\,dx}{\sqrt{X}} + \frac{a-y\,.\,b-y\,.\,dy}{\sqrt{Y}} + \frac{a-z\,.\,b-z\,.\,dz}{\sqrt{Z}},$$

and if we write the single letters A, B, C, D, E, F, G for $\text{al}(u, v, w)_1$, $\text{al}(u, v, w)_2$, $\text{al}(u, v, w)_3$, $\text{al}(u, v, w)_4$, $\text{al}(u, v, w)_5$, $\text{al}(u, v, w)_6$, $\text{al}(u, v, w)_7$ respectively, each of the capital letters thus denoting a function of (u, v, w), the expressions of these functions in terms of (x, y, z) are

$$A = \sqrt{a-x\,.\,b-x\,.\,c-x}, \quad \text{(seven equations)}.$$
$$\vdots \qquad \vdots$$

Similarly, instead of the 21 functions $\text{al}(u, v, w)_{12}, \ldots, \text{al}(u, v, w)_{67}$ writing $AB, \ldots, FG$, each of these binary symbols denoting in like manner a function of (u, v, w), the definition of AB is

$$AB = A\nabla B - B\nabla A,$$

where

$$\nabla = \frac{d}{du} + \frac{d}{dv} + \frac{d}{dw};$$

we have

$$b-c\,.\,c-a\,.\,a-b\,.\,\frac{dx}{\sqrt{X}} = \frac{a-y\,.\,a-z}{x-y\,.\,x-z}(b-c)\,du + \frac{b-y\,.\,b-z}{x-y\,.\,x-z}(c-a)\,dv + \frac{c-y\,.\,c-z}{x-y\,.\,x-z}(a-b)\,dw,$$

$$\text{,,} \qquad \frac{dy}{\sqrt{Y}} = \frac{a-z\,.\,a-x}{y-z\,.\,y-x}(b-c)\,du + \frac{b-z\,.\,b-x}{y-z\,.\,y-x}(c-a)\,dv + \frac{c-z\,.\,c-x}{y-z\,.\,y-x}(a-b)\,dw,$$

$$\text{,,} \qquad \frac{dz}{\sqrt{Z}} = \frac{a-x\,.\,a-y}{z-x\,.\,z-y}(b-c)\,du + \frac{b-x\,.\,b-y}{z-x\,.\,z-y}(c-a)\,dv + \frac{c-x\,.\,c-y}{z-x\,.\,z-y}(a-b)\,dw;$$

hence

$$\frac{b-c\,.\,c-a\,.\,a-b}{\sqrt{X}}\,\nabla x = \frac{a-y\,.\,a-z}{x-y\,.\,x-z}(b-c) + \frac{b-y\,.\,b-z}{x-y\,.\,x-z}(c-a) + \frac{c-y\,.\,c-z}{x-y\,.\,x-z}(a-b),$$

$$= -\frac{b-c\,.\,c-a\,.\,a-b}{x-y\,.\,x-z},$$

that is,

$$\nabla x = \frac{-\sqrt{X}}{x-y\,.\,x-z};$$

and similarly

$$\nabla y = \frac{-\sqrt{Y}}{y-x\,.\,y-z}, \quad \nabla z = \frac{-\sqrt{Z}}{z-x\,.\,z-y}.$$

Hence from the equation

$$A = \sqrt{a-x\,.\,a-y\,.\,a-z}$$

we have

$$\nabla A = -\tfrac{1}{2}A\left(\frac{1}{a-x}\nabla x + \frac{1}{a-y}\nabla y + \frac{1}{a-z}\nabla z\right),$$

that is,

$$\nabla A = \frac{\tfrac{1}{2}A}{y-z\,.\,z-x\,.\,x-y}\left\{\frac{y-z}{a-x}\sqrt{X} + \frac{z-x}{a-y}\sqrt{Y} + \frac{x-y}{a-z}\sqrt{Z}\right\}:$$

and similarly

$$\nabla B = \frac{\tfrac{1}{2}B}{y-z\,.\,z-x\,.\,x-y}\left\{\frac{y-z}{b-x}\sqrt{X} + \frac{z-x}{b-y}\sqrt{Y} + \frac{x-y}{b-z}\sqrt{Z}\right\};$$

consequently

$$AB = \frac{\tfrac{1}{2}(a-b)\,AB}{y-z\,.\,z-x\,.\,x-y}\left\{\frac{(y-z)\sqrt{X}}{a-x\,.\,b-x} + \frac{(z-x)\sqrt{Y}}{a-y\,.\,b-y} + \frac{(x-y)\sqrt{Z}}{a-z\,.\,b-z}\right\},$$

or substituting for A and B their values, and disregarding the constant factor $\frac{1}{2}(a-b)$, this is

$$AB = \frac{1}{y-z.z-x.x-y}\{(y-z)\sqrt{a-y.b-y.a-z.b-z.c-x.d-x.e-x.f-x.g-x}$$
$$+(z-x)\sqrt{a-z.b-z.a-x.b-x.c-y.d-y.e-y.f-y.g-y}$$
$$+(x-y)\sqrt{a-x.b-x.a-y.b-y.c-z.d-z.e-z.f-z.g-z}\}.$$

We have thus in all 21 equations, which exhibit the form of the Weierstrassian functions al $(u, v, w)_{12}$, ..., al $(u, v, w)_{67}$.

To complete the system, there should it is clear be 35 new functions al $(u, v, w)_{123}$, ..., al $(u, v, w)_{567}$, represented by $ABC, \ldots, EFG$, viz. the whole number of functions would then be

$$7 + \frac{7.6}{1.2} + \frac{7.6.5}{1.2.3}(=7+21+35) = 63, \ = 64 - 1,$$

since the functions represent ratios of the ϑ-functions.

II.

Starting now with the radical

$$\sqrt{a-x.b-x.c-x.d-x.e-x.f-x.g-x.h-x}$$

composed of eight linear factors, and writing, as in my "Memoir on the double ϑ-functions," t. LXXXV. (1878), pp. 214—245, [665]; a, b, c, d, e, f, g, h to denote these factors, and similarly $a_1, b_1, c_1, d_1, e_1, f_1, g_1, h_1$ and $a_2, b_2, c_2, d_2, e_2, f_2, g_2, h_2$ to denote $a-y$, $b-y$, etc., and $a-z$, $b-z$, etc., so that $X = \text{abcdefgh}$, $Y = a_1b_1c_1d_1e_1f_1g_1h_1$, $Z = a_2b_2c_2d_2e_2f_2g_2h_2$; then, instead of the Weierstrassian form, the differential equations may be taken to be

$$du = \frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} + \frac{dz}{\sqrt{Z}},$$

$$dv = \frac{x\,dx}{\sqrt{X}} + \frac{y\,dy}{\sqrt{Y}} + \frac{z\,dz}{\sqrt{Z}},$$

$$dw = \frac{x^2\,dx}{\sqrt{X}} + \frac{y^2\,dy}{\sqrt{Y}} + \frac{z^2\,dz}{\sqrt{Z}}.$$

We then have 64 ϑ-functions and an ω-function, viz. writing

$$\theta = y-z.z-x.x-y,$$

and then

$$\sqrt{a} = \sqrt{aa_1a_2} \quad \text{(8 equations)}$$

$$\vdots \qquad \vdots$$

$$\sqrt{abc} = \frac{1}{\theta}\{(y-z)\sqrt{a_1b_1c_1a_2b_2c_2\text{defgh}} + (z-x)\sqrt{a_2b_2c_2\text{abc}d_1e_1f_1g_1h_1} + (x-y)\sqrt{\text{abc}a_1b_1c_1d_2e_2f_2g_2h_2}\}$$

$$\vdots \qquad \vdots \qquad \text{(56 equations)}$$

the equations, which define the ϑ-functions $A, B, \ldots, H, ABC, \ldots, FGH$, and the ω-function Ω, are

$$A = \Omega \sqrt{a} \qquad \text{(8 equations)}$$
$$\vdots \quad \vdots$$
$$ABC = \Omega \sqrt{abc} \qquad \text{(56 equations)}$$
$$\vdots \quad \vdots$$

and one other relation which I have not as yet investigated.

As regards the algebraical relations between the 64 ϑ-functions, it is to be remarked that, selecting in a proper manner 8 of the functions, the square of any one of the other functions can be expressed as a linear function of the squares of the 8 selected functions. To explain this somewhat further, observe that, taking any 5 squares such as $(ABC)^2$, we can with these 5 squares form a linear combination which is rational in x, y, z. We have for instance, writing down the irrational part only,

$$(ABC)^2 = \frac{2}{\theta^2}\{\mathrm{abc}\,(z-x)(x-y)\sqrt{YZ} + \mathrm{a_1b_1c_1}\,(x-y)(y-z)\sqrt{ZX} + \mathrm{a_2b_2c_2}\,(y-z)(z-x)\sqrt{XY}\},$$

and forming in all five such equations, then inasmuch as the coefficients abc,... of $(z-x)(x-y)\sqrt{YZ}$ are each of them a cubic function containing terms in x^0, x^1, x^2, x^3, we have a determinate set of constant factors such that the resulting term in $(z-x)(x-y)\sqrt{YZ}$ will be $=0$; but the coefficients $\mathrm{a_1b_1c_1}, \ldots$ of $(x-y)(y-z)\sqrt{ZX}$ only differ from the first set of coefficients by containing y instead of x, and the same set of constant factors will thus make the resulting term in $(x-y)(y-z)\sqrt{ZX}$ to be $=0$; and similarly the same set of constant factors will make the resulting term in $(y-z)(z-x)\sqrt{XY}$ to be $=0$; viz. we have thus a set of constant factors, such that the whole irrational part will disappear. *It seems to be in general true that the same set of constant factors will make the rational part integral;* viz. the rational part is a function of the form $\frac{1}{\theta^2}$ multiplied by a rational and integral function of x, y, z, and if this rational and integral function divide by θ^2, then the final result will be a rational and integral function, which, being symmetrical in x, y, z, is at once seen to be a linear function of the symmetrical combinations $1, x+y+z, yz+zx+xy, xyz$. Such a function is obviously a linear function of any four squares A^2, B^2, C^2, D^2; or the form is, linear function of five squares $(ABC)^2 =$ linear function of four squares A^2, that is, any one of the five squares is a linear function of 8 squares.

As an instance, consider the *three* squares $(ABC)^2, (ABD)^2, (ABE)^2$, which are such that we have a linear combination which is rational: in fact, we have here in each function the pair of factors ab, which unites itself with $(z-x)(x-y)\sqrt{XY}$, viz. it is only the coefficient of $\mathrm{ab}\,(z-x)(x-y)\sqrt{XY}$ which has to be made $=0$; the required combination is obviously

$$(d-e)(ABC)^2 + (e-c)(ABD)^2 + (c-d)(ABE)^2.$$

Here the irrational part vanishes and the rational part is found to be

$$= \frac{1}{\theta^2}\left[a_1b_1a_2b_2fgh\,(y-z)^2\left\{\begin{array}{l}(d-e)\,c_1c_2de\\+(e-c)\,d_1d_2ce\\+(c-d)\,e_1e_2dc\end{array}\right\}\right.$$

$$+\,a_2b_2abf_1g_1h_1\,(z-x)^2\left\{\begin{array}{l}(d-e)\,c_2cd_1e_1\\+(e-c)\,d_2dc_1e_1\\+(c-d)\,e_2ed_1c_1\end{array}\right\}$$

$$\left.+\,aba_1b_1f_2g_2h_2\,(x-y)^2\left\{\begin{array}{l}(d-e)\,cc_1d_2e_2\\+(e-c)\,dd_1c_2e_2\\+(c-d)\,ee_1d_2c_2\end{array}\right\}\right].$$

The three terms in { } are here $=-(c-d)(d-e)(e-c)$ multiplied by $(z-x)(x-y)$, $(x-y)(y-z)$, $(y-z)(z-x)$ respectively; hence the term in [ ] divides by θ and the result is

$$=-\frac{(c-d)(d-e)(e-c)}{\theta}\,[a_1b_1a_2b_2f\,g\,h\,\,(y-z)$$
$$+\,a_2b_2a\,b\,f_1g_1h_1\,(z-x)$$
$$+\,a\,b\,a_1b_1f_2g_2h_2\,(x-y)],$$

or finally this is

$$=-(c-d)(d-e)(e-c)$$

multiplied by

$$\begin{array}{ll} & \{(a^2+ab+b^2)fgh-(a^2b+ab^2)(fg+fh+gh)+a^2b^2(f+g+h)\}\\ +\;(x+y+z) & \{\;-(a+b)fgh+ab(fg+fh+gh)-a^2b^2\;\}\\ +\;(yz+zx+xy) & \{\;fgh-ab(f+g+h)+a^2b+ab^2\;\}\\ +\;xyz & \{-(fg+fh+gh)+(a+b)(f+g+h)-(a^2+ab+b^2)\;\},\end{array}$$

that is, we have $(d-e)(ABC)^2+(e-c)(ABD)^2+(c-d)(ABE)^2=$ a sum of four squares, viz. we have here a linear relation between 7 squares.

I have not as yet investigated the forms of the relations between the products of pairs of ϑ-functions.

Cambridge, 30 *September*, 1878.