679.

ON THE REGULAR SOLIDS.

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In a regular solid, or say in the spherical figure obtained by projecting such solid, by lines from the centre, on the surface of a concentric sphere, we naturally consider 1° the summits, 2° the centres of the faces, 3° the mid-points of the sides. But, imagining the five regular figures drawn in proper relation to each other on the same spherical surface, the only points which have thus to be considered are 12 points A, 20 points B, 30 points Θ, and 60 points Φ. These may be, in the first instance, described by reference to the dodecahedron; viz. the points A are the centres of the faces, the points B are the summits, the points Θ are the mid-points of the sides, and the points Φ are the mid-points of the diagonals of the faces (viz. there are thus 5 points Φ in each face of the dodecahedron, or in all 60 points Φ). But reciprocally we may describe them in reference to the icosahedron; viz. the points A are the summits, the points B the centres of the faces, the points Θ the mid-points of the sides, (viz. each point Θ is the common mid-point of a side of the dodecahedron and a side of the icosahedron, which sides there intersect at right angles), and the points Φ are points lying by 3's on the faces of the icosahedron, each point Φ of the face being given as the intersection of a perpendicular $A\Theta$ of the face by a line BB, joining the centres of two adjacent faces and intersecting $A\Theta$ at right angles.

The points A lie opposite to each other in pairs in such wise that, taking any two opposite points as poles, the relative situation is as follows:

A	Longitudes.
1	— ,
5	0°, 72°, 144°, 216°, 288°,
5	36°, 108°, 180°, 252°, 324°,
1	— ,

where the points A in the same horizontal line form a zone of points equidistant from the point taken as the North Pole. And the points B lie also opposite to each other in such wise that, taking two opposite points as poles, the relative situation is as follows:

B	Longitudes.
1	— ,
3	0°, 120°, 240° ,
6	(0°, 120°, 240°) $\pm$ 22° 14′,
6	(60°, 180°, 300°) $\pm$ 22° 14′,
3	60°, 180°, 300° ,
1	— ,

where the points B in the same horizontal line form a zone of points equidistant from the point taken as the North Pole. Neglecting the $3+3$ points B which lie adjacent to the poles, the remaining 14 points B may be arranged as follows ($\beta = 22°\ 14'$ as above):

B	Longitudes.	
1	—	
6	β, $120° + \beta$, $240° + \beta$	$-\beta$, $120° - \beta$, $240° - \beta$,
6	$60° + \beta$, $180° + \beta$, $300° + \beta$	$60° - \beta$, $180° - \beta$, $300° - \beta$.
1	—	

And taking the two poles separately with each system of the remaining poles, we have 2 systems each of 8 points B, which are, in fact, the summits of a cube (hexahedron); each point B taken as North Pole thus belongs to two cubes; but inasmuch as the cube has 8 summits, the number of the cubes thus obtained is $20 \times 2 \div 8, = 5$; viz. the 20 points B form the summits of 5 cubes, each point B of course belonging to 2 cubes.

It is to be added that, considering the 5 points B which form a face of the dodecahedron, any diagonal BB of this dodecahedron is a side of a cube. We have thus $12 \times 5, = 60$, the number of the sides of the 5 cubes.

It is at once seen that the centres of the faces of a cube are points Θ, and that the mid-points of the sides of the cube are points Φ.

To each cube there corresponds of course an octahedron, the summits being points Θ, the centres of the faces points B, and the mid-points of the sides points Φ; thus, for the five octahedra the summits are the $5 \times 6, = 30$, points Θ; the centres of the faces are $5 \times 8, = 40$, points B (each point B being thus a centre of face for two octahedra), and the mid-points of the sides being the $5 \times 12, = 60$, points Φ.

Finally, considering the 8 points B which belong to a cube, we can, in four different ways, select thereout 4 points B which are the summits of a tetrahedron;

the remaining 4 points B are then the centres of the faces, and the mid-points of the sides are points Θ: there are thus $5 \times 4, = 20$, tetrahedra having 20×4 summits which are the 20 points B each 4 times; 20×4 centres of faces which are the 20 points B each 4 times; and 20×6 mid-points of sides which are the 30 points Θ each 4 times.

It thus appears that, as mentioned above, the five regular figures depend only on the points A, B, Θ, and Φ.

We might take as poles two opposite points A, B, Θ, or Φ; and in each case determine in reference to these the positions of the other points; but for brevity I consider only the case in which we take as poles two opposite points A. We have the following table:

Poles two opposite points A.

	N. P. D.	Longitudes.
A_0	0°	—
$5A_1$	63° 26′	0°, 72°, 144°, 216°, 288°
$5A_2$	116° 34′	36°, 108°, 180°, 252°, 324°
A_3	180°	—
$5B_1$	37° 22′	36°, 108°, ..., 324°
$5B_2$	79° 12′	36°, 108°, ..., 324°
$5B_3$	100° 48′	0°, 72°, ..., 288°
$5B_4$	142° 38′	0°, 72°, ..., 288°
$5\Theta_1$	31° 43′	0°, 72°, ..., 288°
$5\Theta_2$	58° 77′	36°, 108°, ..., 324°
$10\Theta_3$	90°	(0°, 72°, ..., 288°) ± 18°
$5\Theta_4$	121° 43′	0°, 72°, ..., 288°
$5\Theta_5$	148° 17′	36°, 108°, ..., 324°
$5\Phi_1$	13° 16′	36°, 108°, ..., 324°
$10\Phi_2$	52° 52′	(0°, 72°, ..., 288°) ± 9° 44′
$10\Phi_3$	68° 10′	(0°, 72°, ..., 288°) ± 13° 35′
$5\Phi_4$	76° 42′	0°, 72°, ..., 288°
$5\Phi_5$	103° 18′	36°, 108°, ..., 324°
$10\Phi_6$	111° 50′	(36°, 108°, ..., 324°) ± 13° 35′
$10\Phi_7$	127° 8′	(36°, 108°, ..., 324°) ± 9° 44′
$5\Phi_8$	166° 44′	0°, 72°, ..., 108°.

I add for greater completeness the following results, some of which were used in the calculation of the foregoing table. Considering successively (1) the tetrahedral triangle, summits 3 points B, centre a point B; (2) the hexahedral square, summits 4 points B, centre a point Θ; (3) the octahedral triangle, summits 3 points Θ, centre a point B; (4) the icosahedral triangle, summits 3 points A, centre a point B; (5) the dodecahedral pentagon, summits 5 points, centre a point B; and (6), what may be called the small pentagon, summits 5 points Φ lying within a dodecahedral pentagon, and having therewith the common centre B; we may in each case write s the side, r the radius or distance of the centre from a summit, p the perpendicular or distance of the centre from a side. And the values then are

	s	r	p
Tet. Δ	109° 30′	70° 30′	54° 45′
Hex. square	70 30	54 45	45
Oct. Δ	90	54 45	35 15
Icos. Δ	63 26	37 22	20 55
Dod. pentagon	41 50	37 22	31 43
Small pentagon	15 30	13 16	10 48