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[ADDITION TO MR GLAISHER'S PAPER "PROOF OF STIRLING'S THEOREM."]

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It is easy to extend Mr Glaisher's investigation so as to obtain from it the more approximate value

$$\Pi n = \sqrt{(2\pi)}\, n^{n+\frac{1}{2}}\, e^{-n+\frac{1}{12n}}.$$

We, in fact, have

$$\psi x = e^{2nx+ax^3+bx^5+\dots},$$

where $a, b, \dots$ are given functions of n, viz.

$$a = \tfrac{2}{3}\left\{\frac{1}{3^2}+\frac{1}{5^2}+\dots+\frac{1}{(2n+1)^2}\right\},$$

$$b = \tfrac{3}{5}\left\{\frac{1}{3^4}+\frac{1}{5^4}+\dots+\frac{1}{(2n+1)^4}\right\},$$

&c.

And hence writing $x=1$, we have

$$\psi(1) = \tfrac{1}{2}\,\frac{1}{2^{2n}\,\Pi^2(n)}\,(2n+2)^{2n+1} = e^{2n+a+b+\dots},$$

that is,

$$\Pi n = \left(\frac{2n+2}{2}\right)^{\frac{2n+1}{2}} e^{-n-\frac{1}{2}(a+b+\dots)}$$

$$= (n+1)^{n+\frac{1}{2}}\, e^{-n-\frac{1}{2}(a+b+\dots)}$$

$$= n^{n+\frac{1}{2}}\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} e^{-n-\frac{1}{2}(a+b+\dots)}.$$

Hence for $\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}$ writing $e^{-(n+\frac{1}{2})\log\left(1+\frac{1}{n}\right)}$, the whole exponent of e is

$$(n+\tfrac{1}{2})\log\left(1+\frac{1}{n}\right)-n-\tfrac{1}{2}(a+b+\ldots)$$

$$=(n+\tfrac{1}{2})\left(\frac{1}{n}-\tfrac{1}{2}\frac{1}{n^2}+\tfrac{1}{3}\frac{1}{n^3}-\ldots\right)-n-\tfrac{1}{2}(a+b+\ldots)$$

$$=-n+1+\frac{1}{3.4}\frac{1}{n^2}-\frac{2}{4.6}\frac{1}{n^3}+\frac{3}{5.8}\frac{1}{n^4}-\ldots$$

$$-\tfrac{1}{2}(a+b+\ldots).$$

We have

$$\frac{1}{1^2}+\frac{1}{3^2}+\ldots+\frac{1}{(2n+1)^2}=\text{const.}-\frac{1}{4n}+\text{terms in }\frac{1}{n^2},\ \frac{1}{n^3},\ \&c.$$

(the constant is in fact $=\frac{1}{8}\pi^2$, but the value is not required), hence $a=\text{const.}-\frac{1}{6n}$ + terms in $\frac{1}{n^2}$, $\frac{1}{n^3}$, &c.; as regards b, c, &c., there are no terms in $\frac{1}{n}$, but we have $b=\text{const.}+\text{terms in }\frac{1}{n^2}$, &c., $c=\text{const.}+\text{terms in }\frac{1}{n^3}$, &c. Hence the whole exponent of e is

$$=-n+C+\frac{1}{12n}+\text{terms in }\frac{1}{n^2},\ \&c.$$

As in Mr Glaisher's investigation, it is shown that $e^{-C}=\surd(2\pi)$, and hence neglecting the terms in $\frac{1}{n^2}$, &c., the final result is

$$\Pi n=\surd(2\pi)\,n^{n+\frac{1}{2}}\,e^{-n+\frac{1}{12n}}.$$