671.

ON A SIBI-RECIPROCAL SURFACE.

[From the *Berlin. Akad. Monatsber.*, (1878), pp. 309—313.]

THE question of the generation of a sibi-reciprocal surface—that is, a surface the reciprocal of which is of the same order and has the same singularities as the original surface—was considered by me in the year 1868, see *Proc. London Math. Soc.* t. II. pp. 61—63, [part of 387], where it is remarked that if a surface be considered as the envelope of a quadric surface varying according to given conditions, then the reciprocal surface is given as the envelope of a quadric surface varying according to the reciprocal conditions; whence, if the conditions be sibi-reciprocal, it follows that the surface is a sibi-reciprocal surface. And I gave as instances the surface which is the envelope of a quadric surface touching each of 8 given lines; and also the surface called the "tetrahedroid," which is a homographic transformation of Fresnel's Wave Surface and a particular case of the quartic surface with 16 nodes.

The interesting surface of the order 8, recently considered by Herr Kummer, *Berl. Monatsber.*, Jan. 1878, pp. 25—36, is included under the theory. In fact, if we consider a line L, whereof the six coordinates

$$a,\ b,\ c,\ f,\ g,\ h,$$

satisfy each of the three linear relations

$$f_1a + g_1b + h_1c + a_1f + b_1g + c_1h = 0,$$
$$f_2a + g_2b + h_2c + a_2f + b_2g + c_2h = 0,$$
$$f_3a + g_3b + h_3c + a_3f + b_3g + c_3h = 0,$$

the locus of this line is a quadric surface the equation of which is

$$\begin{aligned} T = {} & (agh)\,x^2 + (bhf)\,y^2 + (cfg)\,z^2 + (abc)\,w^2 \\ & + [(abg) - (cah)]\,xw + [(bfg) + (chf)]\,yz \\ & + [(bch) - (abf)]\,yw + [(cgh) + (afg)]\,zx \\ & + [(caf) - (bcg)]\,zw + [(ahf) + (bgh)]\,xy = 0, \end{aligned}$$

where (agh) is used to denote the determinant $\begin{vmatrix} a_1, & g_1, & h_1 \\ a_2, & g_2, & h_2 \\ a_3, & g_3, & h_3 \end{vmatrix}$, and so for the other symbols. Considering the reciprocal of the line L in regard to the quadric surface $X^2 + Y^2 + Z^2 + W^2 = 0$, the six coordinates of the reciprocal line are

$$f,\ g,\ h,\ a,\ b,\ c,$$

and it is hence at once seen that the locus of the reciprocal line is the quadric surface obtained from the equation $T = 0$ by interchanging therein the symbolical quantities a, b, c and f, g, h: viz. writing also $(\xi, \eta, \zeta, \omega)$ in place of (x, y, z, w), the new equation is

$$\begin{aligned} T' = (fbc)\,\xi^2 + (gca)\,\eta^2 + (hab)\,\zeta^2 + (fgh)\,\omega^2 \\ + [(fgb) - (hfc)]\,\xi\omega + [(fab) + (hca)]\,\eta\zeta \\ + [(ghc) - (fga)]\,\eta\omega + [(gbc) + (fab)]\,\zeta\xi \\ + [(hfa) - (ghb)]\,\zeta\omega + [(hca) + (gbc)]\,\xi\eta = 0; \end{aligned}$$

or, what is the same thing, this equation $T' = 0$ is the equation of the original quadric surface (the locus of L) expressed in terms of the plane-coordinates ξ, η, ζ, ω.

Now considering each of the quantities $a_1, b_1, c_1, f_1, g_1, h_1, a_2, b_2$, etc., a_3, b_3, etc., as a given linear function of a variable parameter λ, say $a_1 = a_1' + a_1''\lambda$, $b_1 = b_1' + b_1''\lambda$, etc., the equation $T = 0$ takes the form

$$A\lambda^3 + 3B\lambda^2 + 3C\lambda + D = 0,$$

where A, B, C, D are given quadric functions of the coordinates x, y, z, w; and the envelope of the quadric surface $T = 0$ is Herr Kummer's surface of the eighth order

$$(AD - BC)^2 - 4(AC - B^2)(BD - C^2) = 0.$$

In like manner the equation $T' = 0$ takes the form

$$A'\lambda^3 + 3B'\lambda^2 + 3C'\lambda + D' = 0,$$

where A', B', C', D' are given functions of the coordinates ξ, η, ζ, ω; and we have

$$(A'D' - B'C')^2 - 4(A'C' - B'^2)(B'D' - C'^2) = 0,$$

as the equation of the reciprocal surface; or (what is the same thing) as that of the original surface, regarding ξ, η, ζ, ω as plane-coordinates.

In regard to the foregoing equation $T = 0$, it is to be noticed that, if $a_1, b_1, c_1, f_1, g_1, h_1$; a_2, b_2, etc., a_3, b_3, etc., instead of being arbitrary coefficients, were the coordinates of three given lines L_1, L_2, L_3 respectively; that is, if we had

$$a_1f_1 + b_1g_1 + c_1h_1 = 0,$$

$$a_2f_2 + b_2g_2 + c_2h_2 = 0,$$

$$a_3f_3 + b_3g_3 + c_3h_3 = 0,$$

then the three linear relations satisfied by (a, b, c, f, g, h) would express that the line L was a line meeting each of the three given lines L_1, L_2, L_3: the locus is therefore the quadric surface which passes through these three lines; and I have in my paper "On the six coordinates of a Line," *Camb. Phil. Trans.*, t. XI. (1869), pp. 290—323, [435], found the equation to be the foregoing equation $T=0$. But it is easy to see that the same equation subsists in the case where the three equations $a_1f_1+b_1g_1+c_1h_1=0$, etc., are not satisfied. For the several coefficients being perfectly general, any one of the three linear relations may be replaced by a linear combination of these equations; that is, in place of a_1, b_1, c_1, f_1, g_1, h_1, we may write a_1', b_1', c_1', f_1', g_1', h_1', where $a_1'=\theta_1a_1+\theta_2a_2+\theta_3a_3$, $b_1'=\theta_1b_1+\theta_2b_2+\theta_3b_3$, etc.; and these factors θ_1, θ_2, θ_3 may be conceived to be such that the condition in question $a_1'f_1'+b_1'g_1'+c_1'h_1'=0$ is satisfied. Similarly the second set of coefficients may be replaced by a_2', b_2', c_2', f_2', g_2', h_2', where $a_2'=\phi_1a_1+\phi_2a_2+\phi_3a_3$, etc., and the condition $a_2'f_2'+b_2'g_2'+c_2'h_2'=0$ is satisfied: and the third set by a_3', b_3', c_3', f_3', g_3', h_3', where $a_3'=\psi_1a_1+\psi_2a_2+\psi_3a_3$, etc., and the condition $a_3'f_3'+b_3'g_3'+c_3'h_3'=0$ is satisfied. We have therefore an equation $0=(a'g'h')\,x^2+\text{etc.}$, which only differs from the equation $T=0$ by having therein the accented letters in place of the unaccented ones: and, substituting for the accented letters their values, the whole divides by the determinant $(\theta\phi\psi)$, and throwing this out we obtain the required equation $T=0$.

But it is easier to obtain the equation $T=0$ directly. We have

$$\begin{aligned} \cdot\quad hy-gz+aw&=0,\\ -hx\quad\cdot\quad+fz+bw&=0,\\ gx-fy\quad\cdot\quad+cw&=0,\\ -ax-by-cz\quad\cdot\quad&=0; \end{aligned}$$

viz. in virtue of the equation $af+bg+ch=0$ which connects the six coordinates, these four equations are equivalent to two independent equations which are the equations of the line (a, b, c, f, g, h): or, what is the same thing, any three of these equations imply the fourth equation and also the relation $af+bg+ch=0$.

We might, from the three linear relations and any three of the last-mentioned four equations, eliminate a, b, c, f, g, h and so obtain the required equation $T=0$; but it is better, introducing the arbitrary coefficients α, β, γ, δ, to employ all the four equations. The result of the elimination is thus given in the form

$$\begin{vmatrix} \alpha, & w, & & & -z, & y \\ \beta, & & w, & z, & & -x \\ \gamma, & & & w, -y, & x, & \\ \delta, & x, & y, & z, & & \\ & f_1, & g_1, & h_1, & a_1, & b_1, & c_1 \\ & f_2, & g_2, & h_2, & a_2, & b_2, & c_2 \\ & f_3, & g_3, & h_3, & a_3, & b_3, & c_3 \end{vmatrix}=0,$$

viz. the left-hand side here contains the factor $-(\alpha x+\beta y+\gamma z+\delta w)$; throwing this out, we obtain the required quadric equation $T=0$. If for the calculation of T we compare the terms containing δ, we have

$$Tw=\begin{vmatrix} w, & & & & -z, & y \\ w, & w, & & z, & & -x \\ & & w, & -y, & x, & \\ f_1, & g_1, & h_1, & a_1, & b_1, & c_1 \\ f_2, & g_2, & h_2, & a_2, & b_2, & c_2 \\ f_3, & g_3, & h_3, & a_3, & b_3, & c_3 \end{vmatrix},$$

where observe that, writing $w=0$, the right-hand side vanishes as containing the factor

$$\begin{vmatrix} & -z, & y \\ z, & & -x \\ -y, & x, & \end{vmatrix}.$$

Hence the right-hand side divides by w; and one of its terms being evidently $w^3(abc)$, T contains as it should do the term $(abc)w^2$: the remaining terms can be found without any difficulty, and the foregoing expression for T is thus verified.