664.

ON THE 16-NODAL QUARTIC SURFACE.

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PROF. BORCHARDT in the Memoir "Ueber die Darstellung u. s. w." *Crelle*, t. LXXXIII. (1877), pp. 234—243, shows that the coordinates x, y, z, w may be taken as proportional to four of the double ϑ-functions, and that the equation of the surface is then Göpel's relation of the fourth order between these four functions: and he remarks at the end of the memoir that it thus appears that the coordinates x, y, z, w of a point on the surface can be expressed as proportional to algebraic functions, involving square roots only, of two arbitrary parameters ξ, ξ'.

It is interesting to develope the theory from this point of view. Writing, as in my paper, "Further investigations on the double ϑ-functions," pp. 220—233, [663],

$$\begin{aligned}
[a] &= aa', \\
[b] &= bb', \\
[c] &= cc', \\
[d] &= dd', \\
[e] &= ee', \\
[f] &= ff', \\
[ab] &= \frac{1}{(\xi-\xi')^2}\left(\sqrt{abfc'd'e'} - \sqrt{a'b'f'cde}\right)^2, \\
&\text{etc.},
\end{aligned}$$

where on the right-hand sides $a, b, \ldots, a', \ldots$ denote $a-\xi$, $b-\xi$, $\ldots$, $a-\xi'$, $\ldots$ (ξ, ξ' being here written in place of the x, x' of my paper), then the sixteen functions are proportional to constant multiples of the square-roots of these expressions; viz. the correspondence is

$$\begin{array}{cccccc} S_2=\vartheta_{13}, & S_1=\vartheta_{24}, & R_1=\vartheta_3, & R=\vartheta_{04}, & Q=\vartheta_1, & Q_2=\vartheta_{02}, \\ i\sqrt[4]{a}\sqrt{[a]}, & i\sqrt[4]{b}\sqrt{[b]}, & i\sqrt[4]{c}\sqrt{[c]}, & i\sqrt[4]{d}\sqrt{[d]}, & i\sqrt[4]{e}\sqrt{[e]}, & i\sqrt[4]{f}\sqrt{[f]}; \\ Q_1=\vartheta_2, & P_1=\vartheta_{34}, & P=\vartheta_{01}, & S=-\vartheta_{14}, & P_2=\vartheta_{12}, & P_3=\vartheta_5, \\ \sqrt[4]{ab}\sqrt{[ab]}, & \sqrt[4]{ac}\sqrt{[ac]}, & \sqrt[4]{ad}\sqrt{[ad]}, & \sqrt[4]{ae}\sqrt{[ae]}, & \sqrt[4]{bc}\sqrt{[bc]}, & \sqrt[4]{bd}\sqrt{[bd]}; \end{array}$$

$$\begin{array}{cccc} S_3=\vartheta_{23}, & Q_3=\vartheta_0, & R_3=\vartheta_4, & R_2=\vartheta_{03}, \\ \sqrt[4]{be}\sqrt{[be]}, & \sqrt[4]{cd}\sqrt{[cd]}, & \sqrt[4]{ce}\sqrt{[ce]}, & \sqrt[4]{de}\sqrt{[de]}; \end{array}$$

where, under the signs $\sqrt[4]{}$, a signifies $bcdef$, that is, $bc\,.\,bd\,.\,be\,.\,bf\,.\,cd\,.\,ce\,.\,cf\,.\,de\,.\,df\,.\,ef$, and ab signifies $abf\,.\,cde$, that is, $ab\,.\,af\,.\,bf\,.\,cd\,.\,ce\,.\,de$, in which expressions bc, bd, ..., ab, af, ... signify the differences $b-c$, $b-d$, ..., $a-b$, $a-f$, ... But in what follows, we are not concerned with the values of these constant multipliers.

Prof. Borchardt's coordinates x, y, z, w are

$$x=\vartheta_0=P;\ \ y=\vartheta_{23}=S_3;\ \ z=\vartheta_{14}=-S;\ \ w=\vartheta_5=P_3;$$

viz. P, S, P_3, S_3 are a set connected by Göpel's relation of the fourth order—and this relation can be found (according to Göpel's method) by showing that Q^2 and R^2 are each of them a linear function of the four squares P^2, P_3^2, S^2, S_3^2, and further that QR is a linear function of PS and P_3S_3; for then, squaring the expression of QR, and for Q^2 and R^2 substituting their values, we have the required relation of the fourth order between P, S, P_3, S_3.

Now we have P, S, P_3, S_3, Q, $R=$ constant multiples of $\sqrt{[ac]}$, $\sqrt{[ab]}$, $\sqrt{[cd]}$, $\sqrt{[bd]}$, $\sqrt{[b]}$, $\sqrt{[c]}$ respectively: and it of course follows that we must have the like relations between these six quantities; viz. we must have $[b]$, $[c]$ each of them a linear function of $[ac]$, $[ab]$, $[cd]$, $[bd]$; and moreover $\sqrt{[b]}\sqrt{[c]}$ a linear function of $\sqrt{[ac]}\sqrt{[ab]}$ and $\sqrt{[bd]}\sqrt{[cd]}$.

As regards this last relation, starting from the formulæ

$$\sqrt{[ac]}=\frac{1}{\xi-\xi'}\{\sqrt{acfb'd'e'}+\sqrt{a'c'f'bde}\},$$

$$\sqrt{[bd]}=\frac{1}{\xi-\xi'}\{\sqrt{bdfa'c'e'}+\sqrt{b'd'f'ace}\},$$

$$\sqrt{[ab]}=\frac{1}{\xi-\xi'}\{\sqrt{abfc'd'e'}+\sqrt{a'b'f'cde}\},$$

$$\sqrt{[cd]}=\frac{1}{\xi-\xi'}\{\sqrt{cdfa'b'e'}+\sqrt{c'd'f'abe}\},$$

we have at once

$$\sqrt{[ac]}\sqrt{[ab]} = \frac{1}{(\xi-\xi')^2}\{(afd'e' + a'f'de)\sqrt{bcb'c'} + (bc' + b'c)\sqrt{adea'd'e'}\},$$

$$\sqrt{[bd]}\sqrt{[cd]} = \frac{1}{(\xi-\xi')^2}\{(dfa'e' + d'f'ae)\sqrt{bcb'c'} + (bc' + b'c)\sqrt{adea'd'e'}\};$$

the difference of these two expressions is

$$= \frac{1}{(\xi-\xi')^2}(ad' - a'd)(fe' - f'e)\sqrt{bcb'c'},$$

where substituting for $a, d, e, f, a', \ldots$ their values $a-\xi, d-\xi, e-\xi, f-\xi, a-\xi', \ldots$ we have $ad' - a'd = (a-d)(\xi-\xi')$, $fe' - f'e = (f-e)(\xi-\xi')$; also $\sqrt{bcb'c'} = \sqrt{[b]}\sqrt{[c]}$; and we have thus the required relation

$$\sqrt{[ac]}\sqrt{[ab]} - \sqrt{[bd]}\sqrt{[cd]} = -(a-d)(e-f)\sqrt{[b]}\sqrt{[c]}.$$

As regards the first mentioned relation, if for greater generality, θ being arbitrary, we write $[\theta] = \theta\theta'$, that is, $= (\theta-\xi)(\theta-\xi')$, then it is easy to see that there exists a relation of the form

$$\nabla [\theta] = A\,[ab] + B\,[ac] + C\,[bd] + D\,[cd],$$

where $A + B + C + D = 0$. The right-hand side is thus a linear function of the differences $[ab]-[ac]$, $[ab]-[bd]$, $[ab]-[cd]$; and each of these, as the irrational terms disappear and the rational terms divide by $(\xi-\xi')^2$, is a mere linear function of 1, $\xi+\xi'$, $\xi\xi'$; whence there is a relation of the form in question. I found without much difficulty the actual formula; viz. this is

$$(a-d)(b-c)(e-f)\begin{vmatrix} 1, & e+f, & ef \\ 1, & b+c, & bc \\ 1, & a+d, & ad \end{vmatrix}[\theta]$$

$$= \begin{vmatrix} 1, & e, & f, & ef \\ 1, & b, & c, & bc \\ 1, & d, & a, & ad \\ 1, & \theta, & \theta, & \theta^2 \end{vmatrix}[ac] - \begin{vmatrix} 1, & e, & f, & ef \\ 1, & c, & b, & bc \\ 1, & d, & a, & ad \\ 1, & \theta, & \theta, & \theta^2 \end{vmatrix}[ab] - \begin{vmatrix} 1, & e, & f, & ef \\ 1, & b, & c, & bc \\ 1, & a, & d, & ad \\ 1, & \theta, & \theta, & \theta^2 \end{vmatrix}[cd] + \begin{vmatrix} 1, & e, & f, & ef \\ 1, & c, & b, & bc \\ 1, & a, & d, & ad \\ 1, & \theta, & \theta, & \theta^2 \end{vmatrix}[bd],$$

where observe that on the right-hand side the last three determinants are obtained from the first one by interchanging b, c: or a, d: or b, c and a, d simultaneously: a single interchange gives the sign $-$, but for two interchanges the sign remains $+$.

Writing successively $\theta = b$ and $\theta = c$, we obtain

$$(a-d)(e-f)\begin{vmatrix} 1, & e+f, & ef \\ 1, & b+c, & bc \\ 1, & a+d, & ad \end{vmatrix}[b]$$

$$= (a-f)(b-d)(b-e)[ac] - (a-b)(b-f)(d-e)[ab]$$
$$+ (a-b)(b-e)(d-f)[cd] - (a-e)(b-d)(b-f)[bd];$$

$$(a-d)(e-f)\begin{vmatrix} 1, & e+f, & ef \\ 1, & b+c, & bc \\ 1, & a+d, & ad \end{vmatrix}[c]$$

$$= -(a-c)(c-f)(d-e)[ac] + (a-f)(c-d)(c-e)[ab]$$
$$-(a-e)(c-d)(c-f)[cd] + (a-c)(c-e)(d-f)[bd];$$

which values of $[b]$ and $[c]$, combined with the foregoing equation

$$(a-d)(e-f)\sqrt{[b]}\sqrt{[c]} = -\sqrt{[ac]}\sqrt{[ab]} + \sqrt{[cd]}\sqrt{[bd]},$$

give the required quartic equation between $\sqrt{[ac]}$, $\sqrt{[ab]}$, $\sqrt{[cd]}$, $\sqrt{[bd]}$.

Cambridge, 2 *August*, 1877.