656.

ON THE THEORY OF PARTIAL DIFFERENTIAL EQUATIONS.

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IN what follows, any letter not otherwise explained denotes a function of certain variables (x, y, p, q), or (x, y, z, p, q, r), &c., as will be stated in each particular case.

An equation $a=$const. denotes that the function a of the variables is, in fact, a constant (viz. by such equation we establish a relation between the variables): and when this is so, we use the same letter a to denote the constant value of the function in question; I find this a very convenient notation.

Thus if the variables are x, y, z, p, q, r and if p, q, r are the differential coefficients in regard to x, y, z respectively of a function V of x, y, z, then H (as a letter not otherwise explained) denotes a function of x, y, z, p, q, r and considering it as a given function,

$$\text{H} = \text{const.}$$

will be a partial differential equation containing the constant H. For instance, if H denote the function $pqr - xyz$, H = const. is the partial differential equation, $pqr - xyz = \text{H}$ (a given constant).

The integration of the partial differential equation, H = const., depends upon that of the linear partial differential equation

$$(\text{H}, \Theta) = 0,$$

where as usual (H, Θ) signifies

$$\frac{\partial(\text{H}, \Theta)}{\partial(p, x)} + \frac{\partial(\text{H}, \Theta)}{\partial(q, y)} + \frac{\partial(\text{H}, \Theta)}{\partial(r, z)}.$$

It can be effected if we know two conjugate solutions a, b of the equation $(\text{H}, \Theta) = 0$, viz. a, b as solutions are such that $(\text{H}, a) = 0$, $(\text{H}, b) = 0$, and (as conjugate solutions) are also such that $(a, b) = 0$; in this case if from the equations

$$\text{H} = \text{const.}, \quad a = \text{const.}, \quad b = \text{const.}$$

we determine p, q, r as functions of x, y, z, the resulting value of $p\,dx + q\,dy + r\,dz$ is an exact differential, and we have

$$V = \lambda + \int (p\,dx + q\,dy + r\,dz),$$

a solution containing three arbitrary constants, λ, a, b, and therefore a complete solution of the proposed partial differential equation $\mathrm{H} = \text{const.}$

But (as is known) there is a different process of integration, for which the conjugate solutions are not required, and which has reference to a system of initial values x_0, y_0, z_0, p_0, q_0, r_0: viz. if the independent solutions of $(\mathrm{H}, \Theta) = 0$, are a, b, c, d, e, and if a_0, b_0, c_0, d_0, e_0 denote respectively the same functions of the initial variables that a, b, c, d, e are of x, y, z, p, q, r, then if from the equations

$$a = a_0,\quad b = b_0,\quad c = c_0,\quad d = d_0,\quad e = e_0,\quad \mathrm{H} = \text{const.}$$

we express p, q, r as functions of x, y, z and of x_0, y_0, z_0, H, these last being regarded as constants, we have $p\,dx + q\,dy + r\,dz$ an exact differential, and

$$V = \lambda + \int (p\,dx + q\,dy + r\,dz),$$

a solution containing the constants λ, x_0, y_0, z_0 (that is, one supernumerary constant), and as such a complete solution.

It is interesting to prove directly that $p\,dx + q\,dy + r\,dz$ is an exact differential.

I consider first the more simple case where the variables are p, q, x, y. Here p, q are to be found from the equations

$$a = a_0,\quad b = b_0,\quad c = c_0,\quad \mathrm{H} = \text{const.}$$

and it is to be shown that $p\,dx + q\,dy$ is an exact differential.

Considering p, q, p_0, q_0 as functions of the independent variables x, y, then differentiating in regard to x, and eliminating $\frac{dp}{dx}$, $\frac{dp_0}{dx}$, $\frac{dq_0}{dx}$, we have

$$\begin{vmatrix} \frac{da}{dx} + \frac{da}{dq}\frac{dq}{dx}, & \frac{da}{dp}, & \frac{da_0}{dp_0}, & \frac{da_0}{dq_0} \\ \frac{db}{dx} + \frac{db}{dq}\frac{dq}{dx}, & \frac{db}{dp}, & \frac{db_0}{dp_0}, & \frac{db_0}{dq_0} \\ \frac{dc}{dx} + \frac{dc}{dq}\frac{dq}{dx}, & \frac{dc}{dp}, & \frac{dc_0}{dp_0}, & \frac{dc_0}{dq_0} \\ \frac{d\mathrm{H}}{dx} + \frac{d\mathrm{H}}{dq}\frac{dq}{dx}, & \frac{d\mathrm{H}}{dp}, & 0, & 0 \end{vmatrix} = 0,$$

or introducing a well-known notation for functional determinants, and expanding the determinant, this is

$$\frac{\partial(a_0,\ b_0)}{\partial(p_0,\ q_0)}\left\{\frac{\partial(\mathrm{H},\ c)}{\partial(p,\ x)}+\frac{\partial(\mathrm{H},\ c)}{\partial(p,\ q)}\frac{dq}{dx}\right\}+\&\text{c.}=0.$$

But in the same way

$$\frac{\partial(a_0,\ b_0)}{\partial(p_0,\ q_0)}\left\{\frac{\partial(\mathrm{H},\ c)}{\partial(q,\ y)}+\frac{\partial(\mathrm{H},\ c)}{\partial(q,\ p)}\frac{dp}{dy}\right\}+\&\text{c.}=0;$$

or adding these, attending to the value of (H, c), and observing that $\frac{\partial(\mathrm{H},\ c)}{\partial(q,\ p)}=-\frac{\partial(\mathrm{H},\ c)}{\partial(p,\ q)}$ we have

$$\frac{\partial(a_0,\ b_0)}{\partial(p_0,\ q_0)}\left\{(\mathrm{H},\ c)+\frac{\partial(\mathrm{H},\ c)}{\partial(p,\ q)}\left(\frac{dq}{dx}-\frac{dp}{dy}\right)\right\}+\&\text{c.}=0,$$

the terms denoted by the &c. being the like terms with b, c, a and c, a, b in place of a, b, c. We have (H, a) = 0, (H, b) = 0, (H, c) = 0, and the equation in fact is

$$\left\{\Sigma\frac{\partial(a_0,\ b_0)}{\partial(p,\ q)}\frac{\partial(\mathrm{H},\ c)}{\partial(p,\ q)}\right\}\left(\frac{dq}{dx}-\frac{dp}{dy}\right)=0;$$

viz. we have $\frac{dq}{dx}-\frac{dp}{dy}=0$, the condition for the exact differential.

Coming now to the case where the variables are x, y, z, p, q, r, and in the six equations treating p, q, r, p_0, q_0, r_0 as functions of the independent variables x, y, z,—then differentiating with regard to x and proceeding as before, we find for $\frac{dr}{dx}$ the equation

$$\frac{\partial(c_0,\ d_0,\ e_0)}{\partial(p_0,\ q_0,\ r_0)}\left\{\frac{dr}{dx}\frac{\partial(a,\ b,\ \mathrm{H})}{\partial(r,\ p,\ q)}+\frac{\partial(a,\ l,\ \mathrm{H})}{\partial(x,\ p,\ q)}\right\}+\&\text{c.}=0.$$

We have, in the same way, for $\frac{dp}{dz}$ the equation

$$\frac{\partial(c_0,\ d_0,\ e_0)}{\partial(p_0,\ q_0,\ r_0)}\left\{\frac{dp}{dz}\frac{\partial(a,\ b,\ \mathrm{H})}{\partial(p,\ r,\ q)}+\frac{\partial(a,\ b,\ \mathrm{H})}{\partial(z,\ r,\ q)}\right\}+\&\text{c.}=0;$$

or, adding the two equations,

$$\frac{\partial(c_0,\ d_0,\ e_0)}{\partial(p_0,\ q_0,\ r_0)}\left\{\left(\frac{dr}{dx}-\frac{dp}{dz}\right)\frac{\partial(a,\ b,\ \mathrm{H})}{\partial(r,\ p,\ q)}+\frac{\partial(a,\ b,\ \mathrm{H})}{\partial(x,\ p,\ q)}+\frac{\partial(a,\ b,\ \mathrm{H})}{\partial(z,\ r,\ q)}\right\}+\&\text{c.}=0,$$

where the terms denoted by the &c. indicate the like terms corresponding to the different partitions of the letters a, b, c, d, e.

The equation may be simplified; we have identically

$$-\frac{da}{dq}(b,\ \mathrm{H})-\frac{db}{dq}(\mathrm{H},\ a)-\frac{d\mathrm{H}}{dq}(a,\ b)=\frac{\partial(a,\ b,\ \mathrm{H})}{\partial(x,\ p,\ q)}+\frac{\partial(a,\ b,\ \mathrm{H})}{\partial(z,\ r,\ q)},$$

or since (H, a) = 0, (b, H) = 0, the left-hand side is simply $-\frac{d\text{H}}{dq}(a, b)$, and the equation becomes

$$\frac{\partial(c_0, d_0, e_0)}{\partial(p_0, q_0, r_0)}\left\{\left(\frac{dr}{dx}-\frac{dp}{dz}\right)\frac{\partial(a, b, \text{H})}{\partial(r, p, q)}-\frac{d\text{H}}{dq}(a, b)\right\}+\&\text{c.}=0.$$

This ought to give $\frac{dr}{dx}-\frac{dp}{dz}=0$, and it will do so if only

$$\Sigma\left\{\frac{\partial(c_0, d_0, e_0)}{\partial(p_0, q_0, r_0)}(a, b)\right\}=0;$$

this is then the equation which has to be proved. By the Poisson-Jacobi theorem, (a, b) is a function of a, b, c, d, e: if we write

$$(a_0, b_0)=\frac{\partial(a_0, b_0)}{\partial(p_0, x_0)}+\frac{\partial(a_0, b_0)}{\partial(q_0, y_0)}+\frac{\partial(a_0, b_0)}{\partial(r_0, z_0)},$$

then (a_0, b_0) is the same function of a_0, b_0, c_0, d_0, e_0; but these are $= a$, b, c, d, e respectively, and we thence have $(a, b) = (a_0, b_0)$, and the theorem to be proved is

$$\Sigma\left\{\frac{\partial(c_0, d_0, e_0)}{\partial(p_0, q_0, r_0)}(a_0, b_0)\right\}=0.$$

But substituting for (a_0, b_0) its value, the function on the left-hand is (it is easy to see) the sum of the three functional determinants

$$\frac{\partial(a_0, b_0, c_0, d_0, e_0)}{\partial(p_0, q_0, r_0, p_0, x_0)}+\frac{\partial(a_0, b_0, c_0, d_0, e_0)}{\partial(p_0, q_0, r_0, q_0, y_0)}+\frac{\partial(a_0, b_0, c_0, d_0, e_0)}{\partial(p_0, q_0, r_0, r_0, z_0)},$$

each of which vanishes as containing the same letter twice in the denominator, that is, as having two identical columns; and the theorem in question is thus proved. And in the same way $\frac{dp}{dy}-\frac{dq}{dx}$, $\frac{dq}{dz}-\frac{dr}{dy}$ are each $= 0$: or we have $p\,dx + q\,dy + r\,dz$ an exact differential.

The proof would fail if the factors multiplying $\frac{dq}{dx}-\frac{dp}{dy}$, &c., or if any one of these factors, were $= 0$; I have not particularly examined this, but the meaning would be, that here the equations in question $a = a_0$, &c., H = const., are such as not to give rise to expressions for p, q, r as functions of x, y, z, x_0, y_0, z_0, H, as assumed in the theorem; whenever such expressions are obtainable, then we have $p\,dx + q\,dy + r\,dz$ an exact differential.

The proof in the case of a greater number of variables, say in the next case where the variables are x, y, z, w, p, q, r, s, would present more difficulty—but I have not proceeded further in the question.

It is worth while to put the two processes into connexion with each other: taking in each case the variables to be x, y, z, p, q, r, and the partial differential equation to be H = const.;

In the one case, a, b being conjugate solutions of $(\mathrm{H}, \Theta) = 0$,

from the equations $\mathrm{H} = \text{const.}$, $a = \text{const.}$, $b = \text{const.}$,

we find p, q, r functions of x, y, z, H, a, b:

and then $p\,dx + q\,dy + r\,dz$ is an exact differential.

In the other case, a, b, c, d, e being the solutions of $(\mathrm{H}, \Theta) = 0$,

from the equations $\mathrm{H} = \text{const.}$, $a = a_0$, $b = b_0$, $c = c_0$, $d = d_0$, $e = e_0$,

we find p, q, r functions of x_0, y_0, z_0, H:

and then $p\,dx + q\,dy + r\,dz$ is an exact differential.

It may be added that, if from the last mentioned equations we determine also p_0, q_0, r_0 as functions of x, y, z, x_0, y_0, z_0, then considering only H as a constant, we ought to have $p\,dx + q\,dy + r\,dz - p_0\,dx_0 - q_0\,dy_0 - r_0\,dz_0$ an exact differential; I have not examined the direct proof.

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