652.

ON A SEXTIC TORSE.

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THE torse having for its edge of regression or cuspidal edge the curve defined by the equations $x = \cos\phi$, $y = \sin\phi$, $z = \cos 2\phi$, is an interesting and convenient one for the construction of a model, and it is here considered partly from that point of view.

The edge is a quadriquadric curve, the intersection of the cylinder $x^2 + y^2 = 1$ with the parabolic hyperboloid $z = x^2 - y^2$; the cylinder regarded as a cone having its vertex at infinity on the line $x = 0$, $y = 0$, viz. the vertex is on the hyperboloid, or the curve is a nodal quadriquadric (the node being thus an isolated point at infinity on the line in question), and the torse is consequently of the order $8 - 2, = 6$, viz. it is a sextic torse.

The edge is a bent oval situate on the cylinder $x^2 + y^2 = 1$, such that, regarding ϕ as the azimuth (or angle measured along the circular base from its intersection with the axis of x), the altitude z is given by the equation $z = \cos 2\phi$; viz. there are in the plane xz, or, say in the planes xz, $x'z$, two maxima altitudes $z = 1$, and in the plane yz, or, say in the planes yz and $y'z$, two minima altitudes $z = -1$. The sections by these principal planes are, as is seen at once, nodal curves on the surface; they are, in fact, the cubic curves $z = 3 - \dfrac{2}{x^2}$, viz. here as x increases from ± 1 to $\pm\infty$, z increases from the before-mentioned value 1 to 3, and $z = -3 + \dfrac{2}{y^2}$, viz. as y increases from ± 1 to $\pm\infty$, z decreases from the before-mentioned value -1 to -3. The two half-sheets (which meet in the cuspidal edge) intersect each other along these nodal lines, in suchwise that the section of the surface by any axial plane (plane through the line $x = 0$, $y = 0$) is a curve having a cusp on the cuspidal edge, and such that when the axial plane coincides with either of the principal planes $x = 0$, $y = 0$, the two half-branches of the curve coincide together with the portions which lie outside the cylinder $x^2+y^2=1$, in fact, the portions referred to above, of the nodal curve in the plane in question; the portions which lie inside the cylinder are acnodal or isolated curves without any real sheet through them. It may be added, in the way of general description, that the section of the surface by any cylinder $x^2+y^2=c^2$ $(c>1)$ is a curve of the form $z=C\cos(2\theta \pm B)$, θ the angle along the base of the cylinder from the intersection with the axis of x; C, B are functions of c; viz. we have for the two half sheets respectively

$$z=C\cos(2\theta+B) \text{ and } z=C\cos(2\theta-B),$$

each curve having thus the two maxima $+C$, and the two minima $-C$; and the two curves intersect each other at the four points in the two principal planes respectively; viz. the points for which $\theta=0, 90°, 180°, 270°$, and $z=C\cos B, -C\cos B, C\cos B, -C\cos B$ accordingly.

Proceeding to discuss the surface analytically, we have for the equations of a generating line

$$\frac{x-\cos\phi}{-\sin\phi}=\frac{y-\sin\phi}{\cos\phi}=\frac{z-\cos 2\phi}{-2\sin 2\phi}, \quad =\rho \text{ suppose},$$

or say

$$\begin{aligned} x &= \cos\ \phi - \ \rho\sin\ \phi, \\ y &= \sin\ \phi + \ \rho\cos\ \phi, \\ z &= \cos 2\phi - 2\rho\sin 2\phi, \end{aligned}$$

which equations, considering therein ρ, ϕ as arbitrary parameters, determine the surface.

Writing $x=0$, we find $y=\dfrac{1}{\sin\phi}$, and then $z=-3+2\sin^2\phi$, viz. we have

$$x=0, \quad z=-3+\frac{2}{y^2}, \text{ for section in plane } yz;$$

and, similarly, writing $y=0$, we find $x=\dfrac{1}{\cos\phi}$, and then $z=3-2\cos^2\phi$, viz.

$$y=0, \quad z=3-\frac{2}{x^2} \text{ for section by plane } xz.$$

By what precedes, these are nodal curves, crunodal for the portions

$$(y=\pm 1 \text{ to } \pm\infty,\ z=-1 \text{ to } -3) \text{ and } (x=\pm 1 \text{ to } \pm\infty,\ z=1 \text{ to } 3)$$

respectively, acnodal for the remaining portions $y<\pm 1$, $x<\pm 1$ respectively.

Writing $x=r\cos\theta$, $y=r\sin\theta$, so that the coordinates of a point on the surface are r, θ, z, where $r=\sqrt{(x^2+y^2)}$ is the projected distance, θ is the azimuth from the axis of x, and z is the altitude, we have

$$\begin{aligned} r\cos\theta &= \cos\ \phi - \ \rho\sin\ \phi, \\ r\sin\theta &= \sin\ \phi + \ \rho\cos\ \phi, \\ z &= \cos 2\phi - 2\rho\sin 2\phi. \end{aligned}$$

We have $r^2 = 1 + \rho^2$; and thence also, if $\tan\alpha = 2\rho$, $= \pm 2\sqrt{(r^2-1)}$, that is,

$$\cos\alpha = \frac{1}{\sqrt{(4r^2-3)}},\quad \sin\alpha = \pm\frac{2\sqrt{(r^2-1)}}{\sqrt{(4r^2-3)}},$$

then

$$z = \sqrt{(4r^2-3)}\cos(2\phi + \alpha),$$

showing that for a given value of r (or section by the cylinder $x^2+y^2=r^2$) the maximum and minimum values of z are $z = \pm\sqrt{(4r^2-3)}$.

But proceeding to eliminate ϕ, we find

$$r^2\cos 2\theta = (1-\rho^2)\cos 2\phi - 2\rho\sin 2\phi,$$
$$r^2\sin 2\theta = 2\rho\cos 2\phi + (1-\rho^2)\sin 2\phi;$$

or multiplying these by $1+3\rho^2$ and $2\rho^3$ and adding

$$r^2\{(1+3\rho^2)\cos 2\theta + 2\rho^3\sin 2\theta\} = (1+\rho^2)^2(\cos 2\phi - 2\rho\sin 2\phi),$$

that is,

$$r^2\{(3r^2-2)\cos 2\theta \pm 2(r^2-1)^{\frac{3}{2}}\sin 2\theta\} = r^4 z;$$

or, finally,

$$r^2 z = (3r^2-2)\cos 2\theta \pm 2(r^2-1)^{\frac{3}{2}}\sin 2\theta,$$

which is the equation of the surface in terms of the coordinates r, θ, z.

Observing that $(3r^2-2)^2 + 4(r^2-1)^3 = r^4(4r^2-3)$, we may write

$$r^2\sqrt{(4r^2-3)}\cos\beta = 3r^2-2,$$
$$r^2\sqrt{(4r^2-3)}\sin\beta = 2(r^2-1)^{\frac{3}{2}},$$

and therefore also

$$\tan\beta = \frac{2(r^2-1)^{\frac{3}{2}}}{3r^2-2},$$

and the equation thus becomes

$$z = \sqrt{(4r^2-3)}\cos(2\theta \mp \beta),$$

where z is the altitude belonging to the azimuth θ in the cylindrical section, radius r. The maxima and minima altitudes are $\pm\sqrt{(4r^2-3)}$, and these correspond to the values $\theta = \pm\frac{1}{2}\beta$, $\frac{1}{2}\pi \pm \frac{1}{2}\beta$, $\pi \pm \frac{1}{2}\beta$, $\frac{3}{2}\pi \pm \frac{1}{2}\beta$; it is to be further noticed that when $r=1$, we have $\beta = 0$, but as r increases and becomes ultimately infinite, β increases to $\frac{1}{2}\pi$, that is, $\frac{1}{2}\beta$ increases from 0 to $\frac{1}{4}\pi$.

It may be noticed that the surface is a peculiar kind of deformation, obtained by giving proper rotations to the several cylindrical sections of the surface $z = \sqrt{(4r^2-3)}\cos 2\theta$; viz. in rectangular coordinates this is $r^2 z = \sqrt{(4r^2-3)}(x^2-y^2)$, that is,

$$(x^2+y^2)^2 z^2 - \{4(x^2+y^2)-3\}(x^2-y^2)^2 = 0.$$

To obtain the equation in rectangular coordinates, we have

$$\left\{r^2 z - \frac{3r^2-2}{r^2}(x^2-y^2)\right\}^2 - 16(r^2-1)^3\frac{x^2y^2}{r^4} = 0,$$

viz. this is

$$r^4z^2 - 2z(3r^2-2)(x^2-y^2) + (3r^2-2)^2\left(1 - \frac{4x^2y^2}{r^4}\right) - 16(r^2-1)^3\frac{x^2y^2}{r^4} = 0,$$

or, what is the same thing, it is

$$r^4z^2 - 2z(3y^2-2)(x^2-y^2) + (3r^2-2)^2 - \frac{4x^2y^2}{r^4}\{4(r^2-1)^3 + (3r^2-2)^2\} = 0,$$

viz. the term in { } being $r^4(4r^2-3)$, this is

$$r^4z^2 - 2z(3r^2-2)(x^2-y^2) + (3r^2-2)^2 - 4x^2y^2(4r^2-3) = 0,$$

or say

$$z^2(x^2+y^2)^2 - 2z(3x^2+3y^2-2)(x^2-y^2) + (3x^2+3y^2-2)^2 - 4x^2y^2(4x^2+4y^2-3) = 0.$$

This may also be written

$$\{z(x^2-y^2) - 3x^2 - 3y^2 + 2\}^2 + 4x^2y^2(z^2 - 4x^2 - 4y^2 + 3) = 0,$$

a form which puts in evidence the nodal curves

$$x = 0,\ xy^2 = -3y^2 + 2, \text{ and } y = 0,\ zx^2 = 3x^2 - 2.$$

It shows also that the quadric cone $z^2 - 4x^2 - 4y^2 + 3 = 0$ touches the surface along the curve of intersection with the surface $z(x^2-y^2) - 3(x^2+y^2) + 2 = 0$. This is, in fact, the curve of maxima and minima of the cylindrical sections, viz. reverting to the form $z = \sqrt{(4r^2-3)}\cos(2\theta \mp \beta)$, or, if for greater clearness, attending only to one sheet of the surface, we write it $z = \sqrt{(4r^2-3)}\cos(2\theta - \beta)$, we have a maximum, $z = \sqrt{(4r^2-3)}$, for $2\theta = \beta$ (or $2\pi + \beta$), giving

$$\cos 2\theta = \cos\beta,\ = \frac{3r^2-2}{r^2\sqrt{(4r^2-3)}},\ = \frac{3r^2-2}{r^2z}:$$

and a minimum, $z = -\sqrt{(4r^2-3)}$, for $2\theta = \pi + \beta$ (or $3\pi + \beta$), giving

$$\cos 2\theta = -\cos\beta = -\frac{3r^2-2}{r^2\sqrt{(4r^2-3)}},\ = \frac{3r^2-2}{r^2z};$$

viz. the locus is $z^2 = 4(r^2-3)$, $z(x^2-y^2) = 3r^2-2$; and for $z = \sqrt{(4r^2-3)}\cos(2\theta+\beta)$ we find the same locus, viz. the equations of the locus are

$$z^2 - 4x^2 - 4y^2 + 3 = 0,\quad z(x^2-y^2) - 3x^2 - 3y^2 + 2 = 0,$$

as above.

To put in evidence the cuspidal edge, write for a moment $\zeta = z - x^2 + y^2$, the equation becomes

$$\{\zeta(x^2-y^2) + (r^2-1)(r^2-2) - 4x^2y^2\}^2 + 4x^2y^2\{\zeta^2 + 2\zeta(x^2-y^2) + (r^2-1)(r^2-3) - 4x^2y^2\} = 0;$$

viz. this is

$$\zeta^2r^4 + 2\zeta(x^2-y^2)(r^2-1)(r^2-2) + (r^2-1)^2(r^2-2)^2 - 4x^2y^2(r^2-1)^2 = 0,$$

or writing the last term thereof in the form

$$-\{r^2 - (x^2-y^2)^2\}(r^2-1)^2,$$

and then putting $r^2 = 1 + U$, the equation is

$$\zeta^2(1 + 2U + U^2) + 2\zeta U(U-1)(x^2-y^2) + U^2(U-1)^2 - U^2\{(U+1)^2 - (x^2-y^2)^2\} = 0;$$

viz. this is

$$\{\zeta - U(x^2-y^2)\}^2 + 2U\{\zeta^2 + \zeta U(x^2-y^2) - 2U^2\} + \zeta^2U^2 = 0,$$

showing the cuspidal edge $\zeta=0$, $U=0$, viz. $z=x^2-y^2$, $x^2+y^2=1$. Moreover, along the cuspidal edge the surface is touched by $\zeta-U(x^2-y^2)=0$, that is, by $z-(x^4-y^4)=0$; and at the points where this tangent surface again meets the surface we have $(x^2-y^2)^2(x^2+y^2+3)-4=0$; viz. the surface contains upon itself the curve represented by this last equation, and $z-(x^4-y^4)=0$.

As a verification, in the form

$$\{z(x^2-y^2)-3x^2-3y^2+2\}^2+4x^2y^2(z^2-4x^2-4y^2+3)=0$$

of the equation of the surface, write $z=x^4-y^4$. If for a moment $x^2+y^2=\lambda$, $x^2-y^2=\mu$, then the value of z is $z=\lambda\mu$, and the equation becomes

$$(\lambda\mu^2-3\lambda+2)^2+(\lambda^2-\mu^2)(\lambda^2\mu^2-4\lambda+3)=0,$$

that is,

$$\mu^2(\lambda^4-6\lambda^2+8\lambda-3)-4\lambda^3+12\lambda^2-12\lambda+4=0;$$

or, what is the same thing,

$$(\lambda-1)^3\{\mu^2(\lambda+3)-4\}=0,$$

so that we have $(\lambda-1)^3=0$, or else $\mu^2(\lambda+3)-4=0$; viz. $(x^2+y^2-1)^3=0$, or else $(x^2-y^2)^2(x^2+y^2+3)-4=0$, agreeing with the former result.

In polar coordinates, the surface is touched along the cuspidal edge by the surface $z=r^4\cos 2\theta$, and where this again meets the surface we have $r^4(r^2+3)\cos^2 2\theta-4=0$.

For the model, taking the unit to be 1 inch, I suppose that for the edge of regression we have

$$x=2\cos\phi,\ y=2\sin\phi,\ z=5+(\cdot 45)\cos 2\phi;$$

viz. the curve is situate on a cylinder radius 2 inches. And I construct in zinc-plate the cylindric sections, or say the templets, for one sheet of the surface, for the several radii 2, 3,.., 8 inches; taking the radius as k inches, the circumference of the cylinder, or entire base of the flattened templet, is $=2k\pi$; and the altitude, writing 2θ in place of $2\theta-\beta$ as above, is given by the formula $z=5+(\cdot 45)\sqrt{(k^2-3)}\cos 2\theta$, so that the half altitude of the wave is $=(\cdot 45)\sqrt{(k^2-3)}$; having this value, the curve is at once constructed geometrically. We have, moreover, $\cos\beta=\dfrac{3k^2-8}{k^2\sqrt{(k^2-3)}}$; the numerical values then are

k	$2k\pi$	$(\cdot 45)\sqrt{(k^2-3)}$	$\dfrac{3k^2-8}{k^2\sqrt{(k^2-3)}}$	$\frac{1}{2}\beta$
2	12·57	0·45	1·00	0°
3	18·85	1·10	·86	15
4	25·13	1·62	·69	23
5	31·42	2·11	·57	$27\frac{1}{2}$
6	37·70	2·59	·48	$30\frac{1}{2}$
7	43·98	3·05	·42	$32\frac{1}{2}$
8	50·27	3·51	·36	34

the altitudes in the successive templets being thus included between the limits $5\pm 0{\cdot}45$, $5\pm 1{\cdot}10$,.., $5\pm 3{\cdot}51$.