632.

ON ARONHOLD'S INTEGRATION-FORMULA.

[From the *Messenger of Mathematics*, vol. v. (1876), pp. 88—90.]

THE fundamental theorem in Aronhold's Memoir, "Ueber eine neue algebraische Behandlungsweise der Integrale...$\Pi(x, y)\,dx$, &c.," *Crelle*, t. LXI. (1863), pp. 95—145, is a theorem of *indefinite* integration. The form is

$$\Lambda\int\frac{dx}{(\alpha x+\beta y+\gamma)(hx+by+f)}=\log\frac{(a\xi+h\eta+g)\,x+(h\xi+b\eta+f)\,y+g\xi+f\eta+c}{\alpha x+\beta y+\gamma},$$

where y is a certain irrational function of x, determined by a quadric equation, and the other symbols denote constants connected by certain relations; viz. writing, for shortness,

$$U=(a,\ b,\ c,\ f,\ g,\ h\between x,\ y,\ 1)^2,\ =(a,\ldots\between x,\ y,\ 1)^2 \text{ for shortness,}$$

that is,

$$=ax^2+2hxy+by^2+2fy+2gx+c\,;$$

$$W=(a,\ b,\ c,\ f,\ g,\ h\between x,\ y,\ 1\between \xi,\ \eta,\ 1),\ =(a,\ldots\between x,\ y,\ 1\between \xi,\ \eta,\ 1),$$

that is,

$$=(ax+hy+g)\,\xi+(hx+by+f)\,\eta+gx+fy+c,$$

or

$$(a\xi+h\eta+g)\,x+(h\xi+b\eta+f)\,y+g\xi+f\eta+c\,;$$

$$(P\ ,\ Q\ ,\ R\ )=(ax+hy+g,\quad hx+by+f,\quad gx+fy+c),$$

$$(P_0,\ Q_0,\ R_0)=(a\xi+h\eta+g,\quad h\xi+b\eta+f,\quad g\xi+f\eta+c),$$

$$\Omega\ =\alpha x+\beta y+\gamma,$$

$$\Omega_0=\alpha\xi+\beta\eta+\gamma,$$

$$(A,\ B,\ C,\ F,\ G,\ H)=(bc-f^2,\ ca-g^2,\ ab-h^2,\ gh-af,\ hf-bg,\ fg-ch),$$

then y is determined as a function of x by the equation $U=0$, that is,

$$(a, b, c, f, g, h \between x, y, 1)^2 = 0;$$

or, what is the same thing,

$$by = -\{hx + f + \sqrt{(-Cx^2 + 2Gx - A)}\};$$

the constants α, β, ξ, η are such that

$$(a, b, c, f, g, h \between \xi, \eta, 1)^2 = 0,$$

$$\alpha\xi + \beta\eta + \gamma = 0,$$

that is,

$$\Omega_0 = 0;$$

and the value of Λ is given by

$$\Lambda^2 = -(A, B, C, F, G, H \between \alpha, \beta, \gamma)^2.$$

The theorem may therefore be written

$$\Lambda \int \frac{dx}{\Omega Q} = \log \frac{W}{\Omega},$$

where the several symbols have the significations explained above.

The verification is as follows. We ought to have

$$\frac{\Lambda\, dx}{\Omega Q} = \frac{P_0 dx + Q_0 dy}{W} - \frac{\alpha\, dx + \beta\, dy}{\Omega},$$

when dx, dy satisfy the relation $P\,dx + Q\,dy = 0$, viz. substituting for dy the value $-\dfrac{P\,dx}{Q}$, the equation becomes

$$\frac{\Lambda}{\Omega} = \frac{P_0 Q - PQ_0}{W} - \frac{\alpha Q - \beta P}{\Omega},$$

that is, substituting for Ω its value,

$$\Lambda W = (P_0 Q - PQ_0)(\alpha x + \beta y + \gamma) - (\alpha Q - \beta P)\, W.$$

On the right-hand side, substituting for W its value,

$$\text{coeff. } \alpha = x(P_0 Q - PQ_0) - Q(P_0 x + Q_0 y + R_0), \quad = Q_0 R - QR_0,$$

$$\text{coeff. } \beta = y(P_0 Q - PQ_0) + P(P_0 x + Q_0 y + R_0), \quad = R_0 P - RP_0,$$

(as at once appears by aid of the relation $U = Px + Qy + R = 0$),

$$\text{coeff. } \gamma \qquad\qquad = P_0 Q - PQ_0.$$

The equation to be verified thus is

$$\Lambda W = \begin{vmatrix} \alpha, & \beta, & \gamma \\ P_0, & Q_0, & R_0 \\ P, & Q, & R \end{vmatrix},$$

which, substituting therein for P, Q, R, P_0, Q_0, R_0, their values, and writing

$$(\lambda,\ \mu,\ \nu) = (\eta - y,\ x - \xi,\ \xi y - \eta x),$$

is in fact

$$\Lambda W = (A, \ldots \between \lambda,\ \mu,\ \nu \between \alpha,\ \beta,\ \gamma).$$

We have identically

$$(a, \ldots \between x,\ y,\ 1)^2 \,.\, (a, \ldots \between \xi,\ \eta,\ 1)^2 - W^2 = (A, \ldots \between \lambda,\ \mu,\ \nu)^2,$$

which, in virtue of $(a, \ldots \between \xi,\ \eta,\ 1)^2 = 0$, gives

$$W^2 = -(A, \ldots \between \lambda,\ \mu,\ \nu)^2;$$

and since $\Lambda^2 = -(A, \ldots \between \alpha,\ \beta,\ \gamma)^2$, the equation is thus

$$\sqrt{\{-(A, \ldots \between \alpha,\ \beta,\ \gamma)^2\}} \,.\, \sqrt{\{-(A, \ldots \between \lambda,\ \mu,\ \nu)^2\}} = (A, \ldots \between \lambda,\ \mu,\ \nu \between \alpha,\ \beta,\ \gamma),$$

that is,

$$(A, \ldots \between \alpha,\ \beta,\ \gamma)^2 \,.\, (A, \ldots \between \lambda,\ \mu,\ \nu)^2 - [(A, \ldots \between \lambda,\ \mu,\ \nu \between \alpha,\ \beta,\ \gamma)]^2 = 0.$$

The left-hand side is here identically

$$= K\,(a, \ldots \between \gamma\mu - \beta\nu,\ \alpha\nu - \gamma\lambda,\ \beta\lambda - \alpha\mu)^2:$$

substituting for λ, μ, ν their values, we find

$$(\gamma\mu - \beta\nu,\ \alpha\nu - \gamma\lambda,\ \beta\lambda - \alpha\mu) = (x\Omega_0 - \xi\Omega,\ y\Omega_0 - \eta\Omega,\ z\Omega_0 - \zeta\Omega);$$

viz. in virtue of $\Omega_0 = 0$, these are $= -\xi\Omega,\ -\eta\Omega,\ -\zeta\Omega$, and the quadric function is $= K\Omega^2\,(a, \ldots \between \xi,\ \eta,\ 1)^2$, vanishing in virtue of the relation $(a, \ldots \between \xi,\ \eta,\ 1)^2 = 0$.

The equation in question

$$\sqrt{\{-(A \ldots \between \alpha,\ \beta,\ \gamma)^2\}} \,.\, \sqrt{\{-(A \ldots \between \lambda,\ \mu,\ \nu)^2\}} = (A \ldots \between \lambda,\ \mu,\ \nu \between \alpha,\ \beta,\ \gamma)$$

is thus verified, and the theorem is proved.