# 630.

## ON AN EXPRESSION FOR $1 \pm \sin(2p+1)u$ IN TERMS OF $\sin u$.

[From the *Messenger of Mathematics*, vol. v. (1876), pp. 7, 8.]

WRITE $\sin u = x$, then we have

$$\begin{aligned} \sin u &= x, & \cos u &= \sqrt{(1-x^2)}, \\ \sin 3u &= 3x - 4x^3, & \cos 3u &= (1-4x^2)\sqrt{(1-x^2)}, \\ \sin 5u &= 5x - 20x^3 + 16x^5, & \cos 5u &= (1-12x^2+16x^4)\sqrt{(1-x^2)}, \\ &\&c. & &\&c. \end{aligned}$$

It is hence clear, that in general

$$\begin{aligned} 1 - \sin(2p+1)u &= (1 \pm x)\{(1,\ \ x)^p\}^2, \\ 1 + \sin(2p+1)u &= (1 \mp x)\{(1,\ -x)^p\}^2, \end{aligned}$$

where $(1,\ x)^p$ denotes a rational and integral function of $x$ of the order $p$, and $(1,\ -x)^p$ the same function of $-x$; for it is only in this manner that we can have

$$\cos^2(2p+1)u = (1-x^2)\{[1,\ x^2]^p\}^2.$$

We, in fact, find

$$\begin{aligned} 1 + \sin\ u &= 1 + x, \\ 1 - \sin 3u &= (1+x)(1-2x)^2, \\ 1 + \sin 5u &= (1+x)(1+2x-4x^2)^2, \\ 1 - \sin 7u &= (1+x)(1-4x-4x^2+8x^3)^2, \\ &\&c. \end{aligned}$$

and it thus appears that the form is

$$1 + (-)^p \sin(2p+1)u = (1+x)\{(1,\ x)^p\}^2.$$

To find herein the expression of the factor $(1, x)^p$, write $u = \frac{1}{2}\pi - \theta$ and consequently $x = \cos\theta$; we have therefore

$$1 + \cos(2p+1)\theta = (1 + \cos\theta)\{(1, x)^p\}^2,$$

where in the second factor on the right-hand side $x$ is retained to stand for its value $\cos\theta$. This gives

$$2\cos^2(p + \tfrac{1}{2})\theta = 2\cos^2\tfrac{1}{2}\theta\{(1, x)^p\}^2,$$

or, what is the same thing,

$$(1, x)^p = \frac{\cos(p + \frac{1}{2})\theta}{\cos\frac{1}{2}\theta},$$

viz. this is

$$= \cos p\theta - \sin p\theta \frac{\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta},$$

which is

$$= \cos p\theta - \sin p\theta \frac{1 - \cos\theta}{\sin\theta}.$$

We have

$$\cos p\theta + i\sin p\theta = \{x + i\sqrt{(1 - x^2)}\}^p$$

$$= X + i\sqrt{(1 - x^2)}\,Y, \text{ suppose,}$$

where $X$, $Y$ are rational and integral functions of $x$ of the orders $p$ and $p - 1$ respectively; that is,

$$\cos p\theta = X, \quad \sin p\theta = \sin\theta \,.\, Y,$$

and we have therefore

$$(1, x)^p = X - Y(1 - x),$$

which is the required expression for $(1, x)^p$. For instance

$$p = 3, \quad X + i\sqrt{(1 - x^2)}\,Y = \{x + i\sqrt{(1 - x^2)}\}^3;$$

that is,

$$X = \quad -3x \qquad + 4x^3$$

$$Y = -1 + 4x^2, \text{ and } \therefore \quad -(1 - x)Y = 1 - \ x - 4x^2 + 4x^3$$

so that

$$X - (1 - x)Y = 1 - 4x - 4x^2 + 8x^3, = (1, x)^2,$$

and hence

$$1 - \sin 7u = (1 + x)(1 - 4x - 4x^2 + 8x^3)^2,$$

which agrees with a result already obtained.

The foregoing value of $(1, x)^p$ may also be written

$$(1, x)^p = \frac{1}{\sin\theta}\{\sin(p + 1)\theta - \sin p\theta\},$$

which however is not practically so convenient.

The formula corresponds to a like formula in elliptic functions, viz. writing $\text{sinam}\, u = x$, the numerator of $1 + (-)^p \text{sinam}(2p + 1)u$ is

$$= (1 + x)\{(1, x)^{2p(p+1)}\}^2,$$

which is $(1 + x)$ multiplied by the square of a rational and integral function of $x$.