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A unified theory of representations for scalar-, vector- and second order tensor-valued anisotropic functions of vectors and second order tensors

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A SUBSTANTIAL generalization of Lokhin–Sedov–Boehler–Liu's isotropic extension method for representations of anisotropic tensor functions is suggested. It is shown that every scalar-, vector- and second order tensor-valued anisotropic tensor function with vector and second order tensor variables can be extended as an isotropic tensor function merely with augmented vector and second order tensor variables through some simple polynomial vector-valued and second order tensor-valued invariant tensor functions characterizing the anisotropy group. This result circumvents the difficulty involved in the usual direct generalization of the aforementioned LSBL method due to the introduction of structural tensor variables of order higher than two, and enables us to derive complete representations for various types of anisotropic tensor functions of vectors and second order tensors directly from the well-known results for isotropic tensor functions of vectors and second order tensors. All anisotropy groups describing symmetries of solid materials, including the thirty-two crystal classes and all infinitely many noncrystal classes, are considered.

Notations

T_k – the space of kth-order tensors. In particular, $T_0 \equiv R$ (the reals), $T_1 \equiv V$,

Orth – the full orthogonal group, being a subset of T_2,

Skw, Sym – the skewsymmetric and symmetric subspaces of T_2,

$D = V^a \times \mathrm{Skw}^b \times \mathrm{Sym}^c$; $E = V^r \times \mathrm{Skw}^s \times \mathrm{Sym}^t$,

$\mathbf{X} = (\mathbf{v}_\alpha; \mathbf{W}_\theta; \mathbf{A}_\sigma) \equiv (\mathbf{v}_1, \dots, \mathbf{v}_a; \mathbf{W}_1, \dots, \mathbf{W}_b; \mathbf{A}_1, \dots, \mathbf{A}_c) \in D$,

$(\mathbf{Q} * \mathbf{T})_{i_1 \dots i_k} = Q_{i_1 j_1} \cdots Q_{i_k j_k} T_{j_1 \dots j_k}$, $(\mathbf{Q} \in \mathrm{Orth}, \mathbf{T} \in T_k)$, $\mathbf{Q} * c = c$, $c \in R$,

$\Gamma(\mathbf{T}) = \{\mathbf{Q} \in \mathrm{Orth} \mid \mathbf{Q} * \mathbf{T} = \mathbf{T}\}$,

$\mathbf{Q} * \mathbf{X} = (\mathbf{Q} * \mathbf{v}_\alpha; \mathbf{Q} * \mathbf{W}_\theta; \mathbf{Q} * \mathbf{A}_\sigma)$,

$G(D, M) = \{\mathbf{F} : D \to M \subset T_k \mid \mathbf{F}(\mathbf{Q} * \mathbf{X}) = \mathbf{Q} * (\mathbf{F}(\mathbf{X})), \forall \mathbf{X} \in D, \mathbf{Q} \in G\}$ $(G \subset \mathrm{Orth})$,

$(\mathbf{Q} * \mathbf{S})(\mathbf{X}) = \mathbf{Q} * (\mathbf{S}(\mathbf{Q}^{\mathrm{T}} * \mathbf{X}))$, $\forall \mathbf{X} \in D$ $(\mathbf{Q} \in \mathrm{Orth},\ \mathbf{S} : D \to E)$,

$\mathbf{S} \cap (\mathbf{Q} * \mathbf{S}) = \{\mathbf{X}_0 \in D \mid (\mathbf{Q} * \mathbf{S})(\mathbf{X}_0) = \mathbf{S}(\mathbf{X}_0)\}$,

$\overset{m}{\otimes} \mathbf{K} = \mathbf{K}_1 \otimes \mathbf{K}_2 \otimes \cdots \otimes \mathbf{K}_m$, $\mathbf{K}_1 = \cdots = \mathbf{K}_m = \mathbf{K} \in T_k$,

$(\mathbf{G} \odot \mathbf{Z})_{i_1 \dots i_p} = G_{i_1 \dots i_p j_1 \dots j_q} Z_{j_1 \dots j_q}$ $(\mathbf{Z} \in T_q,\ \mathbf{G} \in T_{p+q})$,

$\mathbf{G} \odot \mathbf{v} = \mathbf{G}\mathbf{v}$, $\mathbf{v} \in V$; $\mathbf{G} \odot \mathbf{B} = \mathbf{G} : \mathbf{B}$, $\mathbf{B} \in T_2$,

$D(\mathbf{u}) = \{x\mathbf{u} \mid x \in R\}^a \times \{x\mathbf{E}\mathbf{u} \mid x \in R\}^b \times \{x\mathbf{I} + y\mathbf{u} \otimes \mathbf{u} \mid x, y \in R\}^c$ $(\mathbf{0} \neq \mathbf{u} \in V)$,

$\mathbf{e}$, $\mathbf{n}$	two orthonormal vectors,
$\mathbf{I}$, $\mathbf{E}$	the second order identity tensor; the third order Eddington tensor,
$\mathbf{R}_{\mathbf{a}}^{\theta}$	the right-handed rotation through the angle θ about an axis represented by $\mathbf{0} \neq \mathbf{a} \in V$,
$S \setminus T$	the set of all elements that belong to the set S but not to the set T,
$\mathbf{u} \cdot \mathbf{v}$	the scalar product of the vectors $\mathbf{u}, \mathbf{v} \in V$,
$< \mathbf{z}, \mathbf{e} >$	the angle between the vectors $\mathbf{z}$ and $\mathbf{e}$,
$\mathbf{q}(\mathbf{A})$	a vector associated with the symmetry tensor $\mathbf{A} \in \mathrm{Sym}$, refer to (2.9),
$\mathbf{v}^{\circ}$	the perpendicular projection of the vector $\mathbf{v}$ on the $\mathbf{n}$-plane, refer to (2.21).

1. Introduction

SCALAR-, VECTOR- AND SECOND ORDER TENSOR-VALUED FUNCTIONS of vectors and second order tensors provide mathematical models for macroscopic physical behaviour of materials. The principle of material frame-indifference and material symmetry require that such tensor functions modelling material behaviours, i.e. constitutive relations of materials, possess a combined invariance under the material symmetry group. The central problem of theory of representations for tensor functions is to determine general reduced forms of tensor functions that are invariant under various given material symmetry groups and hence, it constitutes a rational basis for a consistent mathematical modelling of complex material behaviours (see RIVLIN [21], TRUESDELL and NOLL [43], MURAKAMI and SAWCZUK [18], TELEGA [42], BOEHLER [7, 8], ERINGEN and MAUGIN [10], KIRAL and ERINGEN [14], BETTEN [3], SMITH [34], and ZHENG [61], *et al.*, for some applications of tensor function representation theory in formulating constitutive equations of materials). In the past decades, representations for isotropic and anisotropic functions of vectors and second order tensors have been extensively studied and many significant results for polynomial and nonpolynomial representations have been obtained (see TRUESDELL and NOLL [43] and SPENCER [38] for the results on polynomial representations up to their respective concerned years; see BOEHLER [7, 8], KIRAL and ERINGEN [14], SMITH [34] and ZHENG [61] for the subsequent development; see RYCHLEWSKI and ZHANG [25] for a comprehensive review and comments). However, most of the established results were confined to *integrity bases* for polynomial scalar-valued functions (see PIPKIN and RIVLIN [20], ADKINS [1, 2], SPENCER and RIVLIN [39, 40, 41], SPENCER [37], SMITH and RIVLIN [36], SMITH [30, 33], SMITH and KIRAL [31], and KIRAL and SMITH [12, 13], *et al.*, for some general results of this aspect; see also SPENCER [38], KIRAL and ERINGEN [14], and SMITH [34] for details). General aspects of representation problems for most types of anisotropic functions remain open, except for isotropic, transversely isotropic and orthotropic functions and for some other particular cases, etc. (see WANG [45], SMITH [32], BOEHLER [5], PENNISI and TROVATO [19], ZHENG [59, 60], JEMIOŁO and TELEGA [11], *et al.*).

The main method in current use for deriving representations of anisotropic functions is the *Lokhin–Sedov–Boehler–Liu isotropic extension method*(¹) (see LOKHIN and SEDOV [17], BOEHLER [6, 8] and LIU [16]). It was through Boehler's and Liu's works that this method became known. According to Boehler and Liu, through some vectors and second order tensors characterizing the anisotropy group, an anisotropic function can be extended as an isotropic function with augmented tensor variables and hence, the representation problem for the former can be reduced to that for the latter. For such simple anisotropy groups as transverse isotropy groups and triclinic, monoclinic and rhombic crystal classes, isotropic extension functions merely with vector and second order tensor variables can be established using the above method, as has been shown by BOEHLER [7, 8] and LIU [16]. Therefore, the well-known results for representations for isotropic functions of vectors and second order tensors (see WANG [45], SMITH [32], BOEHLER [5], and PENNISI and TROVATO [19], *et al.*) can be used to derive the desired results for representations for anisotropic functions of vectors and second order tensors relative to the foregoing anisotropy groups. However, it is known that any set of vectors and second order tensors is not enough to characterize any anisotropy group except those mentioned above, since the symmetry group of any vector or second order tensor involves only two-fold and/or ∞-fold symmetry. In view of this, a direct generalization of the aforementioned LSBL method has been suggested (see ZHANG and RYCHLEWSKI [57] and ZHENG and SPENCER [58]; see the monograph by RYCHLEWSKI [22] for a comprehensive and coherent account of this aspect), which realizes isotropic extension of anisotropic functions by means of additional tensor variables of higher order characterizing the anisotropy group. The latter were introduced earlier as *anisotropic tensors* or *structural tensors* by various authors (see SMITH and RIVLIN [35, 36], SIROTIN [28, 29], SEDOV and LOKHIN [26], *et al.*), and shown to be valid for all anisotropy groups. However, for each anisotropy group other than those mentioned before, such direct generalization of LSBL method results in isotropic extension functions whose variables include tensors of order higher than two, and representation problems for them are difficult (see the comments by ZHANG and RYCHLEWSKI [57] and RYCHLEWSKI and ZHANG [25]). In reality, even for the simplest case of this aspect, i.e. the isotropic scalar-valued function of a single fourth-order tensor, such as the elasticity tensor, a complete functional basis has not been obtained until the recent work by this author (see XIAO [52]; see also RYCHLEWSKI [23], BETTEN and HELISCH [4], and BOEHLER, KIRILLOV and ONAT [9], *et al.*, for some other results; see also the comments by RYCHLEWSKI and ZHANG [25], §5 and RYCHLEWSKI [24], §2).

Recently, this author (see XIAO and GUO [46] and XIAO [49]) has made a substantial extension of the above-mentioned LSBL method. It has been shown

(¹) It seems that the expression *isotropic extension* was first introduced by RYCHLEWSKI and ZHANG in [25], which was followed in [49].

that through some vector-valued and second order tensor-valued invariant tensor functions, an anisotropic function of vector and second order tensor variables can be extended as an isotropic function whose variables consist merely of vectors and second order tensors, and hence the aforesaid difficulty involved in the aforementioned direct generalization is circumvented. In this paper, basing upon a fundamental isotropic extension theorem for anisotropic functions (see XIAO and GUO [46] and XIAO [49] and below), we shall systematically construct isotropic extension functions merely with augmented vector and second order tensor variables for scalar-, vector- and second order tensor-valued anisotropic functions of vector and second order tensor variables relative to all the thirty-two crystal classes and all noncrystal classes. Employing these results and the well-known results for representations for isotropic functions of vectors and second order tensors, one can readily derive complete or even complete irreducible representations for various types of anisotropic functions (see the recent results by this author [47–51, 53–55]).

The early forms of most of the results given in this paper were reported in a summary by this author (see [49]). In the latter, complete proofs for each presented result were sought and moreover, results for the icosahedral class I_h and the infinitely many noncrystal classes D_{2md} and S_{4m}, where $m = 2, 3, \ldots$, were left open. In this article, we present new results for subgroups of the transverse isotropy group $D_{\infty h}$, which simplify the corresponding results given in [49], and moreover, we provide results for the icosahedral class I_h and for all noncrystal classes D_{2md} and S_{4m}. Complete proofs for all these results will be given.

It should be pointed out that the commonly-considered material symmetric groups in solid mechanics are the five classes of transverse isotropy groups, the thirty-two crystal classes and the full orthogonal group etc. (see, e.g., TRUESDELL and NOLL [43] and SPENCER [38]), since for a long time it has been believed that the just-mentioned orthogonal subgroups seem to exhaust symmetries of all known solids. As a result, one may doubt the reality of any noncrystallographic point group other than those just mentioned in describing symmetry of any real solid. For this, we would call attention to the recent advances in modern crystallography, especially the discovery of *quasi-crystals* (see, e.g., VAINSHTEIN [44] and SENECHAL [27] and the references therein).

2. The fundamental isotropic extension theorem and others

Throughout this paper, vector and tensor mean a three-dimensional vector and tensor. The Schoenflies symbol will be used to denote the orthogonal subgroup classes (see SPENCER [38] and VAINSHTEIN [44] for an account of crystal classes and noncrystal classes). Moreover, M will be used to represent any of the sets R, V, Skw and Sym, unless otherwise indicated.

2.1. The fundamental isotropic extension theorem

The succeeding account will be mainly based on the following fact.

THEOREM A. (*ISOTROPIC EXTENSION THEOREM*) *Let* $G \subset \mathrm{Orth}$ *be an anisotropy group, i.e. an orthogonal subgroup other than the full and proper orthogonal groups. Let* $M \subset T_k$ *be a subspace that is invariant under the group* G. *Moreover, let*

$$\mathbf{S} : D \equiv V^a \times \mathrm{Skw}^b \times \mathrm{Sym}^c \to E \equiv V^r \times \mathrm{Skw}^s \times \mathrm{Sym}^t \tag{2.1}$$

be a set of vector-valued and second order tensor-valued functions that are invariant under the group G *and satisfy the following condition*

$$\mathbf{F}(\mathbf{Q}^{\mathrm{T}} * \mathbf{X}_0) = \mathbf{Q}^{\mathrm{T}} * (\mathbf{F}(\mathbf{X}_0)) \qquad (\forall\, \mathbf{F} \in G(D, M), \quad \mathbf{Q} \in \mathrm{Orth}, \quad \mathbf{X}_0 \in \mathbf{S} \cap (\mathbf{Q} * \mathbf{S})). \tag{2.2}$$

Then a tensor function $\mathbf{\Psi} : D \to M \subset T_k$ *is invariant under the group* G *iff there is an isotropic extension function* $\mathbf{\Psi}^e \in \mathrm{Orth}(D \times E, M)$ *such that* $\mathbf{\Psi}$ *is the restriction of* $\mathbf{\Psi}^e$ *on the surface or the graph* $\mathrm{Graph}(\mathbf{S}) \equiv \{(\mathbf{X}, \mathbf{S}(\mathbf{X})) \subset D \times E \mid \mathbf{X} \in D\}$, *i.e.*

$$\mathbf{\Psi}(\mathbf{X}) = \mathbf{\Psi}^e(\mathbf{X}, \mathbf{X}^e) \mid_{\mathbf{X}^e = \mathbf{S}(\mathbf{X})} = \mathbf{\Psi}^e(\mathbf{X}, \mathbf{S}(\mathbf{X})) \qquad (\forall\, \mathbf{X} \in D). \tag{2.3}$$

In the above theorem, the conditions for the set $\mathbf{S}$ of invariant tensor functions are weaker than those given in Xiao and Guo [46] and Xiao [49]. In reality, the conditions for $\mathbf{S}$ in the above theorem are given by (2.2) and

$$\mathbf{Q} \in G \implies \mathbf{Q} * \mathbf{S} = \mathbf{S}, \tag{2.4}$$

while those in [46] and [49] are given by (2.2) and

$$\mathbf{Q} \in G \iff \mathbf{Q} * \mathbf{S} = \mathbf{S}. \tag{2.5}$$

The former merely requires that the set $\mathbf{S}$ of tensor functions be invariant under the group G, while the latter requires that the symmetry group of $\mathbf{S}$ be identical with the group G.

Theorem A can be proved by means of the procedure given in [49] with little change. For a set $\mathbf{S}$ of tensor functions of interest, it is easier to prove whether $\mathbf{S}$ fulfills the invariance condition (2.4) or not, whereas it is not easy to judge whether $\mathbf{S}$ obeys the stronger invariance condition (2.5) or not, since it is not easy to determine the symmetry group of the set $\mathbf{S}$ of tensor functions.

A set $\mathbf{S}$ of tensor functions from D to E (cf. (2.1)) determines a surface in a Euclidean space R^n, where $n = 3(a + b + r + s) + 6(c + t)$, refer to §3.1 in [49] for detail. This fact allows a geometrical interpretation of the above extension theorem. The latter indicates that for every anisotropic tensor function $\mathbf{\Psi}$ relative to the anisotropy group $G \subset \mathrm{Orth}$ with a set of variables pertaining to the space $D = V^a \times \mathrm{Skw}^b \times \mathrm{Sym}^c$, one can find a surface $\mathbf{S}:\ D \to E = V^r \times \mathrm{Skw}^s \times \mathrm{Sym}^t$ in an augmented space $D \times E$ such that $\mathbf{\Psi}$ can be visualized as the restriction of an isotropic tensor function $\mathbf{\Psi}^e$ with a set of variables pertaining to the augmented space $D \times E$ on this surface, i.e. (2.3) holds. Such a surface $\mathbf{S}$ will be referred to as an *isotropic extension surface* for the anisotropic functions in $G(D, M)$ or as an *IES* for $G(D, M)$ for brevity. Necessary and sufficient that a surface $\mathbf{S}:\ D \to E$ is an IES for $G(D, M)$ is the condition that this surface fulfils both the invariance condition (2.4) and the consistency condition (2.2).

From the above theorem it follows that representations for the anisotropic function $\Psi \in G(D, M)$ can be obtained from those for the isotropic function $\mathbf{\Psi}^e \in \mathrm{Orth}(D \times E, M)$ merely by replacing the variables $\mathbf{X}^e \in E$ of the latter with $\mathbf{S}(\mathbf{X})$. In this sense, the above isotropic extension theorem, together with the well-known representation theorems for isotropic functions of vectors and second order tensors, constitutes a unified basis for the theory of representations for anisotropic functions of vectors and second order tensors. In the succeeding sections, for a domain $D = V^a \times \mathrm{Skw}^b \times \mathrm{Sym}^e$ for any given positive integers a, b and c, for each image set $M \in \{R, V, \mathrm{Skw}, \mathrm{Sym}\}$ and for each crystal and noncrystal class G, we shall provide a simple IES for $G(D, M)$.

2.2. A lemma

For each surface $\mathbf{S}$ that will be given, it is required to prove that the conditions (2.2) and (2.4) can be satisfied. The main difficulty arises from the consistency condition (2.2), for even for a given nontrival surface $\mathbf{S}$ it is not easy to determine the intersecting surface $\mathbf{S} \cap (\mathbf{Q} * \mathbf{S})$, let alone the fact that we must find a suitable surface $\mathbf{S}$ such that the conditions (2.2) and (2.4) can be satisfied.

We shall attack the above problem by choosing surfaces $\mathbf{S}$ in such a manner that all nontrivial intersecting surfaces $\mathbf{S} \cap (\mathbf{Q} * \mathbf{S}) \subset D$ are exactly certain prescribed particular subsets of D, which are provided by $D(\mathbf{u})$ or union of such subsets, where $\mathbf{u}$ is a unit vector in the direction of a symmetry axis of the related anisotropy group, since the following fact holds.

LEMMA A. Let $G \in \{C_{mv}, C_{mh}, S_{2m}, D_{mh}, D_{md}\}$, where $m \geq 3$, and let the unit vector $\mathbf{n}$ be in the direction of the principal axis of the group G. Moreover, define the group $D(G)$ by

$$D(G) = \begin{cases} C_{\infty v}, & G = C_{mv}\,, \\ C_{\infty h}, & G = C_{mh}, S_{2m}\,, \\ D_{\infty h}, & G = D_{mh}, D_{md}\,. \end{cases} \tag{2.6}$$

Then we have

$$\mathbf{F}(\mathbf{Q}^{\mathrm{T}} * \mathbf{X}_0) = \mathbf{Q}^{\mathrm{T}} * (\mathbf{F}(\mathbf{X}_0))$$

for any $\mathbf{Q} \in D(G)$, $\mathbf{X}_0 \in D(\mathbf{n})$, $\mathbf{F} \in G(D, M)$ and for each $M \in \{R, V, \mathrm{Skw}, \mathrm{Sym}\}$.

P r o o f. For each group G in question, there is $\mathbf{R}_0 = \mathbf{R}_{\mathbf{n}}^{2\pi/m} \in G$. For such $\mathbf{R}_0$, we have

$$\mathbf{R}_0 * (\mathbf{F}(\mathbf{X}_0)) = \mathbf{F}(\mathbf{R}_0 * \mathbf{X}_0) = \mathbf{F}(\mathbf{X}_0)$$

for each $\mathbf{X}_0 = (a_\alpha \mathbf{n}, b_\theta \mathbf{E}\mathbf{n}, c_\sigma \mathbf{I} + d_\sigma \mathbf{n}\otimes\mathbf{n}) \in D(\mathbf{n})$ and each $\mathbf{F} \in G(D, M)$. From this we derive

$$\mathbf{F}(\mathbf{X}_0) = \begin{cases} a(\mathbf{X}_0)\mathbf{n}, & M = V\,, \\ b(\mathbf{X}_0)\mathbf{E}\mathbf{n}, & M = \mathrm{Skw}\,, \\ c(\mathbf{X}_0)\mathbf{I} + d(\mathbf{X}_0)\mathbf{n}\otimes\mathbf{n}, & M = \mathrm{Sym}\,, \end{cases} \tag{2.7}$$

where for $M = \mathrm{Sym}$ the condition $m \geq 3$ is used. From the latter and the fact that for each $\mathbf{Q} \in D(G)$ there is $\mathbf{Q}_0 \in G$ such that

$$\mathbf{Q}\mathbf{n} = \mathbf{Q}_0\mathbf{n}, \qquad \mathbf{Q} * (\mathbf{E}\mathbf{n}) = \mathbf{Q}_0 * (\mathbf{E}\mathbf{n})\,, \tag{2.8}$$

we conclude that the lemma holds. *Q.E.D.*

2.3. The vector $\mathbf{q}(\mathbf{A})$ and the angle $< \mathbf{q}(\mathbf{A}), \mathbf{e} >$

Symbols $\mathbf{n}$ and $\mathbf{e}$ are used to represent two given orthonormal vectors. For any symmetric tensor $\mathbf{A} \in \mathrm{Sym}$, we introduce the vector $\mathbf{q}(\mathbf{A})$ by

$$\mathbf{q}(\mathbf{A}) = \frac{1}{2}(\mathbf{e}\cdot\mathbf{A}\mathbf{e} - \mathbf{e}'\cdot\mathbf{A}\mathbf{e}')\mathbf{e} + (\mathbf{e}\cdot\mathbf{A}\mathbf{e}')\mathbf{e}'\,. \tag{2.9}$$

Here and hereafter

$$\mathbf{e}' = \mathbf{n}\times\mathbf{e}\,. \tag{2.10}$$

Hence, $(\mathbf{n}, \mathbf{e}, \mathbf{e}')$ constitutes an orthonormal system.

The norm of any vector $\mathbf{v}$ is denoted by $|\mathbf{v}|$. Let $\mathbf{z}$ be a vector on the $\mathbf{n}$-plane. We define the *angle* $< \mathbf{z}, \mathbf{e} >$ *formed by the two vectors* $\mathbf{z}$ *and* $\mathbf{e}$ *on the* $\mathbf{n}$*-plane* as follows:

$$\cos < \mathbf{z}, \mathbf{e} > = \mathbf{z}\cdot\mathbf{e}/|\mathbf{z}|, \qquad \sin < \mathbf{z}, \mathbf{e} > = \mathbf{z}\cdot\mathbf{e}'/|\mathbf{z}|\,, \tag{2.11}$$

for $|\mathbf{z}| \neq 0$ and $< \mathbf{z}, \mathbf{e} > = 0$ for $|\mathbf{z}| = 0$. When $|\mathbf{z}| \neq 0$, it is evident that the angle $< \mathbf{z}, \mathbf{e} >$ is determined by (2.11) within $2k\pi$.

For the vector $\mathbf{q}(\mathbf{A})$ on the $\mathbf{n}$-plane introduced before, when $|\mathbf{q}(\mathbf{A})| \neq 0$ we have

$$\begin{aligned} \cos < \mathbf{q}(\mathbf{A}), \mathbf{e} > &= \frac{1}{2}(\mathbf{e}\cdot\mathbf{A}\mathbf{e} - \mathbf{e}'\cdot\mathbf{A}\mathbf{e}')/|\mathbf{q}(\mathbf{A})|\,, \\ \sin < \mathbf{q}(\mathbf{A}), \mathbf{e} > &= (\mathbf{e}\cdot\mathbf{A}\mathbf{e}')/|\mathbf{q}(\mathbf{A})|. \end{aligned} \tag{2.12}$$

Let $\mathbf{a}$ be a unit vector on the $\mathbf{n}$-plane. Then, applying the equalities

$$\begin{aligned} \mathbf{R}_{\mathbf{n}}^{\theta}\mathbf{e} &= \mathbf{e}\cos\theta + \mathbf{e}'\sin\theta\,, \\ \mathbf{R}_{\mathbf{n}}^{\theta}\mathbf{e}' &= -\mathbf{e}\sin\theta + \mathbf{e}'\cos\theta\,; \end{aligned} \tag{2.13}$$

$$\begin{aligned} \mathbf{R}_{\mathbf{a}}^{\pi}\mathbf{e} &= \mathbf{e}\cos 2 < \mathbf{a}, \mathbf{e} > + \mathbf{e}'\sin 2 < \mathbf{a}, \mathbf{e} >, \\ \mathbf{R}_{\mathbf{a}}^{\pi}\mathbf{e}' &= \mathbf{e}\sin 2 < \mathbf{a}, \mathbf{e} > - \mathbf{e}'\cos 2 < \mathbf{a}, \mathbf{e} >, \end{aligned} \tag{2.14}$$

we derive the transformation formulas (cf. Xiao [54])

$$< \mathbf{q}(\mathbf{Q} * \mathbf{A}), \mathbf{e} > = \begin{cases} 2\theta + < \mathbf{q}(\mathbf{A}), \mathbf{e} >, & \mathbf{Q} = \delta\mathbf{R}_{\mathbf{n}}^{\theta}\,, \\ 4 < \mathbf{a}, \mathbf{e} > - < \mathbf{q}(\mathbf{A}), \mathbf{e} >, & \mathbf{Q} = \delta\mathbf{R}_{\mathbf{a}}^{\pi}\,, \end{cases} \tag{2.15}$$

for $|\mathbf{q}(\mathbf{A})| \neq 0$ and

$$|\mathbf{q}(\mathbf{Q} * \mathbf{A})| = |\mathbf{q}(\mathbf{A})|, \quad \forall\, \mathbf{Q} \in D_{\infty h}\,, \tag{2.16}$$

where $D_{\infty h}$ is the maximal transverse isotropy group with the principal axis $\mathbf{n}$ (cf. (3.1) given later). Moreover, the following formulas hold:

$$< (\mathbf{Q}\mathbf{v})^{\mathrm{o}}, \mathbf{e} > = \begin{cases} (1-\delta)\pi/2 + \theta + < \mathbf{v}^{\mathrm{o}}, \mathbf{e} >, & \mathbf{Q} = \delta\mathbf{R}_{\mathbf{n}}^{\theta}\,, \\ (1+\delta)\pi/2 + 2 < \mathbf{a}, \mathbf{e} > - < \mathbf{v}^{\mathrm{o}}, \mathbf{e} >, & \mathbf{Q} = \delta\mathbf{R}_{\mathbf{a}}^{\pi}\,, \end{cases} \tag{2.17}$$

$$|(\mathbf{Q}\mathbf{v})^{\mathrm{o}}| = |\mathbf{v}^{\mathrm{o}}|, \quad \forall\, \mathbf{Q} \in D_{\infty h}\,, \tag{2.18}$$

for any vector $\mathbf{v} \in V$ and any $\mathbf{Q} \in D_{\infty h}$, and

$$< ((\mathbf{Q} * \mathbf{B})\mathbf{n})^{\mathrm{o}}, \mathbf{e} > = \begin{cases} \theta + < (\mathbf{B}\mathbf{n})^{\mathrm{o}}, \mathbf{e} >, & \mathbf{Q} = \delta\mathbf{R}_{\mathbf{n}}^{\theta}\,, \\ \pi + 2 < \mathbf{a}, \mathbf{e} > - < (\mathbf{B}\mathbf{n})^{\mathrm{o}}, \mathbf{e} >, & \mathbf{Q} = \delta\mathbf{R}_{\mathbf{a}}^{\pi}\,, \end{cases} \tag{2.19}$$

$$|((\mathbf{Q} * \mathbf{B})\mathbf{n})^{\mathrm{o}}| = |(\mathbf{B}\mathbf{n})^{\mathrm{o}}|, \quad \forall\, \mathbf{Q} \in D_{\infty h}\,, \tag{2.20}$$

for any second order tensor $\mathbf{B}$ and any $\mathbf{Q} \in D_{\infty h}$. In the above, $\delta^2 = 1$. Throughout, $\mathbf{v}^{\mathrm{o}}$ is used to designate the perpendicular projection of the vector $\mathbf{v}$ on the $\mathbf{n}$-plane, i.e.

$$\mathbf{v}^{\mathrm{o}} = \mathbf{v} - (\mathbf{v}\cdot\mathbf{n})\mathbf{n}\,. \tag{2.21}$$

For each antisymmetric tensor $\mathbf{W} \in \mathrm{Skw}$, the vector $\mathbf{W}\mathbf{n}$ lies on the $\mathbf{n}$-plane, i.e.

$$(\mathbf{W}\mathbf{n})^{\mathrm{o}} = \mathbf{W}\mathbf{n}\,, \tag{2.22}$$

since the latter is normal to $\mathbf{n}$.

Henceforth, for any two vectors $\mathbf{p}, \mathbf{q} \in V$, $\mathbf{p} \vee \mathbf{q} \in \mathrm{Sym}$ is used to signify the symmetric second order tensor defined by

$$\mathbf{p} \vee \mathbf{q} = \mathbf{p} \otimes \mathbf{q} + \mathbf{q} \otimes \mathbf{p}. \tag{2.23}$$

Moreover, we denote

$$\mathbf{D}_1 = \mathbf{e} \otimes \mathbf{e} - \mathbf{e}' \otimes \mathbf{e}', \qquad \mathbf{D}_2 = \mathbf{e} \vee \mathbf{e}'. \tag{2.24}$$

For any $\mathbf{Q} \in D_{\infty h}$, by using (2.13)–(2.14), we derive the following formulas.

$$\begin{aligned} \mathbf{R}_{\mathbf{n}}^{\theta} * \mathbf{D}_1 &= \mathbf{D}_1 \cos 2\theta + \mathbf{D}_2 \sin 2\theta, \\ \mathbf{R}_{\mathbf{n}}^{\theta} * \mathbf{D}_2 &= -\mathbf{D}_1 \sin 2\theta + \mathbf{D}_2 \cos 2\theta; \end{aligned} \tag{2.25}$$

$$\begin{aligned} \mathbf{R}_{\mathbf{a}}^{\pi} * \mathbf{D}_1 &= \mathbf{D}_1 \cos 4 < \mathbf{a}, \mathbf{e} > + \mathbf{D}_2 \sin 4 < \mathbf{a}, \mathbf{e} >, \\ \mathbf{R}_{\mathbf{a}}^{\pi} * \mathbf{D}_2 &= \mathbf{D}_1 \sin 4 < \mathbf{a}, \mathbf{e} > - \mathbf{D}_2 \cos 4 < \mathbf{a}, \mathbf{e} > . \end{aligned} \tag{2.26}$$

3. Improper subgroups of the transverse isotropy group $D_{\infty h}$

Prior to the succeeding account, we would point out the following fact: According to Theorems 2.1 and 2.2 given in [49], representations for anisotropic functions relative to a *rotation subgroup* $G \subset \mathrm{Orth}$ can be obtained from those for anisotropic functions relative to the centrosymmetric group

$$\bar{G} = \{\pm \mathbf{Q} \mid \mathbf{Q} \in G\}.$$

As a result, henceforth we need only to take the *improper subgroups* of Orth into account.

3.1. Transverse isotropy groups

$$D_{\infty h} = \{\pm \mathbf{R}_{\mathbf{n}}^{\theta},\ \pm \mathbf{R}_{\mathbf{a}}^{\pi} \mid \mathbf{a} = \mathbf{R}_{\mathbf{n}}^{\theta}\mathbf{e},\ \theta \in R\}, \tag{3.1}$$

$$C_{\infty v} = \{\mathbf{R}_{\mathbf{n}}^{\theta}, -\mathbf{R}_{\mathbf{a}}^{\pi} \mid \mathbf{a} = \mathbf{R}_{\mathbf{n}}^{\theta}\mathbf{e},\ \theta \in R\}, \tag{3.2}$$

$$C_{\infty h} = \{\pm \mathbf{R}_{\mathbf{n}}^{\theta} \mid \theta \in R\}. \tag{3.3}$$

According to BOEHLER [6] and LIU [16], the following offer an IES for $G(D, M)$ for each $G \in \{D_{\infty h}, C_{\infty v}, C_{\infty h}\}$.

$$D_{\infty h} : \mathbf{S}(\mathbf{X}) = (\mathbf{n} \otimes \mathbf{n}), \tag{3.4}$$

$$C_{\infty v} : \mathbf{S}(\mathbf{X}) = (\mathbf{n}), \tag{3.5}$$

$$C_{\infty h} : \mathbf{S}(\mathbf{X}) = (\mathbf{E}\mathbf{n}). \tag{3.6}$$

In reality, the following equalities hold (cf. LIU [16]).

$$(3.7)\qquad \Gamma(\mathbf{n}\otimes\mathbf{n}) = D_{\infty h}\,, \qquad \Gamma(\mathbf{n}) = C_{\infty v}\,, \qquad \Gamma(\mathbf{E}\mathbf{n}) = C_{\infty h}\,.$$

Hence, trivially, each surface $\mathbf{S}(\mathbf{X})$ given above satisfies the conditions (2.2) and (2.5).

Only for the anisotropic functions relative to such simple anisotropy groups as the transverse isotropy groups as well as triclinic, monoclinic and rhombic groups, trivial IESes such as those shown above (for the results concerning the latter groups, refer to BOEHLER [6, 8] and LIU [16]), which consist merely of some constant vectors and second order tensors, i.e. trivial vector- and second order tensor-valued tensor functions, can be found. For anisotropic functions concerning any other anisotropy group, nontrivial IESes have to be constructed, as will be done in the succeeding sections.

3.2. Classes D_{2m+1d}, C_{2m+1v} and S_{4m+2} for $m \geq 1$

$$(3.8)\qquad D_{2m+1d} = \{\pm\mathbf{R}_{\mathbf{n}}^{2k\pi/2m+1},\ \pm\mathbf{R}_{\mathbf{l}_k}^{\pi} \mid \mathbf{l}_k = \mathbf{R}_{\mathbf{n}}^{2k\pi/2m+1}\mathbf{e},\ k = 0,1,2,\ldots,2m\},$$

$$(3.9)\qquad C_{2m+1v} = C_{\infty v}\cap D_{2m+1d}; \qquad S_{4m+2} = C_{\infty h}\cap D_{2m+1d}\,.$$

The above classes include the trigonal crystal classes D_{3d}, C_{3v} and S_6 as the particular case when $m = 1$.

Henceforth, we denote

$$(3.10)\qquad \mathbf{D}(G) = \begin{cases} \mathbf{n}, & G = C_{rv}\,, \\ \mathbf{E}\mathbf{n}, & G = C_{rh}, S_{2r}\,, \\ \mathbf{n}\otimes\mathbf{n}, & G = D_{rh}, D_{rd}\,, \end{cases}$$

for each $r \geq 2$.

THEOREM 1. *Let $G \in \{D_{2m+1d}, C_{2m+1v}, S_{4m+2}\}$. Then the surface*

$$(3.11)\qquad \mathbf{S}(\mathbf{X}) = (\mathbf{D}(G); \mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{v}_\alpha^{\circ}); \mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{W}_\theta\mathbf{n}); \mathbf{E}\boldsymbol{\eta}_{2m}((\mathbf{A}_\sigma\mathbf{n})^{\circ}), \mathbf{E}\boldsymbol{\eta}_{m}(\mathbf{q}(\mathbf{A}_\sigma)))$$

is an IES for $G(D,M)$, where $\mathbf{D}(G)$ is given by (3.10) *and*

$$(3.12)\qquad \boldsymbol{\eta}_r(\mathbf{z}) = |\mathbf{z}|^r(\mathbf{e}\cos r < \mathbf{z},\mathbf{e} > -\mathbf{e}'\sin r < \mathbf{z},\mathbf{e} >)$$

for any vector $\mathbf{z}$ on the $\mathbf{n}$-plane and for each integer $r \geq 1$.

P r o o f. First, we prove that the given surface $\mathbf{S}(\mathbf{X})$ obeys the invariance requirement (2.4). Applying the formulas (2.13) and $(2.17)_1$ and (2.18), for $\mathbf{Q} = \pm\mathbf{R}_{\mathbf{n}}^{\theta}$ we infer

$$\begin{aligned}\mathbf{Q}^{\mathrm{T}} * (\mathbf{E}\boldsymbol{\eta}_{2m}((\mathbf{Q}\mathbf{v})^{\circ})) &= |\mathbf{v}^{\circ}|^{2m}\mathbf{Q}^{\mathrm{T}} * (\mathbf{E}(\mathbf{e}\cos(2m\theta + x) - \mathbf{e}'\sin(2m\theta + x))) \\ &= |\mathbf{v}^{\circ}|^{2m}\mathbf{E}(\mathbf{e}\cos((2m+1)\theta + x) - \mathbf{e}'\sin((2m+1)\theta + x))\,,\end{aligned}$$

where $x = 2m < \mathbf{v}^o, \mathbf{e} >$. Hence, we have

$$\mathbf{Q}^T * (\mathbf{E}\boldsymbol{\eta}_{2m}((\mathbf{Q}\mathbf{v})^o)) = \mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{v}^o), \qquad \mathbf{Q} = \pm\mathbf{R}_{\mathbf{n}}^{2\pi/2m+1}.$$

Moreover, applying $(2.17)_2$, (2.18), and (2.14) for $\mathbf{a} = \mathbf{e}$, we deduce

$$\mathbf{R}_{\mathbf{e}}^{\pi} * (\mathbf{E}\boldsymbol{\eta}_{2m}((\mathbf{R}_{\mathbf{e}}^{\pi}\mathbf{v})^o)) = \mathbf{R}_{\mathbf{e}}^{\pi} * (\mathbf{E}(\mathbf{e}\cos(2m\pi - x) - \mathbf{e}'\sin(2m\pi - x))) = \mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{v}^o).$$

From the above facts we derive that the tensor function $\mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{v}^o)$ is invariant under the group D_{2m+1d}, since the three orthogonal tensors $\pm\mathbf{R}_{\mathbf{n}}^{2\pi/2m+1}$ and $\mathbf{R}_{\mathbf{e}}^{\pi}$ can generate the group D_{2m+1d}. Similarly, by applying the formulas (2.13)–(2.16) and (2.19)–(2.20) we can prove that each of the other tensor functions in the given surface $\mathbf{S}(\mathbf{X})$ is also invariant under the group D_{2m+1d}. Thus, the given surface $\mathbf{S}(\mathbf{X})$ obeys (2.4).

Next, we prove that the surface $\mathbf{S}(\mathbf{X})$ satisfies the condition (2.2). We have

$$\mathbf{S} \cap (\mathbf{Q} * \mathbf{S}) = \begin{cases} D, & \mathbf{Q} \in G, \\ \emptyset, & \mathbf{Q} \in \mathrm{Orth}\backslash\Gamma(\mathbf{D}(G)), \end{cases}$$

for each $\mathbf{Q} \in \mathrm{Orth}\backslash(\Gamma(\mathbf{D}(G))\backslash G)$, where the symmetry groups $\Gamma(\mathbf{D}(G))$ are given by (3.10) and (3.7), and moreover $\emptyset$ is used to denote the empty set. Trivially, the condition (2.2) is satisfied for each $\mathbf{Q} \in \mathrm{Orth}\backslash(\Gamma(\mathbf{D}(G))\backslash G)$.

Moreover, for each $\mathbf{Q} \in \Gamma(\mathbf{D}(G)) \setminus G \subset D_{\infty h} \setminus G$, the intersecting point $\mathbf{X}_0 = (\mathbf{v}_\alpha, \mathbf{W}_\theta, \mathbf{A}_\sigma) \in \mathbf{S}\cap(\mathbf{Q}*\mathbf{S})$ is determined by the system of tensor equations of the forms (A.1)–(A.4), where the variables are: $\mathbf{v} = \mathbf{v}_1, \ldots, \mathbf{v}_a$; $\mathbf{W} = \mathbf{W}_1, \ldots, \mathbf{W}_b$; $\mathbf{A} = \mathbf{A}_1, \ldots, \mathbf{A}_c$. By Theorem A.1 we know that $\mathbf{S} \cap (\mathbf{Q} * \mathbf{S}) = D(\mathbf{n})$ for each $\mathbf{Q} \in \Gamma(\mathbf{D}(G)) \setminus G$. Then, from this fact and Lemma A we deduce that the condition (2.2) is also satisfied for each $\mathbf{Q} \in \Gamma(\mathbf{D}(G))\backslash G$. *Q.E.D.*

3.3. Classes D_{2m+2h}, C_{2m+2v} and C_{2m+2h} for $m \geq 1$

(3.13) $$D_{2m+2h} = \{\pm\mathbf{R}_{\mathbf{n}}^{k\pi/m+1}, \ \pm\mathbf{R}_{\mathbf{l}_k}^{\pi} \mid \mathbf{l}_k = \mathbf{R}_{\mathbf{n}}^{k\pi/2m+2}\mathbf{e}, \ k = 0, 1, 2, \ldots, 2m+1\},$$

(3.14) $$C_{2m+2v} = C_{\infty v} \cap D_{2m+2h}, \qquad C_{2m+2h} = C_{\infty h} \cap D_{2m+2h}.$$

The above classes include the tetragonal and hexagonal crystal classes D_{4h}, D_{6h}, C_{4v}, C_{6v}, C_{4h} and C_{6h} as the particular cases when $m = 1, 2$.

THEOREM 2. *Let $G \in \{D_{2m+2h}, C_{2m+2v}, C_{2m+2h}\}$. Then the surface*

(3.15) $$\mathbf{S}(\mathbf{X}) = (\mathbf{D}(G); \boldsymbol{\Phi}_{2m}(\mathbf{v}_\alpha^o); \boldsymbol{\Phi}_{2m}(\mathbf{W}_\theta\mathbf{n}); \boldsymbol{\Phi}_{2m}((\mathbf{A}_\sigma\mathbf{n})^o), \boldsymbol{\Phi}_m(\mathbf{q}(\mathbf{A}_\sigma)))$$

is an IES for $G(D, M)$, where the tensor $\mathbf{D}(G)$ is given by (3.10) *and*

(3.16) $$\boldsymbol{\Phi}_r(\mathbf{z}) = |\mathbf{z}|^r(\mathbf{D}_1 \cos r < \mathbf{z}, \mathbf{e} > -\mathbf{D}_2 \sin r < \mathbf{z}, \mathbf{e} >)$$

for any vector $\mathbf{z}$ *on the* $\mathbf{n}$*-plane and any integer* $r \geq 1$, *and the tensors* $\mathbf{D}_1$ *and* $\mathbf{D}_2$ *are given by* (2.24).

P r o o f. The proof concerning the condition (2.2) is the same as that of Theorem 1, except for the fact that Eqs. (A.7)–(A.10) and Theorem A.2 is used instead of Eqs. (A.1)–(A.4) and Theorem A.1. Hence, in the following we need only to prove that the given surface $\mathbf{S}(\mathbf{X})$ obeys the invariance condition (2.4). In reality, applying the formulas (2.25) and $(2.17)_1$ and (2.18), for $\mathbf{Q} = \pm\mathbf{R}_{\mathbf{n}}^{\theta}$ we deduce

$$\begin{aligned}\mathbf{Q}^{\mathrm{T}} * (\mathbf{\Phi}_{2m}((\mathbf{Q}\mathbf{v})^{\mathrm{o}})) &= |\mathbf{v}^{\mathrm{o}}|^{2m}\mathbf{Q}^{\mathrm{T}} * (\mathbf{D}_1\cos(2m\theta + x) - \mathbf{D}_2\sin(2m\theta + x))\\ &= |\mathbf{v}^{\mathrm{o}}|^{2m}(\mathbf{D}_1\cos((2m+2)\theta + x) - \mathbf{D}_2\sin((2m+2)\theta + x)),\end{aligned}$$

where $x = 2m < \mathbf{v}^{\mathrm{o}}, \mathbf{e} >$. Hence, we have

$$\mathbf{Q}^{\mathrm{T}} * (\mathbf{\Phi}_{2m}((\mathbf{Q}\mathbf{v})^{\mathrm{o}})) = \mathbf{\Phi}_{2m}(\mathbf{v}^{\mathrm{o}}), \qquad \mathbf{Q} = \pm\mathbf{R}_{\mathbf{n}}^{\pi/m+1}.$$

Moreover, applying $(2.17)_2$, (2.18), and (2.26) for $\mathbf{a} = \mathbf{e}$, we infer

$$\mathbf{R}_{\mathbf{e}}^{\pi} * (\mathbf{\Phi}_{2m}((\mathbf{R}_{\mathbf{e}}^{\pi}\mathbf{v})^{\mathrm{o}})) = |\mathbf{v}^{\mathrm{o}}|^{2m}\mathbf{R}_{\mathbf{e}}^{\pi} * (\mathbf{D}_1\cos(2m\pi - x) - \mathbf{D}_2\sin(2m\pi - x)) = \mathbf{\Phi}_{2m}(\mathbf{v}^{\mathrm{o}}).$$

From the above facts we derive that the tensor function $\mathbf{\Phi}_{2m}(\mathbf{v}^{\mathrm{o}})$ is invariant under the group D_{2m+2h}, since the three orthogonal tensors $\pm\mathbf{R}_{\mathbf{n}}^{\pi/m+1}$ and $\mathbf{R}_{\mathbf{e}}^{\pi}$ can generate the group D_{2m+2h}. Similarly, by applying the formulas (2.15)–(2.16), (2.19)–(2.20) and (2.25)–(2.26) we can prove that each of the other tensor functions in the given surface $\mathbf{S}(\mathbf{X})$ is also invariant under the group D_{2m+2h}. Thus, we conclude that the given surface $\mathbf{S}(\mathbf{X})$ obeys (2.4). *Q.E.D.*

3.4. Classes D_{2m+1h} and C_{2m+1h} for $m \geq 1$

$$(3.17)\qquad D_{2m+1h} = \{(-1)^k\mathbf{R}_{\mathbf{n}}^{k\pi/2m+1},\ (-1)^k\mathbf{R}_{\mathbf{l}_k}^{\pi} \mid \mathbf{l}_k = \mathbf{R}_{\mathbf{n}}^{k\pi/4m+2}\mathbf{e},\quad k = 0, 1, 2, \ldots, 4m+1\},$$

$$(3.18)\qquad C_{2m+1h} = C_{\infty h} \cap D_{2m+1h}\,.$$

The above classes include the hexagonal crystal classes D_{3h} and C_{3h} as the particular case when $m = 1$. Note that $\mathbf{e} = \mathbf{l}_0$ is a *two-fold rotation axis* of D_{2m+1h}.

THEOREM 3. *Let $G \in \{D_{2m+1h}, C_{2m+1h}\}$. Then the surface*

$$(3.19)\qquad \mathbf{S}(\mathbf{X}) = (\mathbf{D}(G); \boldsymbol{\eta}_{2m}(\mathbf{v}_{\alpha}^{\mathrm{o}}); \boldsymbol{\eta}_{2m}(\mathbf{W}_{\theta}\mathbf{n}); \boldsymbol{\eta}_{2m}((\mathbf{A}_{\sigma}\mathbf{n})^{\mathrm{o}}), \boldsymbol{\eta}_{m}(\mathbf{q}(\mathbf{A}_{\sigma})))$$

is an IES for $G(D, M)$, where $\mathbf{D}(G)$ is given by (3.10) *and the vector-valued function $\boldsymbol{\eta}_r(\mathbf{z})$ is given by* (3.12) *for any vector $\mathbf{z}$ on the $\mathbf{n}$-plane and each integer $r \geq 1$.*

P r o o f. The proof concerning the condition (2.2) is the same as that of Theorem 1, except for the fact that Eqs. (A.11)–(A.14) and Theorem A.3 are used instead of Eqs. (A.1)–(A.4) and Theorem A.1. In the following, we prove that the given surface obeys the invariance condition (2.4).

Applying the formulas (2.13) and $(2.17)_1$ and (2.18), for $\mathbf{Q} = \delta\mathbf{R}_{\mathbf{n}}^{\theta}$, $\delta^2 = 1$, we deduce

$$\begin{aligned}\mathbf{Q}^{\mathrm{T}}(\boldsymbol{\eta}_{2m}((\mathbf{Q}\mathbf{v})^{\circ})) &= |\mathbf{v}^{\circ}|^{2m}\mathbf{Q}^{\mathrm{T}}(\mathbf{e}\cos(2m\theta + x) - \mathbf{e}'\sin(2m\theta + x)) \\ &= |\mathbf{v}^{\circ}|^{2m}\delta(\mathbf{e}\cos((2m+1)\theta + x) - \mathbf{e}'\sin((2m+1)\theta + x)),\end{aligned}$$

where $x = 2m < \mathbf{v}^{\circ}, \mathbf{e} >$. Hence, we have

$$\mathbf{Q}^{\mathrm{T}}(\boldsymbol{\eta}_{2m}((\mathbf{Q}\mathbf{v})^{\circ})) = \boldsymbol{\eta}_{2m}(\mathbf{v}^{\circ}), \qquad \mathbf{Q} = -\mathbf{R}_{\mathbf{n}}^{\pi/2m+1}.$$

Moreover, applying $(2.17)_2$, (2.18), and (2.14) for $\mathbf{a} = \mathbf{e}$, we infer

$$\mathbf{R}_{\mathbf{e}}^{\pi}(\boldsymbol{\eta}_{2m}((\mathbf{R}_{\mathbf{e}}^{\pi}\mathbf{v})^{\circ})) = \mathbf{R}_{\mathbf{e}}^{\pi}(\mathbf{e}\cos(2m\pi - x) - \mathbf{e}'\sin(2m\pi - x)) = \boldsymbol{\eta}_{2m}(\mathbf{v}^{\circ}).$$

From the above facts we derive that the tensor function $\boldsymbol{\eta}_{2m}(\mathbf{v}^{\circ})$ is invariant under the group D_{2m+1h}, since the two orthogonal tensors $-\mathbf{R}_{\mathbf{n}}^{\pi/2m+1}$ and $\mathbf{R}_{\mathbf{e}}^{\pi}$ can generate the group D_{2m+1h}. Similarly, by applying the formulas (2.13)–(2.16) and (2.19)–(2.20) we can prove that each of the other tensor functions in the given surface $\mathbf{S}(\mathbf{X})$ is also invariant under the group D_{2m+1h}. Thus, we conclude that the given surface $\mathbf{S}(\mathbf{X})$ obeys (2.4). *Q.E.D.*

3.5. Classes D_{2md} for $m \geq 2$

(3.20) $$D_{2md} = \{(-1)^k\mathbf{R}^{k\pi/2m}, (-1)^k\mathbf{R}_{\mathbf{l}_k}^{\pi} \mid \mathbf{l}_k = \mathbf{R}_{\mathbf{n}}^{k\pi/4m}\mathbf{e}, \quad k = 0, 1, 2, \ldots, 4m-1\}.$$

Note that $\mathbf{e} = \mathbf{l}_0$ is a *two-fold rotation axis* of D_{2md}.

THEOREM 4. *The surface*

(3.21) $$\begin{aligned}\mathbf{S}(\mathbf{X}) = (\mathbf{n}\otimes\mathbf{n}; \mathbf{n}\vee\boldsymbol{\eta}_{2m-1}(\mathbf{v}_{\alpha}^{\circ}); \boldsymbol{\eta}_{2m-1}(\mathbf{W}_{\theta}\mathbf{n}); \boldsymbol{\eta}_{2m-1}((\mathbf{A}_{\sigma}\mathbf{n})^{\circ}), f_m(\mathbf{A}_{\sigma})\mathbf{n},\\ \boldsymbol{\Phi}_{2m-1}(\mathbf{q}(\mathbf{A}_{\sigma})); (\mathbf{n}\cdot\mathbf{v}_{\alpha})\boldsymbol{\Phi}_{m-1}(\mathbf{q}(\mathbf{A}_{\sigma})); (\mathbf{e}\cdot\mathbf{W}_{\theta}\mathbf{e}')g_m(\mathbf{A}_{\sigma})\mathbf{n})\end{aligned}$$

is an IES for $D_{2md}(D, M)$, where the tensor-valued function $\boldsymbol{\Phi}_r(\mathbf{z})$ and the vector-valued function $\boldsymbol{\eta}_r(\mathbf{z})$ are given by (3.16) and (3.12) for any vector $\mathbf{z}$ on the $\mathbf{n}$-plane and each integer $r \geq 1$, respectively, and moreover

(3.22) $$\begin{aligned}f_m(\mathbf{A}_{\sigma}) &= |\mathbf{q}(\mathbf{A}_{\sigma})|^m \sin m < \mathbf{q}(\mathbf{A}_{\sigma}), \mathbf{e} >,\\ g_m(\mathbf{A}_{\sigma}) &= |\mathbf{q}(\mathbf{A}_{\sigma})|^m \cos m < \mathbf{q}(\mathbf{A}_{\sigma}), \mathbf{e} > .\end{aligned}$$

P r o o f. First, we prove that the given surface $\mathbf{S}(\mathbf{X})$ obeys the invariance condition (2.4). Applying the formulas (2.13) and $(2.17)_1$ and (2.18), for $\mathbf{Q} = \delta\mathbf{R}_\mathbf{n}^\theta$, $\delta^2 = 1$, we deduce

$$\begin{aligned}\mathbf{Q}^{\mathrm{T}} * (\mathbf{n} \vee \boldsymbol{\eta}_N((\mathbf{Q}\mathbf{v})^{\mathrm{o}})) &= |\mathbf{v}^{\mathrm{o}}|^N \mathbf{Q}^{\mathrm{T}} * \left(\mathbf{n} \vee \left(\mathbf{e}\cos\left(\frac{1-\delta}{2}\pi + N\theta + x\right)\right.\right. \\ &\qquad \left.\left. - \mathbf{e}'\sin\left(\frac{1-\delta}{2}\pi + N\theta + x\right)\right)\right) \\ &= |\mathbf{v}^{\mathrm{o}}|^N \mathbf{n} \vee \left(\mathbf{e}\cos\left(\frac{1-\delta}{2}\pi + 2m\theta + x\right)\right. \\ &\qquad \left. - \mathbf{e}'\sin\left(\frac{1-\delta}{2}\pi + 2m\theta + x\right)\right),\end{aligned}$$

where $N = 2m - 1$ and $x = (2m-1) < \mathbf{v}^{\mathrm{o}}, \mathbf{e} >$. Hence, we have

$$\mathbf{Q}^{\mathrm{T}} * (\mathbf{n} \vee \boldsymbol{\eta}_{2m-1}((\mathbf{Q}\mathbf{v})^{\mathrm{o}})) = \mathbf{n} \vee \boldsymbol{\eta}_{2m-1}(\mathbf{v}^{\mathrm{o}}), \qquad \mathbf{Q} = -\mathbf{R}_\mathbf{n}^{\pi/2m}.$$

Moreover, by applying $(2.17)_2$, (2.18), and (2.14) for $\mathbf{a} = \mathbf{e}$ we infer

$$\begin{aligned}&\mathbf{R}_\mathbf{e}^\pi * (\mathbf{n} \vee \boldsymbol{\eta}_N((\mathbf{R}_\mathbf{e}^\pi \mathbf{v})^{\mathrm{o}})) \\ &\quad = |\mathbf{v}^{\mathrm{o}}|^N \mathbf{R}_\mathbf{e}^\pi * (\mathbf{n} \vee (\mathbf{e}\cos(N\pi - x) - \mathbf{e}'\sin(N\pi - x))) = \mathbf{n} \vee \boldsymbol{\eta}_N(\mathbf{v}^{\mathrm{o}}).\end{aligned}$$

From the above facts we conclude that the tensor function $\mathbf{n} \vee \boldsymbol{\eta}_{2m-1}(\mathbf{v}^{\mathrm{o}})$ is invariant under the group D_{2md}, since the two orthogonal tensors $-\mathbf{R}_\mathbf{n}^{\pi/2m}$ and $\mathbf{R}_\mathbf{e}^\pi$ can generate the group D_{2md}. Similarly, by using the formulas (2.13)–(2.16), (2.19)–(2.20) and (2.25)–(2.26) we can prove that each of the other tensor functions in the given surface $\mathbf{S}(\mathbf{X})$ is also invariant under the group D_{2md}. Thus, we conclude that the given surface obeys (2.4).

Next, we prove that the given surface $\mathbf{S}(\mathbf{X})$ satisfies the condition (2.2). It can be readily verified that the condition (2.2) is satisfied for each $\mathbf{Q} \in \mathrm{Orth} \setminus (D_{\infty h} \setminus D_{2md})$ by using $(3.7)_1$. Thus, the rest is to prove that the condition (2.2) is satisfied for each $\mathbf{Q} \in D_{\infty h} \setminus D_{2md}$. To this end, consider two cases. First, for each $\mathbf{Q} \in D_{\infty h} \setminus D_{4mh}$, the intersection point $\mathbf{X}_0 = (\mathbf{v}_\alpha, \mathbf{W}_\theta, \mathbf{A}_\sigma) \in \mathbf{S} \cap (\mathbf{Q} * \mathbf{S})$ is determined by the system of tensor equations of the forms (A.15)–(A.17) and (A.10), in the latter m being replaced by $2m-1$, and moreover

$$\begin{aligned}f_m(\mathbf{Q}^{\mathrm{T}} * \mathbf{A})\mathbf{Q}\mathbf{n} &= f_m(\mathbf{A})\mathbf{n}, \\ (\mathbf{n}\cdot(\mathbf{Q}^{\mathrm{T}}\mathbf{v}))\mathbf{Q} * (\boldsymbol{\Phi}_{m-1}(\mathbf{q}(\mathbf{Q}^{\mathrm{T}} * \mathbf{A}))) &= (\mathbf{n}\cdot\mathbf{v})\boldsymbol{\Phi}_{m-1}(\mathbf{q}(\mathbf{A})), \\ (\mathbf{e}\cdot(\mathbf{Q}^{\mathrm{T}} * \mathbf{W})\mathbf{e}')g_m(\mathbf{Q}^{\mathrm{T}} * \mathbf{A})\mathbf{Q}\mathbf{n} &= (\mathbf{e}\cdot\mathbf{W}\mathbf{e}')g_m(\mathbf{A})\mathbf{n},\end{aligned}$$

where the variables are: $\mathbf{v} = \mathbf{v}_1, \ldots, \mathbf{v}_a$; $\mathbf{W} = \mathbf{W}_1, \ldots, \mathbf{W}_b$; $\mathbf{A} = \mathbf{A}_1, \ldots, \mathbf{A}_c$. From Theorem A.4 and the proof of Theorem A.2 (cf. $(A.6)_4$) we derive that $\mathbf{S} \cap (\mathbf{Q} * \mathbf{S}) = D(\mathbf{n})$, and furthermore the point $\mathbf{X}_0 \in D(\mathbf{n})$ satisfies the above three equations. Then, by noticing $2m \geq 4$ and applying Lemma A we conclude that the condition (2.2) is satisfied for each $\mathbf{Q}$ in question.

Moreover, for each $\mathbf{Q} \in D_{4mh} \setminus D_{2md}$, by using the fact

$$\mathbf{Q} \in D_{4mh} \setminus D_{2md} \;\Rightarrow\; -\mathbf{Q} \in D_{2md} \tag{3.23}$$

we infer that the intersection point $\mathbf{X}_0 = (\mathbf{v}_\alpha,\ \mathbf{W}_\theta,\ \mathbf{A}_\sigma) \in \mathbf{S} \cap (\mathbf{Q} * \mathbf{S})$ is determined by

$$\begin{aligned} \boldsymbol{\eta}_{2m-1}(\mathbf{z}) = \mathbf{0}, \qquad & f_m(\mathbf{A}_\sigma) = 0, \\ (\mathbf{v}_\alpha \cdot \mathbf{n})\boldsymbol{\Phi}_{m-1}(\mathbf{A}_\sigma) = \mathbf{O}, \qquad & (\mathbf{e} \cdot \mathbf{W}_\theta \mathbf{e}') g_m(\mathbf{A}_\sigma) = 0, \end{aligned}$$

where $\mathbf{z} = \mathbf{v}_\alpha^{\mathrm{o}}, \mathbf{W}_\theta \mathbf{n}, (\mathbf{A}_\sigma \mathbf{n})^{\mathrm{o}}$. The first two equations yield

$$\begin{aligned} \mathbf{v}_\alpha &= a_\alpha \mathbf{n}, \qquad \mathbf{W}_\theta = b_\theta \mathbf{E}\mathbf{n}, \\ \mathbf{A}_\sigma &= c_\sigma \mathbf{I} + d_\sigma \mathbf{n} \otimes \mathbf{n} + h_\sigma (\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2), \end{aligned} \tag{3.24}$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are two orthonormal vectors in the $\mathbf{n}$-plane and will be given later. Substituting the above results into the last two equations given before, we derive

$$a_\alpha h_\sigma = 0, \qquad b_\theta h_\sigma = 0.$$

Thus, for each $\mathbf{Q} \in D_{4mh} \backslash D_{2md}$ the point $\mathbf{X}_0 \in \mathbf{S} \cap (\mathbf{Q} * \mathbf{S})$ is given by $\mathbf{X}_0 \in D(\mathbf{n})$ or

$$\begin{aligned} &\mathbf{v}_\alpha = \mathbf{0}, \quad \mathbf{W}_\theta = \mathbf{O}, \quad \mathbf{A}_\sigma = c_\sigma \mathbf{I} + d_\sigma \mathbf{n} \otimes \mathbf{n} + h_\sigma (\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2), \\ &\mathbf{e}_1 = \mathbf{l}_{2k}\,, \quad \mathbf{e}_2 = \mathbf{n} \times \mathbf{e}_1 = \mathbf{l}_{2k+2m}\,, \quad k = 0, 1, 2, \ldots, m-1\,. \end{aligned} \tag{3.25}$$

For the point $\mathbf{X}_0 \in D(\mathbf{n})$, by noting $2m \geq 4$ and using Lemma A we deduce that the condition (2.2) can be satisfied. For the point $\mathbf{X}_0$ given by (3.25), from the facts

$$\begin{aligned} &\mathbf{R}_0 * \mathbf{X}_0 = \mathbf{X}_0, \quad \mathbf{R}_0 \in \{\mathbf{R}_{\mathbf{e}_1}^\pi, \mathbf{R}_{\mathbf{e}_2}^\pi,\ \mathbf{R}_{\mathbf{n}}^\pi\} \subset D_{2md} \\ &\qquad \Rightarrow\ \forall\, \mathbf{F} \in D_{2md}(D, M):\ \mathbf{F}(\mathbf{X}_0) = \mathbf{F}(\mathbf{R}_0 * \mathbf{X}_0) = \mathbf{R}_0 * (\mathbf{F}(\mathbf{X}_0))\,, \end{aligned}$$

we derive

$$\mathbf{F}(\mathbf{X}_0) = \begin{cases} \mathbf{0}, & M = V, \\ \mathbf{O}, & M = \mathrm{Skw}, \\ c_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + c_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + c_3 \mathbf{n} \otimes \mathbf{n}, & M = \mathrm{Sym}, \end{cases}$$

where $c_i = c_i(\mathbf{X}_0)$. Hence, by means of the above fact and (3.23) we infer

$$\begin{aligned} \mathbf{F}(\mathbf{Q}^{\mathrm{T}} * \mathbf{X}_0) &= \mathbf{F}({\mathbf{Q}_0}^{\mathrm{T}} * \mathbf{X}_0) \qquad (\mathbf{Q}_0 = -\mathbf{Q}) \\ &= {\mathbf{Q}_0}^{\mathrm{T}} * (\mathbf{F}(\mathbf{X}_0)) = \mathbf{Q}^{\mathrm{T}} * (\mathbf{F}(\mathbf{X}_0)) \end{aligned}$$

for any $\mathbf{Q} \in D_{4mh} \backslash D_{2md}$ and any $\mathbf{F} \in D_{2md}(D, M)$, i.e. the condition (2.2) is fulfilled for each $\mathbf{Q} \in D_{4mh} \setminus D_{2md}$. *Q.E.D.*

If the tensor functions of the form $(\mathbf{e}\cdot\mathbf{W}\mathbf{e}')g_m(\mathbf{A})\mathbf{n}$ are removed from the surface $\mathbf{S}(\mathbf{X})$ given by (3.21), then from the above proof we know that the condition (2.2) is still satisfied for each $\mathbf{Q} \in D_{\infty h} \setminus D_{4mh}$. On the other hand, for each $\mathbf{Q} \in D_{4mh} \setminus D_{2md}$, the intersection point $\mathbf{X}_0 \in \mathbf{S} \cap (\mathbf{Q} * \mathbf{S})$ is given by $\mathbf{X}_0 \in D(\mathbf{n})$ or by $(3.25)_2$ and (3.24) with each $a_\alpha = 0$. For the latter, with the aid of (3.23), one may readily verify that the condition (2.2) is fulfilled for scalar-valued and second order tensor-valued functions, i.e. for each image set $M \in \{R, \mathrm{Skw}, \mathrm{Sym}\}$. This shows that the tensor functions mentioned before is needed only for vector-valued functions, i.e. only for the image set $M = V$. Thus, we arrive at the following simplified result.

COROLLARY. The surface

$$\mathbf{S}(\mathbf{X}) = (\mathbf{n}\otimes\mathbf{n}; \mathbf{n}\vee\boldsymbol{\eta}_{2m-1}(\mathbf{v}^{\circ}_{\alpha}); \boldsymbol{\eta}_{2m-1}(\mathbf{W}_\theta\mathbf{n}); \boldsymbol{\eta}_{2m-1}((\mathbf{A}_\sigma\mathbf{n})^{\circ}), \\ f_m(\mathbf{A}_\sigma)\mathbf{n}, \boldsymbol{\Phi}_{2m-1}(\mathbf{q}(\mathbf{A}_\sigma)); (\mathbf{n}\cdot\mathbf{v}_\alpha)\boldsymbol{\Phi}_{m-1}(\mathbf{q}(\mathbf{A}_\sigma))) \tag{3.26}$$

is an IES for $D_{2md}(D, M)$ for each $M \in \{R, \mathrm{Skw}, \mathrm{Sym}\}$.

3.6. Classes S_{4m} for $m \geq 2$

$$S_{4m} = C_{\infty h} \cap D_{2md} = \{(-1)^k \mathbf{R}_{\mathbf{n}}^{k\pi/2m} \mid k = 0, 1, 2, \ldots, 4m-1\}. \tag{3.27}$$

THEOREM 5. *Let* $\mathbf{e}$ *be any given unit vector on the* $\mathbf{n}$*-plane. Then the surface*

$$\mathbf{S}(\mathbf{X}) = (\mathbf{E}\mathbf{n}; \mathbf{n}\vee\boldsymbol{\eta}_{2m-1}(\mathbf{v}^{\circ}_{\alpha}); \boldsymbol{\eta}_{2m-1}(\mathbf{W}_\theta\mathbf{n}); \boldsymbol{\eta}_{2m-1}((\mathbf{A}_\sigma\mathbf{n})^{\circ}), \\ f_m(\mathbf{A}_\sigma)\mathbf{n}, g_m(\mathbf{A}_\sigma)\mathbf{n}), \tag{3.28}$$

is an IES for $S_{4m}(D, M)$, *where* $\boldsymbol{\eta}_r(\mathbf{z})$ *is given by* (3.12) *for any vector* $\mathbf{z}$ *on the* $\mathbf{n}$*-plane and each integer* $r \geq 1$, *and moreover* $f_m(\mathbf{A}_\sigma)$ *and* $g_m(\mathbf{A}_\sigma)$ *are given by* (3.22).

P r o o f. It can easily be verified that the tensor functions $f_m(\mathbf{A})\mathbf{n}$ and $g_m(\mathbf{A})\mathbf{n}$ are invariant under the group S_{4m} by using the formula $(2.15)_1$. Moreover, it is known that each of the tensor functions $\boldsymbol{\eta}_{2m-1}(\mathbf{z})$, $\mathbf{z} = \mathbf{v}^{\circ}$, $\mathbf{W}\mathbf{n}$, $(\mathbf{A}\mathbf{n})^{\circ}$ is invariant under the group $D_{2md}(\supset S_{4m})$ (cf. the former part of the proof for Theorem 4). Thus, we conclude that the given surface $\mathbf{S}(\mathbf{X})$ obeys the invariance condition (2.4).

Now consider the condition (2.2). It is readily verified that the latter can be satisfied for each $\mathbf{Q} \in \mathrm{Orth} \setminus (C_{\infty h} \setminus S_{4m})$ by using $(3.7)_3$. Moreover, for each $\mathbf{Q} \in C_{\infty h} \setminus S_{4m}$, the intersection point $\mathbf{X}_0 = (\mathbf{v}_\alpha, \mathbf{W}_\theta, \mathbf{A}_\sigma) \in \mathbf{S} \cap (\mathbf{Q} * \mathbf{S})$ is determined by the system of tensor equations of the forms (A.15)–(A.17) and

$$f_m(\mathbf{Q}^{\mathrm{T}} * \mathbf{A})\mathbf{Q}\mathbf{n} = f_m(\mathbf{A}), \qquad g_m(\mathbf{Q}^{\mathrm{T}} * \mathbf{A})\mathbf{Q}\mathbf{n} = g_m(\mathbf{A})\mathbf{n},$$

where the variables are: $\mathbf{v} = \mathbf{v}_1, \ldots, \mathbf{v}_a$; $\mathbf{W} = \mathbf{W}_1, \ldots, \mathbf{W}_b$; $\mathbf{A} = \mathbf{A}_1, \ldots, \mathbf{A}_c$. From the latter and Theorems A.4 we infer that $\mathbf{S} \cap (\mathbf{Q} * \mathbf{S}) = D(\mathbf{n})$ for each $\mathbf{Q} \in C_{\infty h} \setminus S_{4m}$. Then, by this fact and Lemma A we infer that the condition (2.2) is satisfied for each $\mathbf{Q} \in C_{\infty h} \setminus S_{4m}$ and each $m \geq 2$. *Q.E.D.*

3.7. The tetragonal crystal class D_{2d}

(3.29) $$D_{2d} = \{(-1)^k \mathbf{R}_{\mathbf{n}}^{k\pi/2},\ (-1)^k \mathbf{R}_{\mathbf{l}_k}^{\pi} \mid \mathbf{l}_k = \mathbf{R}_{\mathbf{n}}^{k\pi/4}\mathbf{e},\ k = 0,1,2,3\}.$$

Note that the orthonormal vectors $\mathbf{l}_0 = \mathbf{e}$ and $\mathbf{l}_2 = \mathbf{e}'$ represent the two two-fold rotation axes of D_{2d}.

THEOREM 6. *The surface*

(3.30) $$\mathbf{S}(\mathbf{X}) = (\mathbf{n}\otimes\mathbf{n};\ \mathbf{n}\vee\boldsymbol{\eta}_1(\mathbf{v}^o) + (\mathbf{v}\cdot\mathbf{n})\mathbf{D}_1;\ \boldsymbol{\eta}_1(\mathbf{W}_\theta\mathbf{n});\ \boldsymbol{\eta}_1((\mathbf{A}\mathbf{n})^o) + f_1(\mathbf{A})\mathbf{n}, \\ \boldsymbol{\Phi}_1(\mathbf{q}(\mathbf{A}_\sigma));\ (\mathbf{e}\cdot\mathbf{W}_\theta\mathbf{e}')g_1(\mathbf{A}_\sigma)\mathbf{n})$$

is an IES for $D_{2d}(D, M)$, where the tensor functions $\boldsymbol{\eta}_1(\mathbf{z})$, $\boldsymbol{\Phi}_1(\mathbf{z})$, $f_1(\mathbf{A})$ and $g_1(\mathbf{A})$ are obtained by taking $m = 1$ in (3.12), (3.16) and (3.22), respectively.

P r o o f. First, we prove that the given surface $\mathbf{S}(\mathbf{X})$ meets the invariance condition (2.4). To this end, it suffices to prove that the tensor function $(\mathbf{v}\cdot\mathbf{n})\mathbf{D}_1$ is invariant under the group D_{2d}, since each other tensor function in the surface $\mathbf{S}(\mathbf{X})$ given here is included in the IES given by Theorem 4, where $m = 1$, and is invariant under the group D_{2d}. By using (2.25) we infer $((\mathbf{Q}^T\mathbf{v})\cdot\mathbf{n})\mathbf{Q} * \mathbf{D}_1 = \delta(\mathbf{v}\cdot\mathbf{n})(\mathbf{D}_1\cos 2\theta + \mathbf{D}_2\sin 2\theta)$ for $\mathbf{Q} = \delta\mathbf{R}_{\mathbf{n}}^{\theta}$, $\delta^2 = 1$. Hence, we have

$$((\mathbf{Q}^T\mathbf{v})\cdot\mathbf{n})\mathbf{Q} * \mathbf{D}_1 = (\mathbf{v}\cdot\mathbf{n})\mathbf{D}_1,\quad \forall\,\mathbf{Q}\in S_4.$$

Moreover, we have

$$((\mathbf{R}_{\mathbf{e}}^{\pi}\mathbf{v})\cdot\mathbf{n})\mathbf{R}_{\mathbf{e}}^{\pi} * \mathbf{D}_1 = (\mathbf{v}\cdot\mathbf{n})\mathbf{D}_1\,.$$

Thus, $(\mathbf{v}\cdot\mathbf{n})\mathbf{D}_1$ is invariant under the group D_{2d}, since S_4 and $\mathbf{R}_{\mathbf{e}}^{\pi}$ generate the latter.

Next, we prove that the given surface obeys the condition (2.2). It can be readily verified that for each $\mathbf{Q}\in \mathrm{Orth}\setminus(D_{\infty h}\setminus D_{2d})$ the condition (2.2) can be satisfied by using $(3.7)_1$.

Moreover, for each $\mathbf{Q}\in D_{\infty h}\setminus D_{4h}$, the intersection point $\mathbf{X}_0 = (\mathbf{v}_\alpha, \mathbf{W}_\theta, \mathbf{A}_\sigma) \in \mathbf{S}\cap(\mathbf{Q} * \mathbf{S})$ is determined by (A.10) (for $m = 1$), (A.15)–(A.17) (for $m = 1$) and

$$\begin{aligned} ((\mathbf{Q}^T\mathbf{v})\cdot\mathbf{n})\mathbf{Q} * \mathbf{D}_1 &= (\mathbf{v}\cdot\mathbf{n})\mathbf{D}_1\,,\\ f_1(\mathbf{Q}^T * \mathbf{A})\mathbf{Q}\mathbf{n} &= f_1(\mathbf{A})\mathbf{n},\\ (\mathbf{e}\cdot(\mathbf{Q}^T\mathbf{W})\mathbf{e}')g_1(\mathbf{Q}^T * \mathbf{A})\mathbf{Q}\mathbf{n} &= (\mathbf{e}\cdot\mathbf{W}\mathbf{e}')g_1(\mathbf{A})\mathbf{n}, \end{aligned}$$

where the variables are: $\mathbf{v} = \mathbf{v}_1,\ldots,\mathbf{v}_a$; $\mathbf{W} = \mathbf{W}_1,\ldots,\mathbf{W}_b$; $\mathbf{A} = \mathbf{A}_1,\ldots,\mathbf{A}_c$. From the first equation above and Theorems A.4 and the proof of Theorem A.2 (cf. $(A.6)_4$) we derive

(3.31) $$\mathbf{v}_\alpha = \mathbf{0},\qquad \mathbf{W}_\theta = b_\theta\mathbf{E}\mathbf{n},\qquad \mathbf{A}_\sigma = c_\sigma\mathbf{I} + d_\sigma\mathbf{n}\otimes\mathbf{n}$$

for each $\mathbf{Q} \in D_{\infty h} \setminus D_{4h}$, and moreover the point $\mathbf{X}_0$ given above satisfies the last two equations given before. Evidently,

$$\mathbf{Q}_0 * \mathbf{X}_0 = \mathbf{X}_0, \qquad \mathbf{Q}_0 = -\mathbf{R}_{\mathbf{n}}^{\pi/2} \in D_{2d}$$

for any point $\mathbf{X}_0 \in \mathbf{S} \cap (\mathbf{Q} * \mathbf{S})$. Then we have

$$\mathbf{Q}_0 * (\mathbf{F}(\mathbf{X}_0)) = \mathbf{F}(\mathbf{Q}_0 * \mathbf{X}_0) = \mathbf{F}(\mathbf{X}_0)$$

for any $\mathbf{F} \in D_{2d}(D, M)$, $\mathbf{X}_0 \in \mathbf{S} \cap (\mathbf{Q} * \mathbf{S})$ and $\mathbf{Q} \in D_{\infty h} \setminus D_{4h}$. Thus, we deduce (2.7) with $a(\mathbf{X}_0) = 0$. Applying the latter fact and the fact that

$$\forall\, \mathbf{Q} \in D_{\infty h}, \exists \mathbf{Q}' \in D_{2d} : \ \mathbf{Q} * (\mathbf{En}) = \mathbf{Q}' * (\mathbf{En}),\ \mathbf{Q} * (\mathbf{n} \otimes \mathbf{n}) = \mathbf{Q}' * (\mathbf{n} \otimes \mathbf{n}),$$

we deduce that the condition (2.2) is satisfied for each $\mathbf{Q} \in D_{\infty h} \setminus D_{4h}$.

Finally, for each $\mathbf{Q} \in D_{4h} \setminus D_{2d}$, by means of (3.23), where $m = 1$, we infer that the intersection point $\mathbf{X}_0 = (\mathbf{v}_\alpha, \mathbf{W}_\theta, \mathbf{A}_\sigma) \in \mathbf{S} \cap (\mathbf{Q} * \mathbf{S})$ is of the form

$$\mathbf{v}_\alpha = \mathbf{0}, \qquad \mathbf{W}_\theta = b_\theta \mathbf{En}, \qquad \mathbf{A}_\sigma = c_\sigma \mathbf{I} + d_\sigma \mathbf{n} \otimes \mathbf{n} + h_\sigma \mathbf{D}_1$$

with $b_\theta h_\sigma = 0$ for $\theta = 1, \ldots, b$ and $\sigma = 1, \ldots, c$. Hence, for each $\mathbf{Q} \in D_{4h} \setminus D_{2d}$, the point $\mathbf{X}_0$ is given by (3.31) or by (3.25) with $\mathbf{e}_1 = \mathbf{e}$ and $\mathbf{e}_2 = \mathbf{e}'$. From the argument given above for the corresponding case, we know that the condition (2.2) is satisfied for the point $\mathbf{X}_0$ given by (3.31). On the other hand, from the latter part of the proof for Theorem 4, we know that the condition (2.2) is also satisfied for the point $\mathbf{X}_0$ given by (3.25). Thus, we conclude that the given surface $\mathbf{S}(\mathbf{X})$ also fulfils the condition (2.2) for each $\mathbf{Q} \in D_{4h} \setminus D_{2d}$. *Q.E.D.*

By virtue of the same argument as that used to derive the corollary of Theorem 4, we arrive at the following simplified result.

COROLLARY. The surface

$$\mathbf{S}(\mathbf{X}) = (\mathbf{n} \otimes \mathbf{n};\ \mathbf{n} \vee \boldsymbol{\eta}_1(\mathbf{v}^{\circ}) + (\mathbf{v} \cdot \mathbf{n})\mathbf{D}_1;\ \boldsymbol{\eta}_1(\mathbf{W}_\theta \mathbf{n}); \\ \boldsymbol{\eta}_1((\mathbf{An})^{\circ}) + f_1(\mathbf{A})\mathbf{n}, \boldsymbol{\Phi}_1(\mathbf{q}(\mathbf{A}_\sigma))) \tag{3.32}$$

is an IES for $D_{2d}(D, M)$ for each $M \in \{R, \mathrm{Skw}, \mathrm{Sym}\}$.

3.8. The tetragonal crystal class S_4

$$S_4 = D_{2d} \cap C_{\infty h} = \{(-1)^k \mathbf{R}_{\mathbf{n}}^{k\pi/2} \mid k = 0, 1, 2, 3\}. \tag{3.33}$$

THEOREM 7. *Let* $\mathbf{e}$ *and* $\mathbf{e}'$ *be any two orthonormal vectors on the* $\mathbf{n}$*-plane. Then the surface*

$$\mathbf{S}(\mathbf{X}) = (\mathbf{En};\ \mathbf{n} \vee \boldsymbol{\eta}_1(\mathbf{v}^{\circ}) + (\mathbf{v} \cdot \mathbf{n})\mathbf{D}_1;\ \boldsymbol{\eta}_1(\mathbf{W}_\theta \mathbf{n}); \\ \boldsymbol{\eta}_1((\mathbf{An})^{\circ}) + f_1(\mathbf{A})\mathbf{n},\ g_1(\mathbf{A}_\sigma)\mathbf{n}) \tag{3.34}$$

is an IES for $S_4(D, M)$*, where each tensor function is given in Theorem 6.*

P r o o f. From the former part of the proof for Theorem 5 we know that each tensor function except $(\mathbf{v}\cdot\mathbf{n})\mathbf{D}_1$ is invariant under the group S_4. Moreover, it is known that the tensor function just indicated is invariant under the group $D_{2d}(\supset S_4)$ (cf. the former part of the proof for Theorem 6). Thus, we conclude that the given surface $\mathbf{S}(\mathbf{X})$ meets the invariance condition (2.4).

In what follows we prove that the given surface $\mathbf{S}(\mathbf{X})$ obeys the condition (2.2). It can be easily verified by using $(3.7)_3$ that for each $\mathbf{Q}\in \mathrm{Orth}\setminus(C_{\infty h}\setminus S_4)$ the condition (2.2) can be satisfied. Moreover, for each $\mathbf{Q}\in C_{\infty h}\setminus S_4$, the intersection point $\mathbf{X}_0=(\mathbf{v}_\alpha,\mathbf{W}_\theta,\mathbf{A}_\sigma)\in \mathbf{S}\cap(\mathbf{Q}*\mathbf{S})$ is determined by (A.15)–(A.17) (for $m=1$) and

$$((\mathbf{Q}^{\mathrm T}\mathbf{v})\cdot\mathbf{n})\mathbf{Q}*\mathbf{D}_1=(\mathbf{v}\cdot\mathbf{n})\mathbf{D}_1\,,$$
$$f_1(\mathbf{Q}^{\mathrm T}*\mathbf{A})\mathbf{Q}\mathbf{n}=f_1(\mathbf{A})\mathbf{n},\qquad g_1(\mathbf{Q}^{\mathrm T}*\mathbf{A})\mathbf{Q}\mathbf{n}=g_1(\mathbf{A})\mathbf{n},$$

where the variables are: $\mathbf{v}=\mathbf{v}_1,\dots,\mathbf{v}_a$; $\mathbf{W}=\mathbf{W}_1,\dots,\mathbf{W}_b$; $\mathbf{A}=\mathbf{A}_1,\dots,\mathbf{A}_c$. From Theorem A.4 we derive

$$\mathbf{v}_\alpha=a_\alpha\mathbf{n},\qquad \mathbf{W}_\sigma=b_\sigma\mathbf{E}\mathbf{n},\qquad \mathbf{A}_\sigma=c_\sigma\mathbf{I}+d_\sigma\mathbf{n}\otimes\mathbf{n}+p_\sigma\mathbf{D}_1+q_\sigma\mathbf{D}_2\,.$$

From the last three equations given before we further derive $a_\alpha=p_\sigma=q_\sigma=0$ and hence the intersection point $\mathbf{X}_0$ is given by (3.31). Thus, from the corresponding case in the proof of Theorem 6 we conclude that the condition (2.2) can be satisfied for each $\mathbf{Q}\in C_{\infty h}\setminus S_4$. *Q.E.D.*

3.9. Remark

In each IES given in this section, each vector and each second order tensor, except the constant tensor $\mathbf{D}(G)$, is a *homogeneous polynomial function* of some components of the vector variable and/or the second order tensor variable concerned. In reality, the trigonometric functions $\cos r\theta$ and $\sin r\theta$ for each integer $r\geq 1$ are associated with the following two kinds of Tschebysheff polynomials.

$$H_r(\cos\theta)=\cos r\theta,\qquad T_r(\sin\theta)=\frac{\sin(r+1)\theta}{\cos\theta}\,. \tag{3.35}$$

Let $C_r(x)\in\{H_r(x),T_r(x)\}$. Then we have

$$C_r(x)=\begin{cases}\sum\limits_{k=0}^{n}c_{2k}x^{2k} & \text{if}\quad r=2n\,,\\ \sum\limits_{k=1}^{n}c_{2k-1}x^{2k-1} & \text{if}\quad r=2n-1\,,\end{cases} \tag{3.36}$$

where each c_k is a constant. Hence, with the aid of the above formulas and (2.11), we infer that for any vector $\mathbf{z}$ on the $\mathbf{n}$-plane, the functions $|\mathbf{z}|^r\cos r<\mathbf{z},\mathbf{e}>$

and $|\mathbf{z}|^r \sin r < \mathbf{z}, \mathbf{e} >$ for each $r \geq 1$, which are used to construct each presented IES, are homogeneous polynomials of degree r in the components $\mathbf{z}\cdot\mathbf{e}$ and $\mathbf{z}\cdot\mathbf{e}'$, where $\mathbf{z} = \mathbf{v}^o$, $\mathbf{Wn}$, $(\mathbf{An})^o$, $\mathbf{q}(\mathbf{A})$.

The results given in this section simplify the corresponding ones given in [49]. In reality, in each IES given here, each tensor function is presented in concise and clear forms, while in each IES given in [49], each tensor function is given in a somewhat implicit and complicated summation form.

Other remarks will be given in Sec. 6.

4. Cubic crystal classes O_h, T_d and T_h

$$O_h = \left(\bigcup_{k=1}^{3} (C_{4h}(\mathbf{n}_k) \cup C_{2h}(\mathbf{p}_k) \cup C_{2h}(\mathbf{q}_k))\right) \cup \left(\bigcup_{t=1}^{4} S_6(\mathbf{r}_t)\right), \tag{4.1}$$

$$T_d = \left(\bigcup_{k=1}^{3} (S_4(\mathbf{n}_k) \cup C_{1h}(\mathbf{p}_k) \cup C_{1h}(\mathbf{q}_k))\right) \cup \left(\bigcup_{t=1}^{4} C_3(\mathbf{r}_t)\right), \tag{4.2}$$

$$T_h = \left(\bigcup_{k=1}^{3} C_{2h}(\mathbf{n}_k)\right) \cup \left(\bigcup_{t=1}^{4} S_6(\mathbf{r}_t)\right), \tag{4.3}$$

where $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ are three orthonormal vectors and

$$\begin{aligned} \sqrt{2}\mathbf{p}_1 &= \mathbf{n}_3 + \mathbf{n}_2, \qquad \sqrt{2}\mathbf{p}_2 = \mathbf{n}_1 + \mathbf{n}_3, \qquad \sqrt{2}\mathbf{p}_3 = \mathbf{n}_2 + \mathbf{n}_1, \\ \sqrt{2}\mathbf{q}_1 &= \mathbf{n}_3 - \mathbf{n}_2, \qquad \sqrt{2}\mathbf{q}_2 = \mathbf{n}_1 - \mathbf{n}_3, \qquad \sqrt{2}\mathbf{q}_3 = \mathbf{n}_2 - \mathbf{n}_1; \end{aligned} \tag{4.4}$$

$$\begin{aligned} \sqrt{3}\mathbf{r}_1 &= \mathbf{n}_1 - \mathbf{n}_2 - \mathbf{n}_3, \qquad \sqrt{3}\mathbf{r}_2 = \mathbf{n}_2 - \mathbf{n}_3 - \mathbf{n}_1, \\ \sqrt{3}\mathbf{r}_3 &= \mathbf{n}_3 - \mathbf{n}_1 - \mathbf{n}_2, \qquad \sqrt{3}\mathbf{r}_4 = \mathbf{n}_1 + \mathbf{n}_2 + \mathbf{n}_3. \end{aligned} \tag{4.5}$$

Each $\mathbf{n}_k$ is called a four-fold axis of either of the groups O_h and T_d or a two-fold axis of the group T_h, and each $\mathbf{r}_t$ is called a three-fold axis of each of the groups O_h and T_d and T_h.

Here and hereafter, for any unit vector $\mathbf{u}$ and each integer $m \geq 2$, $S_{2m}(\mathbf{u})$ and $C_{2mh}(\mathbf{u})$ are used to denote the groups obtained by the replacement of $\mathbf{n}$ with $\mathbf{u}$ in (3.8) and $(3.9)_2$, (3.13) and $(3.14)_2$, and (3.27), respectively. Moreover, $C_3(\mathbf{u})$ is used to denote the rotation subgroup of $S_6(\mathbf{u})$. Finally,

$$C_{1h}(\mathbf{u}) = \{\mathbf{I}, -\mathbf{R}_{\mathbf{u}}^{\pi}\}, \qquad C_{2h}(\mathbf{u}) = \{\pm\mathbf{I}, \pm\mathbf{R}_{\mathbf{u}}^{\pi}\}.$$

4.1. The class O_h

The following fourth-order tensor is invariant under the group O_h:

$$\mathbf{O}_h = \sum_{k=1}^{3} (\overset{4}{\otimes}\, \mathbf{n}_k). \tag{4.6}$$

In this section and the next section, each presented surface $\mathbf{S}(\mathbf{X})$ is formed by tensor functions of the form

$$\mathbf{G} \odot (\overset{s}{\otimes} \mathbf{Z}),$$

where the tensor $\mathbf{G}$ is invariant under the anisotropy group G concerned, and $\mathbf{Z}$ is one of the vector variables and the second order tensor variables. Evidently, each such tensor function is invariant under the group G concerned and therefore the given surface $\mathbf{S}(\mathbf{X})$ meets the invariance condition (2.4). As a result, henceforth only the invariance of the tensor $\mathbf{G}$ is indicated and the invariance condition (2.4) is no longer mentioned.

THEOREM 8. *The surface*

$$\mathbf{S}(\mathbf{X}) = (\mathbf{O}_h : (\overset{2}{\otimes} \mathbf{v}_\alpha);\ \mathbf{O}_h : (\overset{2}{\otimes} (\mathbf{E} : \mathbf{W}_\theta));\ \mathbf{O}_h : \mathbf{A}_\sigma,\ \mathbf{O}_h : \mathbf{A}_\sigma^2) \tag{4.7}$$

is an IES for $O_h(D, M)$.

P r o o f. It is evident that the condition (2.2) can be satisfied for each $\mathbf{Q} \in O_h$. On the other hand, for each $\mathbf{Q} \in \mathrm{Orth}\backslash O_h$, by Theorem A.5 we know that the point $\mathbf{X}_0 = (\mathbf{v}_\alpha, \mathbf{W}_\theta, \mathbf{A}_\sigma) \in \mathbf{S} \cap (\mathbf{Q} * \mathbf{S})$ is given by the following cases.

CASE 1. If there exist $\mathbf{u}$, $\mathbf{v} \in \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ or $\mathbf{u}$, $\mathbf{v} \in \{\mathbf{r}_1, \dots, \mathbf{r}_4\}$ such that $\overset{2}{\otimes} (\mathbf{Q}^{\mathrm{T}}\mathbf{u}) = \overset{2}{\otimes} \mathbf{v}$, then $\mathbf{X}_0 \in D(\mathbf{u})$;

CASE 2. If $\overset{2}{\otimes} (\mathbf{Q}^{\mathrm{T}}\mathbf{u}) \neq \overset{2}{\otimes} \mathbf{v}$ for any $\mathbf{u}, \mathbf{v} \in \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ and any $\mathbf{u}, \mathbf{v} \in \{\mathbf{r}_1, \dots, \mathbf{r}_4\}$, then $\mathbf{v}_\alpha = \mathbf{0}$, $\mathbf{W}_\theta = \mathbf{O}$, $\mathbf{A}_\sigma = c_\sigma \mathbf{I}$.

For Case 1, for each $\mathbf{F} \in O_h(D, M)$ we have

$$\mathbf{R}_0 * (\mathbf{F}(\mathbf{X}_0)) = \mathbf{F}(\mathbf{R}_0 * \mathbf{X}_0) = \mathbf{F}(\mathbf{X}_0),$$

where

$$\mathbf{R}_0 = \mathbf{R}_{\mathbf{u}}^{\theta} \in O_h, \qquad \theta = \begin{cases} \pi/2, & \mathbf{u} \in \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}, \\ 2\pi/3, & \mathbf{u} \in \{\mathbf{r}_1, \dots, \mathbf{r}_4\}. \end{cases}$$

From these we deduce

$$\mathbf{F}(\mathbf{X}_0) = \begin{cases} a(\mathbf{X}_0)\mathbf{u}, & M = V, \\ b(\mathbf{X}_0)\mathbf{E}\mathbf{u}, & M = \mathrm{Skw}, \\ c(\mathbf{X}_0)\mathbf{I} + d(\mathbf{X}_0)\mathbf{u} \otimes \mathbf{u}, & M = \mathrm{Sym}. \end{cases} \tag{4.8}$$

Then, by using the latter and the fact that for each $\mathbf{Q}$ in question, there exists $\mathbf{Q}_0 \in O_h$ such that

$$\mathbf{Q}^{\mathrm{T}}\mathbf{u} = \mathbf{Q}_0{}^{\mathrm{T}}\mathbf{u} \qquad \text{and} \qquad \mathbf{Q}^{\mathrm{T}} * (\mathbf{E}\mathbf{u}) = \mathbf{Q}_0{}^{\mathrm{T}} * (\mathbf{E}\mathbf{u}),$$

we infer

$$\mathbf{F}(\mathbf{Q}^{\mathrm{T}} * \mathbf{X}_0) = \mathbf{F}({\mathbf{Q}_0}^{\mathrm{T}} * \mathbf{X}_0) = {\mathbf{Q}_0}^{\mathrm{T}} * (\mathbf{F}(\mathbf{X}_0)) = \mathbf{Q}^{\mathrm{T}} * (\mathbf{F}(\mathbf{X}_0))$$

for any $\mathbf{F} \in O_h(D, M)$ and therefore the condition (2.2) is satisfied.

Moreover, for Case 2, we have

$$\mathbf{R}_0 * (\mathbf{F}(\mathbf{X}_0)) = \mathbf{F}(\mathbf{R}_0 * \mathbf{X}_0) = \mathbf{F}(\mathbf{X}_0)$$

for any $\mathbf{F} \in O_h(D, M)$ and any $\mathbf{R}_0 \in O_h$. From this we derive

$$\mathbf{F}(\mathbf{X}_0) = \begin{cases} \mathbf{0}, & M = V\,, \\ \mathbf{O}, & M = \mathrm{Skw}\,, \\ c(\mathbf{X}_0)\mathbf{I}, & M = \mathrm{Sym}\,, \end{cases}$$

and hence for any $\mathbf{Q} \in \mathrm{Orth}$,

$$\mathbf{F}(\mathbf{Q}^{\mathrm{T}} * \mathbf{X}_0) = \mathbf{F}(\mathbf{X}_0) = \mathbf{Q}^{\mathrm{T}} * (\mathbf{F}(\mathbf{X}_0)),$$

i.e. the condition (2.2) is fulfilled. *Q.E.D.*

4.2. The class T_d

The following third-order tensor is invariant under the group T_d:

$$\mathbf{T}_d = \sum_{k=1}^{3} \boldsymbol{\omega}_k \otimes \mathbf{n}_k = \sum_{k=1}^{3} \mathbf{n}_k \otimes \boldsymbol{\omega}_k\,, \tag{4.9}$$

where

$$\boldsymbol{\omega}_1 = \mathbf{n}_2 \vee \mathbf{n}_3\,, \qquad \boldsymbol{\omega}_2 = \mathbf{n}_3 \vee \mathbf{n}_1\,, \qquad \boldsymbol{\omega}_3 = \mathbf{n}_1 \vee \mathbf{n}_2\,. \tag{4.10}$$

THEOREM 9. *The surface*

$$\begin{aligned} \mathbf{S}(\mathbf{X}) = (\mathbf{T}_d\mathbf{v}_\alpha,\ \mathbf{O}_h : (\overset{2}{\otimes}\, \mathbf{v}_\alpha); \mathbf{T}_d : (\overset{2}{\otimes}\, (\mathbf{E} : \mathbf{W}_\theta)), \\ \mathbf{O}_h : (\overset{2}{\otimes}\, (\mathbf{E} : \mathbf{W}_\theta));\ \mathbf{T}_d : \mathbf{A}_\sigma,\ \mathbf{O}_h : \mathbf{A}_\sigma;\ \mathbf{T}_d : (\mathbf{W}_\theta \mathbf{A}_\sigma)) \end{aligned} \tag{4.11}$$

is an IES for $T_d(D, M)$.

P r o o f. It is evident that for each $\mathbf{Q} \in T_d$ the condition (2.2) can be satisfied. In what follows we prove that the condition (2.2) can also be satisfied for each $\mathbf{Q} \in \mathrm{Orth} \setminus T_d$. First, for each $\mathbf{Q} \in \mathrm{Orth} \setminus O_h$, by using Theorem A.6 we know that the point $\mathbf{X}_0 = (\mathbf{v}_\alpha,\ \mathbf{W}_\theta,\ \mathbf{A}_\sigma) \in \mathbf{S} \cap (\mathbf{Q} * \mathbf{S})$ is given by the following cases.

CASE 1. $\mathbf{v}_\alpha = 0$, $\mathbf{W}_\sigma = b_\theta \mathbf{E}\mathbf{u}$, $\mathbf{A}_\sigma = c_\sigma \mathbf{I} + d_\sigma \mathbf{u} \otimes \mathbf{u}$ if

$$\exists\, \mathbf{u}, \mathbf{v} \in \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\} : \overset{2}{\otimes} (\mathbf{Q}^{\mathrm{T}}\mathbf{u}) = \overset{2}{\otimes} \mathbf{v};$$

CASE 2. $\mathbf{X}_0 \in D(\mathbf{u})$ if

$$\exists\, \mathbf{u},\ \mathbf{v} \in \{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4\} :\ \mathbf{Q}^{\mathrm{T}}\mathbf{u} = \mathbf{v}\,; \tag{4.12}$$

CASE 3. $\mathbf{v}_\alpha = \mathbf{0}$, $\mathbf{W}_\theta = \mathbf{O}$, $\mathbf{A}_\sigma = c_\sigma \mathbf{I}$, if $\mathbf{Q}$ obeys

$$\forall\, \mathbf{u},\ \mathbf{v} \in \{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4\} :\ \mathbf{Q}^{\mathrm{T}}\mathbf{u} \neq \mathbf{v} \quad \text{and} \quad \forall\, \mathbf{u}, \mathbf{v} \in \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\} : \overset{2}{\otimes} (\mathbf{Q}^{\mathrm{T}}\mathbf{u}) \neq \overset{2}{\otimes} \mathbf{v}\,. \tag{4.13}$$

It is readily verfied that the condition (2.2) can be satisfied for Case 3 and Case 1. For Case 2, we have

$$\mathbf{R}_0 * (F(\mathbf{X}_0)) = F(\mathbf{R}_0 * \mathbf{X}_0) = F(\mathbf{X}_0), \qquad \mathbf{R}_0 = \mathbf{R}_{\mathbf{u}}^{2\pi/3} \in T_d$$

for each $\mathbf{F} \in T_d(D, M)$. From this we derive (4.8). Then by using the fact that for each $\mathbf{Q}$ satisfying (4.12) there is $\mathbf{Q}_0 \in T_d$ such that

$$\mathbf{Q}^{\mathrm{T}}\mathbf{u} = \mathbf{Q}_0{}^{\mathrm{T}}\mathbf{u}, \qquad \mathbf{Q}^{\mathrm{T}} * (\mathbf{E}\mathbf{u}) = \mathbf{Q}_0^{\mathrm{T}} * (\mathbf{E}\mathbf{u})\,,$$

we infer

$$\mathbf{F}(\mathbf{Q}^{\mathrm{T}} * \mathbf{X}_0)) = \mathbf{F}(\mathbf{Q}_0^{\mathrm{T}} * \mathbf{X}_0) = \mathbf{Q}_0^{\mathrm{T}} * (\mathbf{F}(\mathbf{X}_0)) = \mathbf{Q}^{\mathrm{T}} * (\mathbf{F}(\mathbf{X}_0)).$$

Thus the condition (2.2) is satisfied for Case 2.

Next, for each $\mathbf{Q} \in O_h \setminus T_d$, the point $\mathbf{X}_0 = (\mathbf{v}_\alpha,\ \mathbf{W}_\theta,\ \mathbf{A}_\sigma) \in \mathbf{S} \cap (\mathbf{Q} * \mathbf{S})$ is determined by

$$\mathbf{T}_d \mathbf{v}_\alpha = \mathbf{O}; \qquad \mathbf{T}_d : (\overset{2}{\otimes} (\mathbf{E} : \mathbf{W}_\theta)) = \mathbf{0}; \qquad \mathbf{T}_d : \mathbf{A}_\sigma = \mathbf{0}, \qquad \mathbf{T}_d : (\mathbf{W}_\theta \mathbf{A}_\sigma)) = 0,$$

where $\alpha = 1, 2, \ldots, a$, $\theta = 1, 2, \ldots, b$, $\sigma = 1, 2, \ldots, c$. The first three equations yield

$$\mathbf{v}_\alpha = \mathbf{0}, \qquad \mathbf{W}_\theta = b_\theta \mathbf{E}\mathbf{u}, \qquad \mathbf{u} \in \{\mathbf{n}_1,\ \mathbf{n}_2,\ \mathbf{n}_3\},$$
$$\mathbf{A}_\sigma = a_\sigma \mathbf{n}_1 \otimes \mathbf{n}_1 + b_\sigma \mathbf{n}_2 \otimes \mathbf{n}_2 + c_\sigma \mathbf{n}_3 \otimes \mathbf{n}_3\,;$$

and the last equation further produces

$$\mathbf{v}_\alpha = \mathbf{0}, \qquad \mathbf{W}_\theta = b_\theta \mathbf{E}\mathbf{u}, \qquad \mathbf{A}_\sigma = c_\sigma \mathbf{I} + d_\sigma \mathbf{u} \otimes \mathbf{u}, \qquad \mathbf{u} \in \{\mathbf{n}_1,\ \mathbf{n}_2,\ \mathbf{n}_3\}\,;$$

or

$$\mathbf{v}_\alpha = \mathbf{0}, \qquad \mathbf{W}_\theta = \mathbf{O}, \qquad \mathbf{A}_\sigma = a_\sigma \mathbf{n}_1 \otimes \mathbf{n}_1 + b_\sigma \mathbf{n}_2 \otimes \mathbf{n}_2 + c_\sigma \mathbf{n}_3 \otimes \mathbf{n}_3\,.$$

For the former, we have

$$\mathbf{R}_0 * (F(\mathbf{X}_0)) = F(\mathbf{R}_0 * \mathbf{X}_0) = F(\mathbf{X}_0), \qquad \mathbf{R}_0 = -\mathbf{R}_{\mathbf{u}}^{\pi/2} \subset T_d\,,$$

for each $F \in T_d(D, M)$. From the above we derive (4.8) with $a(\mathbf{X}_0) = 0$. Then by using the latter and the fact that $\mathbf{Q}_0 = -\mathbf{Q} \in T_d$ for each $\mathbf{Q} \in O_h \setminus T_d$ we infer that the condition (2.2) is satisfied for the case at issue. For the latter case for $\mathbf{X}_0$ we have

$$\mathbf{R}_0 * (\mathbf{F}(\mathbf{X}_0)) = \mathbf{F}(\mathbf{R}_0 * \mathbf{X}_0) = \mathbf{F}(\mathbf{X}_0), \qquad \mathbf{R}_0 \in \{\mathbf{R}_{\mathbf{n}_1}^{\pi}, \mathbf{R}_{\mathbf{n}_2}^{\pi}, \mathbf{R}_{\mathbf{n}_3}^{\pi}\} \subset T_d$$

for each $\mathbf{F} \in T_d(D, M)$. From this we derive

$$F(X_0) = \begin{cases} \mathbf{0}, & M = V\,, \\ \mathbf{O}, & M = \mathrm{Skw}\,, \\ a(\mathbf{X}_0)\mathbf{n}_1 \otimes \mathbf{n}_1 + b(\mathbf{X}_0)\mathbf{n}_2 \otimes \mathbf{n}_2 + c(\mathbf{X}_0)\mathbf{n}_3 \otimes \mathbf{n}_3\,, & M = \mathrm{Sym}\,. \end{cases}$$

Then, using the latter and the fact that $-\mathbf{Q} \in T_d$ for each $\mathbf{Q} \in O_h \setminus T_d$, one may easily deduce that the condition (2.2) is also satisfied for the case in question. *Q.E.D.*

4.3. The class T_h

The following two fourth-order tensors are invariant under the group T_h:

$$\mathbf{T}_h^a = \sum_{k=1}^{3} \mathbf{E}\mathbf{n}_k \otimes \boldsymbol{\omega}_k\,, \tag{4.14}$$

$$\begin{aligned} \mathbf{T}_h^s &= (\mathbf{N}_2 - \mathbf{N}_3) \otimes \mathbf{N}_1 + (\mathbf{N}_3 - \mathbf{N}_1) \otimes \mathbf{N}_2 + (\mathbf{N}_1 - \mathbf{N}_2) \otimes \mathbf{N}_3\,, \\ \mathbf{N}_k &= \mathbf{n}_k \otimes \mathbf{n}_k\,, \qquad k = 1, 2, 3\,, \end{aligned} \tag{4.15}$$

where $\boldsymbol{\omega}_k$, $k = 1, 2, 3$, are given by (4.10).

THEOREM 10. *The surface*

$$\mathbf{S}(\mathbf{X}) = \Big(\mathbf{T}_h^a : (\overset{2}{\otimes}\, \mathbf{v}_\alpha), \mathbf{T}_h^s : (\overset{2}{\otimes}\, \mathbf{v}_\alpha);\ \mathbf{T}_h^a : (\overset{2}{\otimes}\, (\mathbf{E} : \mathbf{W}_\theta)), \mathbf{T}_h^s : (\overset{2}{\otimes}\, (\mathbf{E} : \mathbf{W}_\theta)); \mathbf{T}_h^a : \mathbf{A}_\sigma, \mathbf{T}_h^s : \mathbf{A}_\sigma\Big) \tag{4.16}$$

is an IES for $T_h(D, M)$.

P r o o f. It is evident that the condition (2.2) can be satisfied for each $\mathbf{Q} \in T_h$. Moreover, for each $\mathbf{Q} \in \mathrm{Orth}\setminus T_h$, by Theorem A.7 we infer that the point $\mathbf{X}_0 \in \mathbf{S} \cap (\mathbf{Q} * \mathbf{S})$ is given by the two cases

CASE 1. $\mathbf{X}_0 \in D(\mathbf{u})$ if $\exists \mathbf{u}, \mathbf{v} \in \{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4\}$: $\mathbf{Q}^T\mathbf{u} = (\det \mathbf{Q})\mathbf{v}$;

CASE 2. $\mathbf{v}_\alpha = \mathbf{0}, \mathbf{W}_\theta = \mathbf{O}, \mathbf{A}_\sigma = c_\sigma \mathbf{I}$ if $\forall \mathbf{u}, \mathbf{v} \in \{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4\}$: $\mathbf{Q}^T\mathbf{u} \neq (\det \mathbf{Q})\mathbf{v}$.

It can be easily proved that the condition (2.2) can be satisfied for Case 2. Moreover, by means of the similar procedure used in the proof for Case 2 of Theorem 9, it can be verified that the condition (2.2) can also be satisfied for Case 1 shown above. *Q.E.D.*

5. The icosahedral group I_h

The icosahedral group I_h is the most complicated yet intriguing one in all subgroups of Orth, which characterizes the symmetry of the icosahedron. In a famous lecture delivered in 1884, F. KLEIN [15] presented a comprehensive account of the icosahedron and the icosahedral group. According to classical crystallography, there exists no solid whose symmetry is described by the icosahedral group or any other non-crystallographic point group except the transverse isotropy groups and the full and proper orthogonal groups. However, such a traditional viewpoint has been proved to be too narrow by the recent discovery of quasicrystals (cf. Vainshtein [44] and Senechal [27] and the related literature therein). The latter possess symmetries forbidden by the classical crystallography rule, such as five-, eight-, and ten-fold symmetries etc. Of them, the icosahedral quasicrystal is the one which has received much attention.

The icosahedral group I_h is of the form

$$I_h = \left(\bigcup_{s=1}^{6} S_{10}(\mathbf{n}_s) \right) \cup \left(\bigcup_{t=1}^{10} S_6(\mathbf{r}_t) \right) \cup \left(\bigcup_{c=1}^{15} C_{2h}(\mathbf{a}_c) \right), \tag{5.1}$$

where the groups $S_{10}(\mathbf{u})$, $S_6(\mathbf{u})$ and $C_{2h}(\mathbf{u})$ for any unit vector $\mathbf{u}$ are indicated at the start of §4.

The unit vectors $\mathbf{n}_\alpha$, $\mathbf{r}_\sigma$, $\mathbf{a}_\tau$, $\alpha = 1, \ldots, 6$; $\sigma = 1, \ldots, 10$; $\tau = 1, \ldots, 15$ are used to represent the six five-fold axes, the ten three-fold axes and the fifteen two-fold axes, respectively. Let $\mathbf{n}$ and $\mathbf{e}$ be two orthonormal vectors. Then the six five-fold axes of I_h are expressible in the form (cf. XIAO [53])

$$\begin{aligned} \mathbf{n}_6 &= \mathbf{n}, \\ \mathbf{n}_k &= (\mathbf{n} + 2\mathbf{l}_k)/\sqrt{5} = \mathbf{R}_\mathbf{n}^{2k\pi/5}\mathbf{n}_5, \\ \mathbf{l}_k &= \mathbf{R}_\mathbf{n}^{2k\pi/5}\mathbf{e}, \qquad k = 1, \ldots, 5, \end{aligned} \tag{5.2}$$

with the property

$$(\mathbf{n}_i \cdot \mathbf{n}_j)^2 = \frac{1}{5} + \frac{4}{5}\delta_{ij}, \qquad i, j = 1, \ldots, 6. \tag{5.3}$$

Moreover, each three-fold axis $\mathbf{r}_t$ and each two-fold axis $\mathbf{l}_c$ can be determined by the five-fold axes $\mathbf{n}_s$, refer to XIAO [52] for detail.

The follwing three tensors are invariant under the group I_h:

$$\mathbf{I}_h^r = \sum_{k=1}^{6} \left(\overset{2r+4}{\otimes} \mathbf{n}_k \right), \qquad r = 1, 2, 3. \tag{5.4}$$

THEOREM 11. *The surface*

$$\mathbf{S}(\mathbf{X}) = \Big(\mathbf{I}_h^1 \odot (\overset{4}{\otimes} \mathbf{v}_\alpha), \mathbf{I}_h^2 \odot (\overset{6}{\otimes} \mathbf{v}_\alpha), \mathbf{I}_h^3 \odot (\overset{8}{\otimes} \mathbf{v}_\alpha); \mathbf{I}_h^1 \odot (\overset{4}{\otimes} (\mathbf{E} : \mathbf{W}_\theta)), \tag{5.5}$$

$$\mathbf{I}_h^2 \odot (\overset{6}{\otimes} (\mathbf{E} : \mathbf{W}_\theta)), \mathbf{I}_h^3 \odot (\overset{8}{\otimes} (\mathbf{E} : \mathbf{W}_\theta)); \mathbf{I}_h^1 \odot (\overset{2}{\otimes} \mathbf{A}_\sigma), \mathbf{I}_h^2 \odot (\overset{3}{\otimes} \mathbf{A}_\sigma), \mathbf{I}_h^3 \odot (\overset{4}{\otimes} \mathbf{A}_\sigma)\Big)$$

is an IES for $I_h(D, M)$.

P r o o f. It suffices to prove that (2.2) holds for each $\mathbf{Q} \in \mathrm{Orth} \setminus I_h$. For $\mathbf{X}_0 = (\mathbf{v}_\alpha, \mathbf{W}_\theta, \mathbf{A}_\sigma) \in \mathbf{S} \cap (\mathbf{Q} * \mathbf{S})$, $\mathbf{Q} \in \mathrm{Orth} \setminus I_h$, by applying Theorem A.8 we infer that $\mathbf{X}_0 \in D(\mathbf{u})$ if $\mathbf{Q}$ satisfies (A.62) and that $\mathbf{v}_\alpha = 0$, $\mathbf{W}_\theta = \mathbf{O}$, $\mathbf{A}_\sigma = c_\sigma \mathbf{I}$ if $\mathbf{Q}$ satisfies (A.63). By means of these facts and the procedure used in the proof of Theorem 8, it can be proved that the condition (2.2) is satisfied. *Q.E.D.*

6. Examples and concluding remarks

Employing the results presented in the previous sections as well as the well-known representation theorems for isotropic functions of vectors and second order tensors, one can derive complete representations for any type of scalar-, vector- and second order tensor-valued anisotropic functions of vectors and second order tensors merely replacing some variables of the former with $\mathbf{S}(\mathbf{X})$ (cf. (2.3)). It should be noted, however, that representations obtained in this manner are generally not irreducible. To obtain complete irreducible representations, further effort should be made. Recently, the general results given here have been used to investigate various kinds of anisotropic functions of vectors and second order tensors. Simple irreducible functional bases and generating sets for scalar-valued and symmetric second order tensor-valued anisotropic functions of a single symmetric second order tensor have been obtained for all thirty-two crystal classes (cf. XIAO [47, 50, 51] and all noncrystal classes (cf. XIAO [53, 54]). Moreover, irreducible representations for scalar-, vector- and second order tensor-valued anisotropic functions of any finite number of vectors and second order tensors have been derived for all kinds of subgroups of the transverse isotropy group $C_{\infty h}$ (cf. XIAO [55]).

The extension theorems presented in the previous sections are concerned with anisotropic functions with an arbitrary number of vector and second order tensor variables. Recently, this author (see XIAO [48]) has proved that representation problems for rth-order tensor-valued isotropic or anisotropic tensor functions with an arbitrary number of vector and second order tensor variables can be reduced to those for certain rth-order tensor-valued isotropic or anisotropic tensor functions merely with not more than three (for $r \geq 1$) or four (for $r = 0$) vector and/or second order tensor variables (see [56] for further results). According to this fact, to derive a complete representation for any given type of anisotropic functions of vectors and second order tensors, it suffices to apply the corresponding extension theorem given here to treat the related anisotropic functions of not more than three or four vectors and/or second order tensors.

As an example, we apply Theorem 3 to derive irreducible nonpolynomial representations for scalar-valued and vector-valued anisotropic functions of any finite number of vectors relative to the group D_{2m+1h} for each integer $m \geq 1$.

According to Theorem A and Theorem 3, anisotropic functions of the a vector variables $\mathbf{v}_1, \ldots, \mathbf{v}_a$ relative to the group D_{2m+1h} can be extended as isotropic functions of the extended variables $(\mathbf{v}_\alpha, \boldsymbol{\eta}_{2m}(\mathbf{v}^{\circ}_\alpha), \mathbf{N})$, where $\mathbf{N} = \mathbf{n} \otimes \mathbf{n}$. Thus, applying the well-known results for representations of isotropic functions (cf. Wang [45] and Smith [32], *et al.*), we obtain a functional basis and a generating set for scalar-valued and vector-valued anisotropic functions of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_a$ relative to the group D_{2m+1h} as follows.

Functional basis:

$$|\mathbf{v}|^2,\ \mathbf{u}\cdot\mathbf{v},\ \mathbf{v}\cdot\mathbf{N}\mathbf{v},\ \mathbf{v}\cdot\mathbf{N}^2\mathbf{v}, \mathbf{u}\cdot\mathbf{N}\mathbf{v},\ \mathbf{u}\cdot\mathbf{N}^2\mathbf{v};$$

Generating set:

$$\mathbf{v}, \mathbf{N}\mathbf{v}, \mathbf{N}^2\mathbf{v},$$

where $\mathbf{u}, \mathbf{v} = \mathbf{v}_1, \ldots, \mathbf{v}_a, \boldsymbol{\eta}_{2m}(\mathbf{v}^{\circ}_1), \ldots, \boldsymbol{\eta}_{2m}(\mathbf{v}^{\circ}_a)$, $\mathbf{u} \neq \mathbf{v}$ and $\boldsymbol{\eta}_{2m}(\mathbf{v}^{\circ})$ is given by (3.12).

Since

$$\mathbf{N}\boldsymbol{\eta}_{2m}(\mathbf{v}^{\circ}) = \mathbf{0}, \qquad \mathbf{N}^2 = \mathbf{N},$$

either of the above two sets includes a large number of obviously redundant elements. Removing the latter and noticing the identity

$$|\mathbf{v}|^2 = |\mathbf{v}^{\circ}|^2 + (\mathbf{v}\cdot\mathbf{n})^2,$$

we arrive at the following simplified results.

Functional basis:

$$(\mathbf{u}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{n}), \mathbf{u}^{\circ}\cdot\mathbf{v}^{\circ}, |\mathbf{u}^{\circ}|\cdot|\mathbf{v}^{\circ}|^{2m}\cos(<\mathbf{u}^{\circ},\mathbf{e}> + 2m <\mathbf{v}^{\circ},\mathbf{e}>). \tag{6.1}$$

Generating set:

$$(\mathbf{v}\cdot\mathbf{n})\mathbf{n}, \mathbf{v}^{\circ}, |\mathbf{v}^{\circ}|^{2m}(\mathbf{e}\cos 2m <\mathbf{v}^{\circ},\mathbf{e}> - \mathbf{e}'\sin 2m <\mathbf{v}^{\circ},\mathbf{e}>). \tag{6.2}$$

In the above, $\mathbf{u}, \mathbf{v} = \mathbf{v}_1, \ldots, \mathbf{v}_a$, the unit vector $\mathbf{e}$ may be any two-fold rotation axis of D_{2m+1h} and $\mathbf{e}'$ is given by (2.10). In deriving the former, the invariants of the form

$$\boldsymbol{\eta}_{2m}(\mathbf{u}^o)\cdot\boldsymbol{\eta}_{2m}(\mathbf{v}^o) = (|\mathbf{u}^o|\cdot|\mathbf{v}^o|)^{2m}\cos 2m < \mathbf{u}^o, \mathbf{v}^o >,$$

which seems not obviously to be redundant, are also removed. In reality, by virtue of $(3.36)_1$ we know that the above invariant is expressible as a polynomial of degree $2m$ in $\mathbf{u}^o\cdot\mathbf{v}^o$ and $|\mathbf{u}|^o\cdot|\mathbf{v}^o|$ with constant coefficients. It can easily be proved that the functional basis given is irreducible, and moreover, that the generating set given is minimal.

It is worthwhile to point out the fact that the results derived above are valid for all *infinitely many* classes D_{2m+1h}. They provide all the desired representations *in a unified form*, while usually each anisotropy group has to be dealt with separately. The fact just indicated is also true for other kinds of subgroups of $D_{\infty h}$. Thus, as far as infinitely many classes of subgroups of $D_{\infty h}$ are concerned, *universal representations* may be derived by applying the extension theorems given in §3, as is done in the above and in [53–55].

Appendix A. General solutions to some related systems of polynomial tensor equations

In this appendix, we offer general solutions to some systems of polynomial tensor equations associated with the isotropic extension surfaces given in the previous sections. These results are used to determine the intersecting surface $\mathbf{S}\cap(\mathbf{Q}*\mathbf{S})\subset D$ for each presented IES $\mathbf{S}$.

Henceforth, δ is used to represent $+1$ or -1, i.e. $\delta^2 = 1$; m is used to signify any given positive integer; and $\mathbf{v}, \mathbf{x} \in V$, $\mathbf{W} \in \mathrm{Skw}$ and $\mathbf{A} \in \mathrm{Sym}$ are used to designate vector variable, skewsymmetric tensor variable and symmetric tensor variable, respectively.

A.1. Polynomial tensor equations: subgroups of $D_{\infty h}$

THEOREM A.1. *Let* $\boldsymbol{\eta}_r(\mathbf{z})$ *be the vector-valued function given by* (3.12) *for any vector* $\mathbf{z}$ *on the* $\mathbf{n}$*-plane and each integer* $r \geq 1$. *Then, for each* $\mathbf{Q} \in D_{\infty h} \setminus D_{2m+1d}$, *the general solution to the system of tensor equations*

(A.1) $$\mathbf{Q}*(\mathbf{E}\boldsymbol{\eta}_{2m}((\mathbf{Q}^T\mathbf{v})^o)) = \mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{v}^o),$$

(A.2) $$\mathbf{Q}*(\mathbf{E}\boldsymbol{\eta}_{2m}((\mathbf{Q}^T*\mathbf{W})\mathbf{n})) = \mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{W}\mathbf{n}),$$

(A.3) $$\mathbf{Q}*(\mathbf{E}\boldsymbol{\eta}_{2m}(((\mathbf{Q}^T*\mathbf{A})\mathbf{n})^o)) = \mathbf{E}\boldsymbol{\eta}_{2m}((\mathbf{A}\mathbf{n})^o),$$

(A.4) $$\mathbf{Q}*(\mathbf{E}\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{Q}^T*\mathbf{A}))) = \mathbf{E}\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A})),$$

is given by

(A.5) $$\mathbf{v} = x\mathbf{n}, \qquad \mathbf{W} = y\mathbf{E}\mathbf{n}, \qquad \mathbf{A} = z\mathbf{I} + w\mathbf{n}\otimes\mathbf{n} \quad (\forall\, x, y, z, w \in R).$$

P r o o f. Let $\mathbf{Q} = \delta\mathbf{R}_{\mathbf{n}}^{\theta}$. Then, by applying the formulas (2.13), $(2.15)_1$, $(2.17)_1$ and $(2.18)_1$ we convert Eqs. (A.1)–(A.4) to

$$|\mathbf{z}|^{2m}(\mathbf{e}\cos\Theta - \mathbf{e}'\sin\Theta) = |\mathbf{z}|^{2m}(\mathbf{e}\cos 2m < \mathbf{z},\mathbf{e} > - \mathbf{e}'\sin 2m < \mathbf{z},\mathbf{e} >),$$
$$|\mathbf{q}(\mathbf{A})|^{m}(\mathbf{e}\cos\Theta' - \mathbf{e}'\sin\Theta') = |\mathbf{q}(\mathbf{A})|^{m}(\mathbf{e}\cos m < \mathbf{q}(\mathbf{A}),\mathbf{e} > -\mathbf{e}'\sin m < \mathbf{q}(\mathbf{A}),\mathbf{e} >),$$

where

$$\Theta = -(2m+1)\theta + 2m < \mathbf{z},\mathbf{e} >, \qquad \mathbf{z} = \mathbf{v}^{\circ}, \mathbf{Wn}, (\mathbf{An})^{\circ},$$
$$\Theta' = -(2m+1)\theta + m < \mathbf{q}(\mathbf{A}),\mathbf{e} > .$$

Since $\mathbf{Q} \notin D_{2m+1d}$, we have $(2m+1)\theta \neq 2k\pi$. Then we derive

$$\text{(A.6)} \qquad |\mathbf{v}^{\circ}| = |\mathbf{Wn}| = |(\mathbf{An})^{\circ}| = |\mathbf{q}(\mathbf{A})| = 0.$$

Hence (A.5) holds for each $\mathbf{Q} = \delta\mathbf{R}_{\mathbf{n}}^{\theta} \in D_{\infty h} \setminus D_{2m+1d}$.

Next, let $\mathbf{Q} = \delta\mathbf{R}_{\mathbf{a}}^{\pi}$. Then, by applying the formulas (2.14), $(2.15)_2$, $(2.17)_2$ and $(2.18)_2$ we recast Eqs. (A.1)–(A.4) in the form

$$|\mathbf{z}|^{2m}(\mathbf{e}\cos\Theta + \mathbf{e}'\sin\Theta) = |\mathbf{z}|^{2m}(\mathbf{e}\cos 2m < \mathbf{z},\mathbf{e} > - \mathbf{e}'\sin 2m < \mathbf{z},\mathbf{e} >),$$
$$|\mathbf{q}(\mathbf{A})|^{m}(\mathbf{e}\cos\Theta' + \mathbf{e}'\sin\Theta') = |\mathbf{q}(\mathbf{A})|^{m}(\mathbf{e}\cos m < \mathbf{q}(\mathbf{A}),\mathbf{e} > -\mathbf{e}'\sin m < \mathbf{q}(\mathbf{A}),\mathbf{e} >),$$

where

$$\Theta = (4m+2) < \mathbf{a},\mathbf{e} > -2m < \mathbf{z},\mathbf{e} >, \qquad \mathbf{z} = \mathbf{v}^{\circ}, \mathbf{Wn}, (\mathbf{An})^{\circ},$$
$$\Theta' = (4m+2) < \mathbf{a},\mathbf{e} > -m < \mathbf{q}(\mathbf{A}),\mathbf{e} > .$$

Since $\mathbf{Q} \notin D_{2m+1d}$, we have $(4m+2) < \mathbf{a},\mathbf{e} > \neq 2k\pi$. Then we derive (A.6). Hence (A.5) also holds for each $\mathbf{Q} = \delta\mathbf{R}_{\mathbf{a}}^{\pi} \in D_{\infty h} \setminus D_{2m+1d}$. *Q.E.D.*

THEOREM A.2. *Let $\mathbf{\Phi}_r(\mathbf{z})$ be the symmetric second order tensor-valued function given by* (3.16) *for any vector $\mathbf{z}$ on the $\mathbf{n}$-plane and each integer $r \geq 1$. Then, for each $\mathbf{Q} \in D_{\infty h} \setminus D_{2m+2h}$, the general solution to the system of tensor equations*

$$\text{(A.7)} \qquad \mathbf{Q} * (\mathbf{\Phi}_{2m}((\mathbf{Q}^{\mathrm{T}}\mathbf{v})^{\circ})) = \mathbf{\Phi}_{2m}(\mathbf{v}^{\circ}),$$
$$\text{(A.8)} \qquad \mathbf{Q} * (\mathbf{\Phi}_{2m}((\mathbf{Q}^{\mathrm{T}} * \mathbf{W})\mathbf{n})) = \mathbf{\Phi}_{2m}(\mathbf{Wn}),$$
$$\text{(A.9)} \qquad \mathbf{Q} * (\mathbf{\Phi}_{2m}(((\mathbf{Q}^{\mathrm{T}} * \mathbf{A})\mathbf{n})^{\circ})) = \mathbf{\Phi}_{2m}((\mathbf{An})^{\circ}),$$
$$\text{(A.10)} \qquad \mathbf{Q} * (\mathbf{\Phi}_{m}(\mathbf{q}(\mathbf{Q}^{\mathrm{T}} * \mathbf{A}))) = \mathbf{\Phi}_{m}(\mathbf{q}(\mathbf{A})),$$

is given by (A.5).

P r o o f. Let $\mathbf{Q} = \delta\mathbf{R}_{\mathbf{n}}^{\theta}$. By using the formulas (2.13), $(2.15)_1$, $(2.17)_1$ and $(2.18)_1$, we convert Eqs. (A.7)–(A.10) to the form

$$|\mathbf{z}|^{2m}(\mathbf{D}_1\cos\Theta - \mathbf{D}_2\sin\Theta) = |\mathbf{z}|^{2m}(\mathbf{D}_1\cos 2m < \mathbf{z},\mathbf{e} > -\mathbf{D}_2\sin 2m < \mathbf{z},\mathbf{e} >),$$
$$|\mathbf{q}(\mathbf{A})|^{m}(\mathbf{D}_1\cos\Theta' - \mathbf{D}_2\sin\Theta') = |\mathbf{q}(\mathbf{A})|^{m}(\mathbf{D}_1\cos m < \mathbf{q}(\mathbf{A}),\mathbf{e} > - \mathbf{D}_2\sin m < \mathbf{q}(\mathbf{A}),\mathbf{e} >),$$

where

$$\Theta = -(2m+2)\theta + 2m < \mathbf{z}, \mathbf{e} >, \qquad \mathbf{z} = \mathbf{v}^{\circ}, \mathbf{Wn}, (\mathbf{An})^{\circ},$$
$$\Theta' = -(2m+2)\theta + m < \mathbf{q}(\mathbf{A}), \mathbf{e} > .$$

Since $\mathbf{Q} \in D_{2m+2h}$, we have $(2m+2)\theta \neq 2k\pi$. Then we derive (A.6) and therefore (A.5) holds for each $\mathbf{Q} = \delta \mathbf{R}_{\mathbf{n}}^{\theta} \in D_{\infty h} \setminus D_{2m+2h}$.

Next, let $\mathbf{Q} = \delta \mathbf{R}_{\mathbf{a}}^{\pi}$. Then, by applying the formulas (2.14), $(2.15)_2$, $(2.17)_2$ and $(2.18)_2$ we recast Eqs. (A.7)–(A.10) in the form

$$|\mathbf{z}|^{2m}(\mathbf{D}_1 \cos\Theta + \mathbf{D}_2 \sin\Theta) = |\mathbf{z}|^{2m}(\mathbf{D}_1 \sin 2m < \mathbf{z}, \mathbf{e} > - \mathbf{D}_2 \sin 2m < \mathbf{z}, \mathbf{e} >),$$
$$|\mathbf{q}(\mathbf{A})|^{m}(\mathbf{D}_1 \cos\Theta' + \mathbf{D}_2 \sin\Theta') = |\mathbf{q}(\mathbf{A})|^{m}(\mathbf{D}_1 \sin m < \mathbf{q}(\mathbf{A}), \mathbf{e} > - \mathbf{D}_2 \sin m < \mathbf{q}(\mathbf{A}), \mathbf{e} >),$$

where

$$\Theta = (4m+4) < \mathbf{a}, \mathbf{e} > -2m < \mathbf{z}, \mathbf{e} >, \qquad \mathbf{z} = \mathbf{v}^{\circ}, \mathbf{Wn}, (\mathbf{An})^{\circ},$$
$$\Theta' = (4m+4) < \mathbf{a}, \mathbf{e} > -m < \mathbf{q}(\mathbf{A}), \mathbf{e} > .$$

Since $\mathbf{Q} \notin D_{2m+2h}$, we have $(4m+4) < \mathbf{a}, \mathbf{e} > \neq 2k\pi$. Then we derive (A.6). Hence (A.5) also holds for each $\mathbf{Q} = \delta \mathbf{R}_{\mathbf{a}}^{\pi} \in D_{\infty h} \setminus D_{2m+2h}$. *Q.E.D.*

THEOREM A.3. *Let* $\boldsymbol{\eta}_r(\mathbf{z})$ *be the vector-valued function given by* (3.12). *Then, for each* $\mathbf{Q} \in D_{\infty h} \setminus D_{2m+1h}$, *the general solution to the system of tensor equations*

$$\mathbf{Q}(\boldsymbol{\eta}_{2m}((\mathbf{Q}^{\mathrm{T}}\mathbf{v})^{\circ})) = \boldsymbol{\eta}_{2m}(\mathbf{v}^{\circ}), \tag{A.11}$$
$$\mathbf{Q}(\boldsymbol{\eta}_{2m}((\mathbf{Q}^{\mathrm{T}} * \mathbf{W})\mathbf{n})) = \boldsymbol{\eta}_{2m}(\mathbf{Wn}), \tag{A.12}$$
$$\mathbf{Q}(\boldsymbol{\eta}_{2m}(((\mathbf{Q}^{\mathrm{T}} * \mathbf{A})\mathbf{n})^{\circ})) = \boldsymbol{\eta}_{2m}((\mathbf{An})^{\circ}), \tag{A.13}$$
$$\mathbf{Q}(\boldsymbol{\eta}_{m}(\mathbf{q}(\mathbf{Q}^{\mathrm{T}} * \mathbf{A}))) = \boldsymbol{\eta}_{m}(\mathbf{q}(\mathbf{A})), \tag{A.14}$$

is given by (A.5).

The proof of this theorem is similar to that of Theorem A.1, except for the fact that the factor δ plays no role in the latter, while it comes into play in the former (cf. the proof for the next theorem).

THEOREM A.4. *Let* $\boldsymbol{\eta}_r(\mathbf{z})$ *be the vector-valued function given by* (3.12) *for any vector* $\mathbf{z}$ *on the* $\mathbf{n}$*-plane and each integer* $r \geq 1$. *Then, for each* $\mathbf{Q} \in D_{\infty h} \setminus D_{2md}$, *the general solution to the system of tensor equations*

$$\mathbf{Q} * (\mathbf{n} \vee \boldsymbol{\eta}_{2m-1}((\mathbf{Q}^{\mathrm{T}}\mathbf{v})^{\circ})) = \mathbf{n} \vee \boldsymbol{\eta}_{2m-1}(\mathbf{v}^{\circ}), \tag{A.15}$$
$$\mathbf{Q}^{\mathrm{T}}(\boldsymbol{\eta}_{2m-1}((\mathbf{Q} * \mathbf{W})\mathbf{n})) = \boldsymbol{\eta}_{2m-1}(\mathbf{Wn}), \tag{A.16}$$
$$\mathbf{Q}^{\mathrm{T}}(\boldsymbol{\eta}_{2m-1}(((\mathbf{Q} * \mathbf{A})\mathbf{n})^{\circ})) = \boldsymbol{\eta}_{2m-1}((\mathbf{An})^{\circ}), \tag{A.17}$$

is given by

(A.18) $$\mathbf{v} = x\mathbf{n}, \qquad \mathbf{W} = y\mathbf{E}\mathbf{n}, \qquad (\mathbf{A}\mathbf{n})^{\circ} = \mathbf{0}.$$

P r o o f. Let $\mathbf{Q} = \delta \mathbf{R}_{\mathbf{n}}^{\theta}$. Then, by using the formulas (2.13), $(2.17)_1$, $(2.19)_1$, (2.18) and (2.20) we infer

$$|\mathbf{z}|^{2m-1}(\mathbf{e}\cos\Theta - \mathbf{e}'\sin\Theta) = |\mathbf{z}|^{2m-1}(\mathbf{e}\cos\Theta_0 - \mathbf{e}'\sin\Theta_0),$$

where

$$\Theta_0 = (2m-1) < \mathbf{z}, \mathbf{e} >, \qquad \Theta = \Theta_0 - 2m\theta - \frac{1}{2}(1-\delta)\pi,$$
$$\mathbf{z} = \mathbf{v}^{\circ}, \ \ \mathbf{W}\mathbf{n}, \ \ (\mathbf{A}\mathbf{n})^{\circ}.$$

Since $\mathbf{Q} \notin D_{2md}$, we have

$$2m\theta + \frac{1}{2}(1-\delta)\pi \neq 2k\pi.$$

Hence, we deduce $\mathbf{z} = \mathbf{0}$, $\mathbf{z} = \mathbf{v}^{\circ}$, $\mathbf{W}\mathbf{n}$, $(\mathbf{A}\mathbf{n})^{\circ}$, i.e. (A.18) holds for each $\mathbf{Q} = \delta \mathbf{R}_{\mathbf{n}}^{\theta} \in D_{\infty h} \setminus D_{2md}$.

Let $\mathbf{Q} = \delta \mathbf{R}_{\mathbf{a}}^{\pi}$. Then, by using the formulas (2.14), $(2.17)_2$, $(2.19)_2$, (2.18) and (2.20) we infer

$$|\mathbf{z}|^{2m-1}(\mathbf{e}\cos\Theta + \mathbf{e}'\sin\Theta) = |\mathbf{z}|^{2m-1}(\mathbf{e}\cos\Theta_0 - \mathbf{e}'\sin\Theta_0),$$

where

$$\Theta_0 = (2m-1) < \mathbf{z}, \mathbf{e} >, \qquad \Theta = 4m < \mathbf{a}, \mathbf{e} > + \frac{1}{2}(1-\delta)\pi - \Theta_0\,,$$
$$\mathbf{z} = \mathbf{v}^{\circ}, \ \ \mathbf{W}\mathbf{n}, \ \ (\mathbf{A}\mathbf{n})^{\circ}.$$

Since $\mathbf{Q} \notin D_{2md}$, we have

$$4m < \mathbf{a}, \mathbf{e} > + (1-\delta)\pi \neq 2k\pi.$$

Hence, we deduce $\mathbf{z} = \mathbf{0}$, $\mathbf{z} = \mathbf{v}^{\circ}$, $\mathbf{W}\mathbf{n}$, $(\mathbf{A}\mathbf{n})^{\circ}$, i.e. (A.18) also holds for each $\mathbf{Q} = \delta \mathbf{R}_{\mathbf{a}}^{\pi} \in D_{\infty h} \setminus D_{2md}$. *Q.E.D.*

A.2. Polynomial tensor equations: cubic crystal classes

THEOREM A.5. *Let* $\mathbf{O}_h$ *be the tensor given by* (4.6), *which is invariant under the group* O_h. *Then for each* $\mathbf{Q} \in \mathrm{Orth} \setminus O_h$, *the solution to the system of polynomial tensor equations*

(A.19) $$(\mathbf{Q} * \mathbf{O}_h) : (\mathbf{x} \otimes \mathbf{x}) = \mathbf{O}_h : (\mathbf{x} \otimes \mathbf{x}),$$

(A.20) $$(\mathbf{Q} * \mathbf{O}_h) : (\overset{2}{\otimes} (\mathbf{E} : \mathbf{W})) = \mathbf{O}_h : (\overset{2}{\otimes} (\mathbf{E} : \mathbf{W})),$$

(A.21) $$(\mathbf{Q} * \mathbf{O}_h) : \mathbf{A} = \mathbf{O}_h : \mathbf{A},$$

(A.22) $$(\mathbf{Q} * \mathbf{O}_h) : \mathbf{A}^2 = \mathbf{O}_h : \mathbf{A}^2,$$

are as follows:

CASE 1. *If there are* $\mathbf{u}, \mathbf{v} \in \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ *or* $\mathbf{u}, \mathbf{v} \in \{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4\}$ *such that*

$$\overset{2}{\otimes}(\mathbf{Q}^{\mathrm{T}}\mathbf{u}) = \overset{2}{\otimes}\mathbf{v}, \tag{A.23}$$

then

$$\mathbf{x} = a\mathbf{u}, \qquad \mathbf{W} = b\mathbf{E}\mathbf{u}, \qquad \mathbf{A} = c\mathbf{I} + d\mathbf{u}\otimes\mathbf{u} \quad (\forall\, a,\ b,\ c,\ d \in R). \tag{A.24}$$

CASE 2. *If for any* $\mathbf{u}, \mathbf{v} \in \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ *and* $\mathbf{u}, \mathbf{v} \in \{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4\}$,

$$\overset{2}{\otimes}(\mathbf{Q}^{\mathrm{T}}\mathbf{u}) \neq \overset{2}{\otimes}\mathbf{v}, \tag{A.25}$$

then

$$\mathbf{x} = \mathbf{0}, \qquad \mathbf{W} = \mathbf{O}, \qquad \mathbf{A} = c\mathbf{I}. \tag{A.26}$$

In the above, each $\mathbf{n}_k$ *and each* $\mathbf{r}_i$ *are a four-fold axis and a three-fold axis of* O_h, *respectively (cf.* (4.1) *and* (4.5)*).*

P r o o f. First, suppose that there be $\mathbf{u}, \mathbf{v} \in \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ such that (A.23) holds. Then there are permutations $\sigma, \tau \in P_3$, where P_3 is the symmetric group on three letters, such that

$$\begin{aligned} \mathbf{Q}\mathbf{n}_{\sigma(1)} &= \mathbf{n}_{\tau(1)}\cos\theta + \mathbf{n}_{\tau(2)}\sin\theta, \\ \mathbf{Q}\mathbf{n}_{\sigma(2)} &= -\mathbf{n}_{\tau(1)}\sin\theta + \mathbf{n}_{\tau(2)}\cos\theta, \\ \mathbf{Q}\mathbf{n}_{\sigma(3)} &= r\mathbf{n}_{\tau(3)}, \qquad r^2 = 1. \end{aligned} \tag{A.27}$$

Substituting the above into the equivalent form of (A.21):

$$\sum_{k=1}^{3}\left(\hat{\mathbf{n}}_{\sigma(k)}\cdot\mathbf{A}\hat{\mathbf{n}}_{\sigma(k)}\right)\hat{\mathbf{n}}_{\sigma(k)}\otimes\hat{\mathbf{n}}_{\sigma(k)} = \sum_{k=1}^{3}\left(\mathbf{n}_{\tau(k)}\cdot\mathbf{A}\mathbf{n}_{\tau(k)}\right)\mathbf{n}_{\tau(k)}\otimes\mathbf{n}_{\tau(k)} (\equiv \mathbf{C}), \tag{A.28}$$

where $\hat{\mathbf{n}}_{\sigma(k)} = \mathbf{Q}\mathbf{n}_{\sigma(k)}$, we derive

$$\begin{aligned} (A_{\tau(2)\tau(2)} - A_{\tau(1)\tau(1)})\sin^2 2\theta + A_{\tau(1)\tau(2)}\sin 4\theta &= 0, \\ (A_{\tau(2)\tau(2)} - A_{\tau(1)\tau(1)})\sin 4\theta - 4A_{\tau(1)\tau(2)}\sin^2 2\theta &= 0. \end{aligned}$$

Since $\mathbf{Q} \notin O_h$, i.e. $\theta \neq k\pi/2$, the above system of homogeneous equations has merely a trivial solution, i.e.

$$A_{\tau(1)\tau(2)} = 0, \qquad A_{\tau(1)\tau(1)} = A_{\tau(2)\tau(2)}, \tag{A.29}$$

where $A_{ij} = \mathbf{n}_i\cdot\mathbf{A}\mathbf{n}_j = A_{ji}$. Consequently, the equations (A.19) and (A.20) yield

$$(\mathbf{x}\cdot\mathbf{n}_{\tau(1)})(\mathbf{x}\cdot\mathbf{n}_{\tau(2)}) = 0, \qquad (\mathbf{x}\cdot\mathbf{n}_{\tau(1)})^2 = (\mathbf{x}\cdot\mathbf{n}_{\tau(2)})^2,$$

$$(\mathbf{y}\cdot\mathbf{n}_{\tau(1)})(\mathbf{y}\cdot\mathbf{n}_{\tau(2)}) = 0, \qquad (\mathbf{y}\cdot\mathbf{n}_{\tau(1)})^2 = (\mathbf{y}\cdot\mathbf{n}_{\tau(2)})^2, \qquad \mathbf{y} = \mathbf{E} : \mathbf{W},$$

and the equations (A.21) and (A.22) produce (A.29) and

$$B_{\tau(1)\tau(2)} = 0, \qquad B_{\tau(1)\tau(1)} = B_{\tau(2)\tau(2)}, \qquad \mathbf{B} = \mathbf{A}^2.$$

From these and the fact stated at the end of this proof we infer that the solution of Eqs. (A.19)–(A.22) is provided by (A.24) for each $\mathbf{Q} \in \mathrm{Orth} \setminus O_h$ satisfying (A.23) for $\mathbf{u}, \mathbf{v} \in \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$.

Next, suppose that for any $\mathbf{u},\ \mathbf{v} \in \{\mathbf{n}_1,\ \mathbf{n}_2,\ \mathbf{n}_3\}$, (A.25) holds. Since (A.21), i.e. (A.28) offers two spectral representations of the same symmetric tensor $\mathbf{C} \in \mathrm{Sym}$; we infer that the two sets of eigenvalues, $\{\hat{\mathbf{n}}_k\cdot\mathbf{A}\hat{\mathbf{n}}_k\}$ and $\{\mathbf{n}_k\cdot\mathbf{A}\mathbf{n}_k\}$, coincide and their subordinate eigenprojections coincide. Taking this fact and the condition

(A.30) $$\overset{2}{\otimes}\, \mathbf{Q}^{\mathrm{T}}\mathbf{u} \neq \overset{2}{\otimes}\, \mathbf{v} \qquad (\forall\, \mathbf{u},\ \mathbf{v} \in \{\mathbf{n}_1,\ \mathbf{n}_2,\ \mathbf{n}_3\})$$

into account, we infer that $\mathbf{C} = \bar{c}\mathbf{I}$ and hence that

(A.31) $$(\mathbf{Q}\mathbf{n}_k)\cdot\mathbf{A}(\mathbf{Q}\mathbf{n}_k) = \mathbf{n}_k\cdot\mathbf{A}\mathbf{n}_k = \bar{c}, \qquad k = 1,2,3.$$

Moreover, letting the symmetric tensor $\mathbf{A} \in \mathrm{Sym}$ take the particular forms $\mathbf{x}\otimes\mathbf{x}$ and $\mathbf{y}\otimes\mathbf{y}$, $\mathbf{y} = \mathbf{E} : \mathbf{W}$, respectively, we infer that Eqs. (A.19), (A.20) and (A.22) yield

(A.32) $$(\mathbf{x}\cdot(\mathbf{Q}\mathbf{n}_k))^2 = (\mathbf{x}\cdot\mathbf{n}_k)^2 = \bar{a}^2, \qquad k = 1,2,3,$$

(A.33) $$(\mathbf{y}\cdot(\mathbf{Q}\mathbf{n}_k))^2 = (\mathbf{y}\cdot\mathbf{n}_k)^2 = \bar{b}^2, \qquad k = 1,2,3,$$

(A.34) $$(\mathbf{Q}\mathbf{n}_k)\cdot\mathbf{A}^2(\mathbf{Q}\mathbf{n}_k) = \mathbf{n}_k\cdot\mathbf{A}^2\mathbf{n}_k = \bar{d}^2, \qquad k = 1,2,3.$$

From (4.5), (A.31)–(A.34) and $\mathbf{Q} \notin O_h$ and the facts

$$\mathbf{p}\cdot(\mathbf{Q}\mathbf{q}) = (\mathbf{Q}^{\mathrm{T}}\mathbf{p})\cdot\mathbf{q}; \qquad (\mathbf{Q}\mathbf{p})\cdot\mathbf{B}(\mathbf{Q}\mathbf{q}) = \mathbf{p}\cdot(\mathbf{Q}^{\mathrm{T}} * \mathbf{B})\mathbf{q},$$

$$\mathbf{n}_k\cdot\mathbf{B}\mathbf{n}_k = c \ \ \&\ \ \mathbf{n}_k\cdot\mathbf{B}^2\mathbf{n}_k = d^2 \neq 0, \quad k = 1,2,3$$
$$\Longrightarrow \quad \exists\,\mathbf{u} \in \{\mathbf{r}_1, \ldots, \mathbf{r}_4\} : \ \mathbf{B} = x\mathbf{I} + y\mathbf{u}\otimes\mathbf{u}, \quad y \neq 0,$$

for any $\mathbf{p}, \mathbf{q} \in V$ and $\mathbf{B} \in \mathrm{Sym}$, we derive (A.24) if $\bar{a}^2 + \bar{b}^2 + \bar{d}^2 \neq 0$ holds, i.e. there are $\mathbf{u},\ \mathbf{v} \in \{\mathbf{r}_1,\ \mathbf{r}_2,\ \mathbf{r}_3, \mathbf{r}_4\}$ such that (A.23) holds. Moreover, we derive (A.26) if (A.25) holds for any $\mathbf{u},\ \mathbf{v} \in \{\mathbf{n}_1,\ \mathbf{n}_2,\ \mathbf{n}_3\}$ and any $\mathbf{u},\ \mathbf{v} \in \{\mathbf{r}_1,\ \mathbf{r}_2,\ \mathbf{r}_3,\ \mathbf{r}_4\}$, i.e. $\bar{a} = \bar{b} = \bar{d} = 0$ holds. In deriving the former, the following fact is used: if an orthogonal tensor $\mathbf{Q}$ transforms any two given three-fold axes of O_h into three-fold axes of O_h, then $\mathbf{Q} \in O_h$. *Q.E.D.*

THEOREM A.6. *Let* $\mathbf{T}_d$ *and* $\mathbf{O}_h$ *be the tensors given by* (4.9)–(4.10) *and* (4.6), *which are invariant under the groups* $T_d \subset O_h$ *and* O_h, *respectively. Then for each* $\mathbf{Q} \in \mathrm{Orth} \setminus O_h$, *the solution to the system of polynomial tensor equations*

$$(\mathbf{Q} * \mathbf{T}_d)\mathbf{x} = \mathbf{T}_d\mathbf{x}; \tag{A.35}$$

$$(\mathbf{Q} * \mathbf{O}_h) : (\mathbf{x} \otimes \mathbf{x}) = \mathbf{O}_h : (\mathbf{x} \otimes \mathbf{x}); \tag{A.36}$$

$$(\mathbf{Q} * \mathbf{T}_d) : (\overset{2}{\otimes} (\mathbf{E} : \mathbf{W})) = \mathbf{T}_d : (\overset{2}{\otimes} (\mathbf{E} : \mathbf{W})); \tag{A.37}$$

$$(\mathbf{Q} * \mathbf{O}_h) : (\overset{2}{\otimes} (\mathbf{E} : \mathbf{W})) = \mathbf{O}_h : (\overset{2}{\otimes} (\mathbf{E} : \mathbf{W})); \tag{A.38}$$

$$(\mathbf{Q} * \mathbf{T}_d) : \mathbf{A} = \mathbf{T}_d : \mathbf{A}; \tag{A.39}$$

$$(\mathbf{Q} * \mathbf{O}_h) : \mathbf{A} = \mathbf{O}_h : \mathbf{A}; \tag{A.40}$$

are as follows:

CASE 1. $\mathbf{x} = \mathbf{0}$, $\mathbf{W} = b\mathbf{E}\mathbf{n}$, $\mathbf{A} = c\mathbf{I} + d\mathbf{n} \otimes \mathbf{n}$ *if* $\exists\ \mathbf{u},\ \mathbf{v} \in \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$: $\overset{2}{\otimes} (\mathbf{Q}^{\mathrm{T}}\mathbf{u}) = \overset{2}{\otimes} \mathbf{v}$.

CASE 2. *If* $\exists \mathbf{u},\ \mathbf{v} \in \{\mathbf{r}_1, \ldots, \mathbf{r}_4\}$: $\mathbf{Q}^{\mathrm{T}}\mathbf{u} = \mathbf{v}$, *then the solutions are given by* (A.24).

CASE 3. *If*

$$\forall\, \mathbf{u},\ \mathbf{v} \in \{\mathbf{r}_1, \ldots, \mathbf{r}_4\}\ :\ \mathbf{Q}^{\mathrm{T}}\mathbf{u} \neq \mathbf{v} \quad \&\quad \forall\, \mathbf{u},\ \mathbf{v} \in \{\mathbf{n}_1,\ \mathbf{n}_2,\ \mathbf{n}_3\} :\ \overset{2}{\otimes} (\mathbf{Q}^{\mathrm{T}}\mathbf{u}) \neq \overset{2}{\otimes} \mathbf{v}, \tag{A.41}$$

then the solution is given by (A.26).

P r o o f. Consider two cases. First, for each $\mathbf{Q}$ given by (A.27), from the proof of Theorem A.5 we know that Eqs. (A.40), (A.38) and (A.36) yield (A.29) and

$$\mathbf{x} = a\mathbf{n}_{\tau(3)}, \qquad \mathbf{W} = b\mathbf{E}\mathbf{n}_{\tau(3)}. \tag{A.42}$$

Moreover, for each $\mathbf{Q}$ given by (A.27), Eqs. (A.35) and (A.39) further yield (Eq. (A.37) provides no further restriction for $\mathbf{W}$)

$$a \sin 2\theta = 0, \qquad a(1 - r\cos 2\theta) = 0,$$
$$A_{\tau(2)\tau(3)} \sin 2\theta + A_{\tau(1)\tau(3)}(\cos 2\theta - r) = 0,$$
$$A_{\tau(2)\tau(3)}(\cos 2\theta - r) - A_{\tau(1)\tau(3)} \sin 2\theta = 0.$$

Thus, by using $\mathbf{Q} \notin O_h$, i.e. $\theta \neq k\pi/2$ we infer

$$a = 0, \qquad A_{\tau(1)\tau(1)} - A_{\tau(2)\tau(2)} = A_{\tau(i)(\tau(j)} = 0, \qquad i,\ j = 1, 2, 3, \quad i \neq j,$$

where (A.29) is incorporated. Hence Case 1 holds.

Next, for each $\mathbf{Q}$ satisfying (A.30), from the proof of Theorem A.5 we know that Eqs. (A.36), (A.38) and (A.40) yield (A.31)–(A.33). Substituting (A.32) and (A.33) into (A.35) and (A.37) respectively, we obtain

$$\bar{a}(\hat{r}_1\hat{\boldsymbol{\omega}}_1 + \hat{r}_2\hat{\boldsymbol{\omega}}_2 + \hat{r}_3\hat{\boldsymbol{\omega}}_3) = \bar{a}(r_1\boldsymbol{\omega}_1 + r_2\boldsymbol{\omega}_2 + r_3\boldsymbol{\omega}_3), \qquad \hat{\boldsymbol{\omega}}_k = \mathbf{Q} * \boldsymbol{\omega}_k,$$

i.e.

$$\begin{gathered}\bar{a}\hat{f}(\overset{2}{\otimes}(\hat{r}_1\hat{\mathbf{n}}_1 + \hat{r}_2\hat{\mathbf{n}}_2 + \hat{r}_3\hat{\mathbf{n}}_3)) = \bar{a}f(\overset{2}{\otimes}(r_1\mathbf{n}_1 + r_2\mathbf{n}_2 + r_3\mathbf{n}_3)),\\ \hat{f} = \hat{r}_1\hat{r}_2\hat{r}_3, \qquad f = r_1r_2r_3,\end{gathered} \tag{A.43}$$

and

$$\begin{gathered}\bar{b}^2\hat{g}(\hat{s}_1\hat{\mathbf{n}}_1 + \hat{s}_2\hat{\mathbf{n}}_2 + \hat{s}_3\hat{\mathbf{n}}_3) = \bar{b}^2g(s_1\mathbf{n}_1 + s_2\mathbf{n}_2 + s_3\mathbf{n}_3), \qquad \hat{\mathbf{n}}_k = \mathbf{Q}\mathbf{n}_k,\\ \hat{g} = \hat{s}_1\hat{s}_2\hat{s}_3, \qquad g = s_1s_2s_3,\end{gathered} \tag{A.44}$$

where $\boldsymbol{\omega}_k$ are given by (4.10) and moreover

$$\begin{aligned}\mathbf{x}\cdot\mathbf{n}_k &= \bar{a}r_k, & \mathbf{x}\cdot(\mathbf{Q}\mathbf{n}_k) &= \bar{a}\hat{r}_k, & r_k^2 = \hat{r}_k^2 &= 1, & k &= 1,2,3,\\ (\mathbf{E}:\mathbf{W})\cdot\mathbf{n}_k &= \bar{b}s_k, & (\mathbf{E}:\mathbf{W})\cdot(\mathbf{Q}\mathbf{n}_k) &= \bar{b}\hat{s}_k, & s_k^2 = \hat{s}_k^2 &= 1, & k &= 1,2,3.\end{aligned}$$

By using (A.43)–(A.44), (4.5) and the fact that

$$\mathbf{Q}\mathbf{u},\ \mathbf{Q}\mathbf{v} \in \{\mathbf{r}_1, \ldots, \mathbf{r}_4\} \iff \mathbf{Q} \in T_d$$

for any given $\mathbf{u}, \mathbf{v} \in \{\mathbf{r}_1, \ldots, \mathbf{r}_4\}$ and $\mathbf{u} \neq \mathbf{v}$, we infer that $\mathbf{x} = a\mathbf{u}$, $\mathbf{W} = b\mathbf{E}\mathbf{u}$, if there exist $\mathbf{u}$, $\mathbf{v} \in \{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4\}$ such that $\mathbf{Q}^T\mathbf{u} = \mathbf{v}$; and that $\mathbf{x} = 0$, $\mathbf{W} = 0$, if (A.41) holds.

On the other hand, let $w_k = \mathbf{A} : \boldsymbol{\omega}_k$ and $\hat{w}_k = \mathbf{A} : \hat{\boldsymbol{\omega}}_k$. Then Eqs. (A.39) and (A.40) may be rewritten in the forms

$$\sum_{k=1}^{3} \hat{w}_k\hat{\mathbf{n}}_k = \sum_{k=1}^{3} w_k\mathbf{n}_k \quad \text{i.e.} \quad \hat{\mathbf{q}} = \mathbf{q}, \tag{A.45}$$

$$\sum_{k=1}^{3} \hat{w}_k\hat{\boldsymbol{\omega}}_k = \sum_{k=1}^{3} w_k\boldsymbol{\omega}_k \quad \text{i.e.} \quad \hat{\mathbf{B}} = \mathbf{B}. \tag{A.46}$$

For the latter, the identity

$$\mathbf{Q} * \left(\mathbf{O}_h + \frac{1}{2}\sum_{k=1}^{3} \boldsymbol{\omega}_k \otimes \boldsymbol{\omega}_k\right) = \mathbf{O}_h + \frac{1}{2}\sum_{k=1}^{3} \boldsymbol{\omega}_k \otimes \boldsymbol{\omega}_k \quad (\forall\, \mathbf{Q} \in \mathrm{Orth}) \tag{A.47}$$

is used. From $\hat{\mathbf{q}}\cdot\hat{\mathbf{B}}\hat{\mathbf{q}} = \mathbf{q}\cdot\mathbf{B}\mathbf{q}$ and $\overset{2}{\otimes}(\hat{\mathbf{B}}\hat{\mathbf{q}}) = \overset{2}{\otimes}(\mathbf{B}\mathbf{q})$ we derive

$$\sum_{k=1}^{3} (\hat{C}_k)^2 \hat{\mathbf{n}}_k \otimes \hat{\mathbf{n}}_k = \sum_{k=1}^{3} (C_k)^2 \mathbf{n}_k \otimes \mathbf{n}_k, \tag{A.48}$$

where

$$C_1 = w_2 w_3\,, \qquad C_2 = w_3 w_1\,, \qquad C_3 = w_1 w_2\,;$$
$$\hat{C}_1 = \hat{w}_2 \hat{w}_3\,, \qquad \hat{C}_2 = \hat{w}_3 \hat{w}_1\,, \qquad \hat{C}_3 = \hat{w}_1 \hat{w}_2\,.$$

By (A.30) and (A.48) we infer

$$(\hat{C}_k)^2 = (C_k)^2 = c, \qquad k = 1, 2, 3\,, \tag{A.49}$$

and then by the latter and (A.45) we infer that $\mathbf{A} = c\mathbf{I} + d\mathbf{u} \otimes \mathbf{u}$ if there exist $\mathbf{u}$, $\mathbf{v} \in \{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4\}$ such that $\mathbf{Q}^{\mathrm{T}}\mathbf{u} = \mathbf{v}$, and that $\mathbf{A} = c\mathbf{I}$ if (A.41) holds.

Finally, combining the facts derived above and the property of the group T_d stated before, we conclude that Theorem A.6 holds. *Q.E.D.*

THEOREM A.7. *Let* $\mathbf{T}_h^a$ *and* $\mathbf{T}_h^s$ *be the two tensors given by* (4.14) *and* (4.15), *which are invariant under the group* T_h. *Then for each* $\mathbf{Q} \in \mathrm{Orth}\backslash T_h$, *the solution to the system of polynomial tensor* equations

$$(\mathbf{Q} * \mathbf{T}_h^a) : (\mathbf{x} \otimes \mathbf{x}) = \mathbf{T}_h^a : (\mathbf{x} \otimes \mathbf{x}), \tag{A.50}$$
$$(\mathbf{Q} * \mathbf{T}_h^s) : (\mathbf{x} \otimes \mathbf{x}) = \mathbf{T}_h^s : (\mathbf{x} \otimes \mathbf{x}); \tag{A.51}$$
$$(\mathbf{Q} * \mathbf{T}_h^a) : (\overset{2}{\otimes} (\mathbf{E} : \mathbf{W})) = \mathbf{T}_h^a : (\overset{2}{\otimes} (\mathbf{E} : \mathbf{W})), \tag{A.52}$$
$$(\mathbf{Q} * \mathbf{T}_h^s) : (\overset{2}{\otimes} (\mathbf{E} : \mathbf{W})) = \mathbf{T}_h^s : (\overset{2}{\otimes} (\mathbf{E} : \mathbf{W})); \tag{A.53}$$
$$(\mathbf{Q} * \mathbf{T}_h^a) : \mathbf{A} = \mathbf{T}_h^a : \mathbf{A}, \tag{A.54}$$
$$(\mathbf{Q} * \mathbf{T}_h^s) : \mathbf{A} = \mathbf{T}_h^s : \mathbf{A}; \tag{A.55}$$

are as follows:

CASE 1. *If*

$$\exists\, \mathbf{u},\ \mathbf{v} \in \{\mathbf{r}_1,\ \mathbf{r}_2,\ \mathbf{r}_3,\ \mathbf{r}_4\}\ :\ \mathbf{Q}^{\mathrm{T}}\mathbf{u} = (\det \mathbf{Q})\mathbf{v}\,, \tag{A.56}$$

then the solutions are given by (A.24).

CASE 2. *If*

$$\forall\, \mathbf{u},\ \mathbf{v} \in \{\mathbf{r}_1,\ \mathbf{r}_2,\ \mathbf{r}_3,\ \mathbf{r}_4\}\ :\ \mathbf{Q}^{\mathrm{T}}\mathbf{u} \neq (\det \mathbf{Q})\mathbf{v}\,, \tag{A.57}$$

then the solutions are given by (A.26).

Proof. First, for each $\mathbf{Q}$ given by (A.27), Eqs. (A.54)–(A.55) yield

$$\begin{aligned}
(A_{\tau(2)\tau(2)} - A_{\tau(1)\tau(1)}) \sin 2\theta + 2A_{\tau(1)\tau(2)}(\cos 2\theta - 1) &= 0,\\
(A_{\tau(2)\tau(2)} - A_{\tau(1)\tau(1)})(1 - \cos 2\theta) + 2A_{\tau(1)\tau(2)} \sin 2\theta &= 0,\\
A_{\tau(2)\tau(3)}(\cos 2\theta - 1) - A_{\tau(1)\tau(3)} \sin 2\theta &= 0,\\
A_{\tau(2)\tau(3)} \sin 2\theta + A_{\tau(1)\tau(3)}(\cos 2\theta - 1) &= 0,\\
(A_{\tau(2)\tau(2)} - A_{\tau(3)\tau(3)})(\cos 2\theta - 1) - 2A_{\tau(1)\tau(2)} \sin 2\theta &= 0,\\
(A_{\tau(3)\tau(3)} - A_{\tau(1)\tau(1)})(\cos 2\theta - 1) - 2A_{\tau(1)\tau(2)} \sin 2\theta &= 0.
\end{aligned}$$

By using $\mathbf{Q} \notin T_h$, i.e. $\theta \neq k\pi$, from the above we derive

$$A_{11} = A_{22} = A_{33}\,, \qquad A_{12} = A_{23} = A_{31} = 0.$$

Moreover, letting the tensor $\mathbf{A} \in \mathrm{Sym}$ take the particular forms $\mathbf{x} \otimes \mathbf{x}$ and $\overset{2}{\otimes}(\mathbf{E} : \mathbf{W})$, respectively, from Eqs. (A.50)–(A.53) we derive

$$x_1^2 = x_2^2 = x_3^2\,, \qquad x_1x_2 = x_2x_3 = x_3x_1 = 0,$$

$$y_1^2 = y_2^2 = y_3^2\,, \qquad y_1y_2 = y_2y_3 = y_3y_1 = 0, \qquad \mathbf{y} = \mathbf{E} : \mathbf{W}.$$

Thus, we conclude that the Case 2 holds for each $\mathbf{Q} \in \mathrm{Orth}\backslash T_h$ satisfying (A.27).

Next, for each $\mathbf{Q}$ satisfying (A.30), since the two sides of Eq. (A.55) provide two spectral representations of the same symmetric second order tensor, we deduce that either of the two involved sets of eigenvalues must be triply coalescent, or else (A.30) will be violated. Hence, we have

$$A_{11} = A_{22} = A_{33} = \bar{A}_{11} = \bar{A}_{22} = \bar{A}_{33} = c.$$

From these and the identity (A.47) we infer that (A.46) holds. Moreover, (A.54) can be recast in the form

$$\text{(A.58)} \qquad \sum_{k=1}^{3} w_k \mathbf{n}_k = (\det \mathbf{Q}) \sum_{k=1}^{3} \hat{w}_k \hat{\mathbf{n}}_k\,.$$

By using the same procedure as that used in deriving (A.48), from (A.46) and (A.58) we can derive (A.48) again. Thus (A.30) and (A.48) yield (A.49). From (A.49) and (A.58) we infer that $\mathbf{A} = c\mathbf{I} + d\mathbf{u} \otimes \mathbf{u}$ for each $\mathbf{Q}$ obeying (A.56) and (A.30) or $\mathbf{A} = c\mathbf{I}$ for each $\mathbf{Q}$ satisfying (A.57) and (A.30). Finally, using the results for Eqs. (A.54)–(A.55) just derived and noticing the fact that for an orthogonal tensor $\mathbf{Q} \in \mathrm{Orth}$, if there are $\mathbf{u}, \mathbf{v} \in \{\mathbf{r}_1, \ldots, \mathbf{r}_4\}$, $\mathbf{u} \neq \mathbf{v}$, such that

$$\mathbf{Qu}, \mathbf{Qv} \in \{(\det \mathbf{Q})\mathbf{r}_1, \ldots, (\det \mathbf{Q})\mathbf{r}_4\},$$

then $\mathbf{Q} \in T_h$, we conclude that Theorem A.7 also holds for each $\mathbf{Q} \in \mathrm{Orth} \setminus T_h$ satisfying (A.30). *Q.E.D.*

A.3. Polynomial tensor equations: the icosahedral group I_h

THEOREM A.8. *Let $\mathbf{I}_h^1$, $\mathbf{I}_h^2$ and $\mathbf{I}_h^3$ be the three tensors given by (5.4), which are invariant under I_h. Then for each $\mathbf{Q} \in \mathrm{Orth}\backslash I_h$, the system of polynomial tensor equations*

$$\text{(A.59)} \qquad (\mathbf{Q} * \mathbf{I}_h^r) \odot (\overset{2r+2}{\otimes}\mathbf{x}) = \mathbf{I}_h^r \odot (\overset{2r+2}{\otimes}\mathbf{x}), \qquad r = 1, 2, 3;$$

$$\text{(A.60)} \qquad (\mathbf{Q} * \mathbf{I}_h^r) \odot (\overset{2r+2}{\otimes}(\mathbf{E} : \mathbf{W})) = \mathbf{I}_h^r \odot (\overset{2r+2}{\otimes}(\mathbf{E} : \mathbf{W})), \qquad r = 1, 2, 3;$$

$$(A.61)\qquad (\mathbf{Q} * \mathbf{I}_h^r) \odot (\overset{r+1}{\otimes} \mathbf{A}) = \mathbf{I}_h^r \odot (\overset{r+1}{\otimes} \mathbf{A}), \qquad r = 1, 2, 3;$$

has the following solutions:

CASE 1. $\mathbf{x} = a\mathbf{u}$, $\mathbf{W} = b\mathbf{E}\mathbf{u}$, $\mathbf{A} = c\mathbf{I} + d\mathbf{u} \otimes \mathbf{u}$, $\forall\, a, b, c, d \in R$, *if*

$$(A.62)\qquad \exists\, \mathbf{u}, \mathbf{v} \in \{\mathbf{n}_1, \ldots, \mathbf{n}_6\} \text{ or } \mathbf{u}, \mathbf{v} \in \{\mathbf{r}_1, \ldots, \mathbf{r}_{10}\} : \overset{2}{\otimes} (\mathbf{Q}^T\mathbf{u}) = \overset{2}{\otimes} \mathbf{v};$$

CASE 2. $\mathbf{x} = \mathbf{0}$, $\mathbf{W} = \mathbf{O}$, $\mathbf{A} = c\mathbf{I}$, *if*

$$(A.63)\qquad \forall\, \mathbf{u}, \mathbf{v} \in \{\mathbf{n}_1, \ldots, \mathbf{n}_6\} \text{ and } \mathbf{u}, \mathbf{v} \in \{\mathbf{r}_1, \ldots, \mathbf{r}_{10}\} : \overset{2}{\otimes} (\mathbf{Q}^T\mathbf{u}) \neq \overset{2}{\otimes} \mathbf{v}.$$

To prove the above theorem, some facts concerning the symmetry axes of the icosahedral group I_h are needed.

LEMMA A.1. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \{\mathbf{n}_1, \ldots, \mathbf{n}_6\}$ be any four different five-fold axes of the group I_h. Then for $\mathbf{G} \in$ Orth and any $p, q, r \in R$, the conditions

$$\begin{aligned} \mathbf{G} * (\mathbf{a} \otimes \mathbf{a} + p\mathbf{d} \otimes \mathbf{d}) &= \mathbf{a} \otimes \mathbf{a} + p\mathbf{d} \otimes \mathbf{d}, \\ \mathbf{G} * (\mathbf{b} \otimes \mathbf{b} + q\mathbf{d} \otimes \mathbf{d}) &= \mathbf{b} \otimes \mathbf{b} + q\mathbf{d} \otimes \mathbf{d}, \\ \mathbf{G} * (\mathbf{c} \otimes \mathbf{c} + r\mathbf{d} \otimes \mathbf{d}) &= \mathbf{c} \otimes \mathbf{c} + r\mathbf{d} \otimes \mathbf{d}, \end{aligned}$$

imply $\mathbf{G} \in I_h$.

P r o o f. Consider two cases. First, let at least two of p, q and r, say $p = q = 0$, be zero. Then by means of the conditions

$$\mathbf{G} * (\mathbf{a} \otimes \mathbf{a}) = \mathbf{a} \otimes \mathbf{a}, \qquad \mathbf{G} * (\mathbf{b} \otimes \mathbf{b}) = \mathbf{b} \otimes \mathbf{b}, \qquad \mathbf{a} \cdot \mathbf{b} \neq 0$$

we infer

$$\mathbf{G} \in \{\pm\mathbf{I}, \pm\mathbf{R}^{\pi}_{\mathbf{a} \times \mathbf{b}}\} \subset I_h,$$

where $\mathbf{a} \times \mathbf{b}$ be a two-fold axis of I_h (cf. Proposition 7.2 in [53]).

Next, let two of p, q and r be nonvanishing, e.g. $pq \neq 0$. Then the two tensors $\mathbf{a} \otimes \mathbf{a} + p\mathbf{d} \otimes \mathbf{d}$ and $\mathbf{b} \otimes \mathbf{b} + q\mathbf{d} \otimes \mathbf{d}$ have no eigenline in common and therefore the first two conditions in the above lemma imply $\mathbf{G} = \pm\mathbf{I} \in I_h$ (see Lemma 3.1.1 given in [48]). In reality, $\mathbf{a} \times \mathbf{d}$ and $\mathbf{b} \times \mathbf{d}$ offer two eigenlines of the just-mentioned two tensors, respectively, and the other eigenlines of the two tensors lie on the two planes perpendicular to these two eigenlines, respectively. Hence, the intersecting line of the two planes is the only possible commom eigenline of the aforementioned two tensors. The former is just $\mathbf{d}$ and can not be an eigenline of any of the aforementioned tensors. *Q.E.D.*

LEMMA A.2. Let $\mathbf{n}_i, \mathbf{n}_j$ and $\mathbf{n}_k$ be any three noncoplanar five-fold axes of the group I_h. Then the following equality holds.

$$(A.64)\qquad \mathbf{n}_i \otimes \mathbf{n}_i + \mathbf{n}_j \otimes \mathbf{n}_j + \mathbf{n}_k \otimes \mathbf{n}_k = x\mathbf{I} + y\mathbf{u} \otimes \mathbf{u}, \qquad y \neq 0,$$

where

$$\mathbf{u} = (\mathbf{n}_j \cdot \mathbf{n}_k)\mathbf{n}_i + (\mathbf{n}_k \cdot \mathbf{n}_i)\mathbf{n}_j + (\mathbf{n}_i \cdot \mathbf{n}_j)\mathbf{n}_k \tag{A.65}$$

represents a three-fold axis of the group I_h.

P r o o f. In terms of any three noncoplaner three-fold axes $(\mathbf{n}_i, \mathbf{n}_j, \mathbf{n}_k)$ of I_h, the second order identity tensor $\mathbf{I}$ is expressible as (cf. the formula (7.11) in [53])

$$\mathbf{I} = f(\mathbf{n}_i \otimes \mathbf{n}_i + \mathbf{n}_j \otimes \mathbf{n}_j + \mathbf{n}_k \otimes \mathbf{n}_k) \\ + g \overset{2}{\otimes} ((\mathbf{n}_j \cdot \mathbf{n}_k)\mathbf{n}_i + (\mathbf{n}_k \cdot \mathbf{n}_i)\mathbf{n}_j + (\mathbf{n}_i \cdot \mathbf{n}_j)\mathbf{n}_k), \quad fg \neq 0\,.$$

From the above equality we derive (A.64). Moreover, from Proposition 7.1 in [53] we know that the vector $\mathbf{u}$ given by (A.65) represents a three-fold axis of I_h. *Q.E.D.*

The proof for Theorem A.8 is as follows. By using (5.3) we deduce that $\det((\mathbf{n}_i \cdot \mathbf{n}_j)^2) = 2(\frac{4}{5})^5 \neq 0$ and hence that $\{\mathbf{n}_i \otimes \mathbf{n}_i\}$ offers a basis of the space Sym. In terms of this basis each $\mathbf{A} \in$ Sym is expressible as (cf. Proposition 7.4 given in [53])

$$\mathbf{A} = \frac{5}{4}\sum_{k=1}^{6} A_k \mathbf{N}_k - \frac{1}{2}(\mathrm{tr}\mathbf{A})\mathbf{I}\,, \tag{A.66}$$

where

$$\mathbf{N}_k = \mathbf{n}_k \otimes \mathbf{n}_k\,, \qquad A_k = \mathbf{n}_k \cdot \mathbf{A}\mathbf{n}_k\,, \qquad k = 1, \ldots, 6.$$

Utilizing (A.66) we infer that the following identities hold.

$$\sum_{k=1}^{6} \mathbf{N}_k = \sum_{k=1}^{6} \mathbf{Q} * \mathbf{N}_k \ (= 2\mathbf{I}),$$
$$\sum_{k=1}^{6} A_k \mathbf{N}_k = \sum_{k=1}^{6} A'_k \mathbf{Q} * \mathbf{N}_k\,,$$

for any $\mathbf{Q} \in$ Orth, where $A'_k = (\mathbf{Q}\mathbf{n}_k) \cdot \mathbf{A}(\mathbf{Q}\mathbf{n}_k)$. The above two identities and the equations (A.61) may be combined into

$$\sum_{k=1}^{6} (A_k)^r \mathbf{N}_k = \sum_{k=1}^{6} (A'_k)^r \mathbf{Q} * \mathbf{N}_k\,, \qquad r = 0, 1, 2, 3, 4\,. \tag{A.67}$$

Let

$\mathbf{A}_r \equiv$ the left-hand side of (A.67); $\qquad \mathbf{A}'_r \equiv$ the right-hand side of (A.67).

Then $\mathrm{tr}\mathbf{A}_r = \mathrm{tr}\mathbf{A}'_r$, $r = 0,1,2,3,4$, yield

$$\sum_{k=1}^{6}(A_k)^r = \sum_{k=1}^{6}(A'_k)^r, \qquad r = 0,1,2,3,4.$$

Here and hereafter $\mathrm{tr}\mathbf{B}$ is used to represent the trace of the tensor $\mathbf{B} \in T_2$. Furthermore, from (5.3) and the following equalities

$$(\mathrm{tr}\mathbf{A}_s)(\mathrm{tr}\mathbf{A}_t) = (\mathrm{tr}\mathbf{A}'_s)(\mathrm{tr}\mathbf{A}'_t), \qquad \mathrm{tr}(\mathbf{A}_s\mathbf{A}_t) = \mathrm{tr}(\mathbf{A}'_s\mathbf{A}'_t),$$

we derive

$$\sum_{k=1}^{6}(A_k)^{s+t} = \sum_{k=1}^{6}(A'_k)^{s+t}. \tag{A.68}$$

Let P_6 be the symmetric group on six letters. Then (A.68) yields

$$A'_k = A_{\sigma(k)}\,, \qquad k = 1,\ldots,6\,; \qquad \sigma \in P_6\,. \tag{A.69}$$

Hence the five equations for $\mathbf{A} \in \mathrm{Sym}$ given by (A.67) can be recast in the form

$$\sum_{k=1}^{6}(A_{\sigma(k)})^r\mathbf{Q} * \mathbf{N}_k = \sum_{k=1}^{6}(A_k)^r\mathbf{N}_k\,, \qquad r = 0,1,2,3,4.$$

Since for any given $\sigma \in P_6$ there is $\mathbf{R} \in I_h$ such that

$$\mathbf{R}^{\mathrm{T}} * \mathbf{N}_k = \mathbf{N}_k\,, \qquad k = 1,\ldots,6,$$

the above system of equations for $\mathbf{A}$ can be rewritten as

$$\sum_{k=1}^{6} A_k^r\mathbf{G}{*}\mathbf{N}_k = \sum_{k=1}^{6} A_k^r\mathbf{N}_k\,, \qquad \mathbf{G} = \mathbf{QR}, \qquad \mathbf{R} \in I_h\,, \quad r = 0,1,2,3,4. \tag{A.70}$$

Suppose that all A_k are pairwise distinct. Reformulating (A.70) in matrix notation as follows:

$$VX' + Y' = VX + Y,$$

where V is the 5×5 Vandermonde matrix of $A_1 \ \cdots \ A_5$, the sth row of which is given by $(A_1^{s-1} \ \cdots \ A_5^{s-1})$, and moreover, X, Y, X', and Y' are the following 5×1 column matrices:

$$X = (\mathbf{N}_1 \ \cdots \ \mathbf{N}_5)^{\mathrm{T}}, \qquad X' = (\mathbf{G} * \mathbf{N}_1 \ \cdots \ \mathbf{G} * \mathbf{N}_5)^{\mathrm{T}},$$

$$Y = (\mathbf{N}_6 \ A_6\mathbf{N}_6 \ \cdots \ (A_6)^4\mathbf{N}_6)^{\mathrm{T}}, \qquad Y' = (\mathbf{G} * \mathbf{N}_6 \ A_6\mathbf{G} * \mathbf{N}_6 \ \cdots \ (A_6)^4\mathbf{G} * \mathbf{N}_6)^{\mathrm{T}}.$$

Since the matrix V is invertible, we obtain

$$X' + V^{-1}Y' = X + V^{-1}Y,$$

i.e.

$$\mathbf{G} * (\mathbf{N}_k + x_k\mathbf{N}_6) = \mathbf{N}_k + x_k\mathbf{N}_6\,, \qquad k = 1,2,3,4,5.$$

Then by Lemma A.1 we infer that $\mathbf{G} \in I_h$ and therefore that $\mathbf{Q} = \mathbf{G}\mathbf{R}^{\mathrm{T}} \in I_h$, which violates the condition $\mathbf{Q} \notin I_h$.

Suppose that some of $A_1, \ldots, A_6$ coincide. By means of the similar procedure as that just used, we infer that the following facts hold.

(i) If there are $i, j \in \{1, \ldots, 6\}$, $i \neq j$, such that $A_i \neq A_j$ and $A_k \neq A_i$, A_j for all $k \in \{1, \ldots, 6\}$, $k \neq i, j$, then

$$\mathbf{G} * \mathbf{N}_i = \mathbf{N}_i\,, \qquad \mathbf{G} * \mathbf{N}_j = \mathbf{N}_j.$$

(ii) If there are $i, j \in \{1, \ldots, 6\}$, $i \neq j$, such that $A_i = A_j$ and $A_k \neq A_i$ for all $k \in \{1, \ldots, 6\}$, $k \neq i, j$, then

$$\mathbf{G} * (\mathbf{N}_i + \mathbf{N}_j) = \mathbf{N}_i + \mathbf{N}_j.$$

(iii) If $A_i = A_j = A_k \neq A_l = A_m = A_n$, where $(i, \ldots, n)$ is a permutation of $1, \cdots, 6$, then

$$\mathbf{A} = c'\mathbf{I} + d'(\mathbf{N}_i + \mathbf{N}_j + \mathbf{N}_k), \qquad d' \neq 0\,,$$
$$\mathbf{G} * (\mathbf{N}_i + \mathbf{N}_j + \mathbf{N}_k) = \mathbf{N}_i + \mathbf{N}_j + \mathbf{N}_k\,,$$

i.e. (cf. Lemma A.2)

$$\mathbf{A} = c\mathbf{I} + d\mathbf{u} \otimes \mathbf{u}, \qquad d \neq 0\,,$$
$$\overset{2}{\otimes}(\mathbf{Q}^{\mathrm{T}}\mathbf{u}) = \overset{2}{\otimes}\mathbf{v}\,,$$

where $\mathbf{v} = \mathbf{R}\mathbf{u}$ represents a three-fold axis of I_h, since $\mathbf{R} \in I_h$ and $\mathbf{u}$ represents a three-fold axis of I_h;

(iv) If $A_i \neq A_j = A_k = A_l = A_m = A_n$, then

$$\mathbf{A} = c\mathbf{I} + d\mathbf{N}_i\,, \qquad d \neq 0\,,$$
$$\mathbf{Q}^{\mathrm{T}} * \mathbf{N}_i = \mathbf{R} * \mathbf{N}_i\,, \qquad \mathbf{R} \in I_h\,.$$

(v) If $A_1 = \cdots = A_6 = c$, then $\mathbf{A} = c\mathbf{I}$.

In the last two cases, the identity (A.66) for $\mathbf{A} = \mathbf{I}$ has been used.

The cases (i) – (v) exhaust all the cases when $A_1, \ldots, A_6$ are not pairwise distinct. For the first two cases, we have

$$\mathbf{G} \in \{\pm\mathbf{I}, \pm\mathbf{R}^{\pi}_{\mathbf{n}_i+\mathbf{n}_j}, \pm\mathbf{R}^{\pi}_{\mathbf{n}_i-\mathbf{n}_j}, \pm\mathbf{R}^{\pi}_{\mathbf{n}_i\times\mathbf{n}_j}\}.$$

Since the vectors $\mathbf{n}_i + \mathbf{n}_j$, $\mathbf{n}_i - \mathbf{n}_j$ and $\mathbf{n}_i \times \mathbf{n}_j$ give three two-fold axes of I_h (cf. Proposition 7.2 in [53]), we infer that $\mathbf{G} \in I_h$ and hence $\mathbf{Q} = \mathbf{G}\mathbf{R}^{\mathrm{T}} \in I_h$ for the first two cases, which violates the condition $\mathbf{Q} \notin I_h$. Thus, the first two cases are excluded. On the other hand, the latter three cases yield three kinds of solutions to the polynomial tensor equations (A.61) for $\mathbf{A} \in \mathrm{Sym}$, and from them the solutions to the polynomial tensor equations (A.59) and (A.60) for $\mathbf{x} \in V$ and $\ddot{\mathbf{W}} \in \mathrm{Skw}$ can be derived immediately, since both $\mathbf{x} \otimes \mathbf{x}$ and $\overset{2}{\otimes} (\mathbf{E} : \mathbf{W})$ can be visualized as two particular forms of the symmetric second order tensor $\mathbf{A}$. It is evident that the solutions thus obtained agree with those given by the two cases in Theorem A.8. *Q.E.D.*

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Fabric tensor and constitutive equations for a class of plastic and locking orthotropic materials

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THE AIM of the paper is to provide a general and common mathematical structure for a class of orthotropic materials undergoing plastic deformations or exhibiting locking behaviour. The orthotropy is included by using a fabric tensor. The tensorial constitutive relationships are studied from the point of view of tensor functions representations. Specific cases are also discussed.

1. Introduction

IN THE PAPER [18] the tensorial structure of the constitutive relationships for isotropic, perfectly locking materials was examined in detail. It was revealed that the general structure of equations is similar to those of isotropic, perfect plasticity, since they are time-independent.

The main aim of the present paper is to include orthotropy into such a general framework. This has been achieved by using a fabric tensor **H**, which is a particular case of structural tensors characterising anisotropic materials. It was introduced by COWIN [8-10] as a symmetric, positive definite tensor, a square root of the inverse of the mean intercept length tensor **M**. By using the classical spectral theorem, the interrelations between those tensors have been examined in Sec. 2.

In Sec. 3 the general structure of constitutive relationships involving two symmetric tensors and a scalar, common to plastic and locking materials, has been introduced. A different interpretation of the tensor **C** appearing in such a general relationships has been provided. For instance, **C** may be the tensor of plastic deformation or the locking stress tensor. In this manner, plastic hardening and/or softening and non-perfectly locking behaviour may be taken into account. Perfectly plastic and perfectly locking orthotropic materials are characterized by $\mathbf{C} = \mathbf{H}$.

In Sec. 4 the tensorial constitutive relationship introduced in Sec. 3 has been discussed from the point of view of tensor functions representations. Both polynomial and nonpolynomial representations have been investigated.

Our considerations of perfectly plastic and perfectly locking materials have deliberately been restricted to the orthotropy since then the available representation theorems are well developed. The same cannot be said about the case where anisotropy is described by a fourth-order tensor. Specific cases of constitutive relationships, yield and locking conditions have been studied in Secs. 5 and 6.

Throughout this paper the summation convention applies to repeated indices, unless otherwise stated.

2. The fabric tensor

Some materials such as wood, granular materials, bones and plastics exhibit elastic, plastic and locking behaviour under compressive stresses. The stress-deformation curves are then strongly influenced by the density of a material, cf. Figs. 10.3 and 11.5 in [12].

The aim of the present paper is not a study of such particular materials, which should be performed within a framework of elastic-plastic-locking behaviour. In the papers [18, 19] we have noticed a formal similarity between isotropic perfectly plastic and perfectly locking materials. In the present contribution we shall provide a general framework for perfectly plastic and perfectly locking *orthotropic* materials, provided that structural anisotropy is described by a second-order tensor, called the fabric tensor, cf. [1, 8, 13, 14, 20, 25, 42].

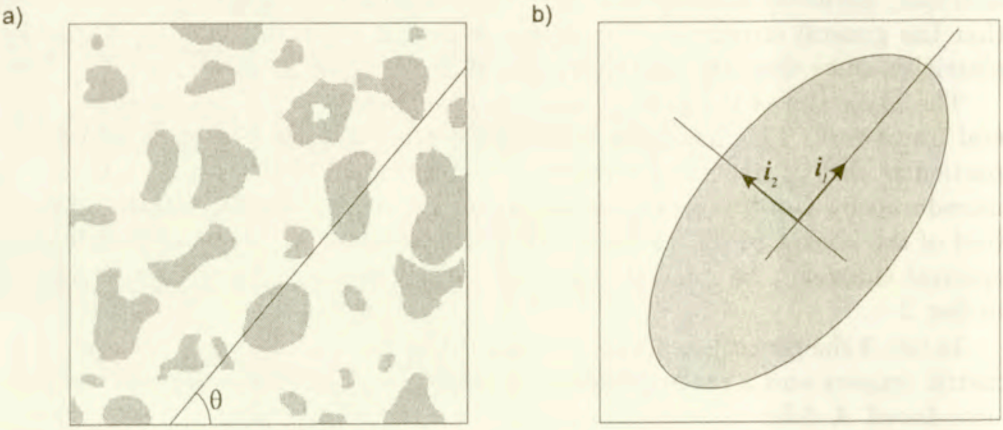

FIG. 1. The mean intercept length ellipse and its construction: a) test lines superimposed on a cellular material specimen. The test lines are oriented at angle Θ, which is varied to obtain the mean intercept length $L(\Theta)$, b) the ellipse constructed according to Eq. (2.1).

Let us introduce this tensor. Firstly, however, following WHITEHOUSE [40] we recall the notion of the mean intercept length L. This author measured L in cancellous bone as a function of direction on polished plane sections. Then L is the distance between two bone/marrow interfaces measured along a line. The value of L is a function of the slope Θ of the line along which the measurement is made. WHITEHOUSE [40] showed that when $L(\Theta)$ is plotted in the polar coordinates then the polar diagram produces ellipses, cf. Fig. 1. If the test lines are rotated through several values of Θ and the corresponding values of $L(\Theta)$ are measured, the data is found to fit the following equation of an ellipse:

$$\frac{1}{L(\Theta)} = M_{11}\cos^2\Theta + M_{22}\sin^2\Theta + 2M_{12}\sin\Theta\cos\Theta\,, \tag{2.1}$$

where M_{11}, M_{22} and M_{12} are constants, provided that the reference line from which the angle Θ is measured is constant.

HARRIGAN and MANN [14] extended Whitehouse's approach to the three-dimensional case and showed that $L(\mathbf{n})$, as a function of a direction $\mathbf{n}$, would be represented by ellipsoids and would therefore be equivalent to a positive definite second-order tensor $\mathbf{M}$ defined by

$$\frac{1}{L(\Theta)} = M_{ij}n_i n_j\,, \tag{2.2}$$

where $\mathbf{n}$ is a unit vector in the direction of the test line.

COWIN [8 – 10] defined a fabric tensor of cancellous bone to be the inverse square root of the mean intercept length tensor $\mathbf{M}$:

$$\mathbf{H} = \frac{1}{\sqrt{\mathbf{M}}}\,. \tag{2.3}$$

Obviously, $\mathbf{H}$ is well defined because $\mathbf{M}$ is a positive definite and symmetric tensor, cf. MARSDEN and and HUGES [21], pp. 52–55. The components of $\mathbf{M}$ or the mean intercept ellipsoid can be measured by using the techniques described by HARRIGAN and MAN [14] for a cubic specimen.

A specific form of the fabric tensor $\mathbf{H}$ is not required for our subsequent developments. The only assumption is that $\mathbf{H}$ should be a positive definite and symmetric second order tensor.

An alternative approach to the fabric tensor has been discussed by ZYSSET and CURNIER [42]. These authors decompose the fabric tensor $\mathbf{H}$ as follows: $\mathbf{H} = g\mathbf{I} + \mathbf{G}$.

An elementary microstructural description is contained in a single scalar property such as relative density, while material anisotropy requires fabric tensors of higher even rank [20]. KANATANI'S [20] approach can be applied to a class of materials with strictly positive morphological properties that are radially symmetric. In these situations we can use a scalar-valued orientation distribution function $h(\mathbf{N}) > 0$, where $\mathbf{N} = \mathbf{n}\otimes\mathbf{n}$ is the tensor product of the unit vector $\mathbf{n}$ specifying the orientation. Assuming the function to be square integrable it can be expanded in a convergent Fourier series:

$$h(\mathbf{N}) = g(\mathbf{N})1 + \mathbf{G}\cdot\mathbf{F}(\mathbf{N}) + \mathbb{G}\cdot\mathbb{F}(\mathbf{N}) + \ldots\,, \tag{2.4}$$

where 1, $\mathbf{F}(\mathbf{N})$ and $\mathbb{F}(\mathbf{N})$ are even rank tensorial basis functions and g, $\mathbf{G}$ and $\mathbb{G}$ – the corresponding even rank fabric tensors [20]. In bone mechanics we can use an approximation based on a scalar and a symmetric, traceless second rank fabric tensor. Then the first tensorial basis function is: $\mathbf{F} - \frac{1}{3}\mathbf{I}$, while the tensorial coefficients are calculated by

$$g = \frac{1}{4\pi} \int_S h(\mathbf{N})\, dS, \qquad \mathbf{G} = \frac{15}{8\pi} \int_S h(\mathbf{N})\, \mathbf{F}(\mathbf{N})\, dS, \tag{2.5}$$

where S is the surface of the unit sphere. For the particular case of an ellipsoidal distribution function we have

$$h(\mathbf{N}) = \frac{1}{\sqrt{\mathbf{N}\cdot\mathbf{M}}}\,. \tag{2.6}$$

GOULET *et al.* [13] applied the concept of the mean intercept length to investigate the relationships between the structural parameters for cancellous bone, to determine their correlation to the mechanical properties, and to evaluate which parameters are important for maintaining bone strength and integrity.

The fabric tensor $\mathbf{H}$, as defined by (2.3), is an isotropic tensor function of $\mathbf{M}$, say $\widehat{\mathbf{H}}(\mathbf{M})$, cf. TING [36]. It means that

$$\forall\, \mathbf{Q} \in O(3) \qquad \mathbf{Q}\widehat{\mathbf{H}}(\mathbf{M})\mathbf{Q}^T = \widehat{\mathbf{H}}\left(\mathbf{Q}\mathbf{M}\mathbf{Q}^T\right) = \mathbf{Q}\frac{1}{\sqrt{\mathbf{M}}}\mathbf{Q}^T. \tag{2.7}$$

Here $O(3)$ stands for the full orthogonal group:

$$O(3) \equiv \{\mathbf{Q} : \quad \mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I}\}, \tag{2.8}$$

where $\mathbf{I}$ is the identity tensor while $\mathbf{Q}^T$ is the transpose of $\mathbf{Q}$.

Let us pass to the determination of the function

$$\mathbf{H} = \widehat{\mathbf{H}}(\mathbf{M}) = \frac{1}{\sqrt{\mathbf{M}}}\,. \tag{2.9}$$

Since $\mathbf{M}$ is a symmetric, positive definite tensor, therefore by applying the spectral theorem we may write

$$\mathbf{M} = M_1 \mathbf{i}_1 \otimes \mathbf{i}_1 + M_2 \mathbf{i}_2 \otimes \mathbf{i}_2 + M_3 \mathbf{i}_3 \otimes \mathbf{i}_3\,, \tag{2.10}$$

where M_j $(j = 1, 2, 3)$ are eigenvalues of the tensor $\mathbf{M}$, and $\mathbf{i}_j$ its eigenvectors. It is assumed that $M_1 \geq M_2 \geq M_3$, where

$$M_i = \frac{1}{3} I_M + \frac{2}{3}\sqrt{I_M^2 - 3II_M}\, \cos\left[\frac{2}{3}\pi(i-1) - \varphi\right], \qquad i = 1, 2, 3 \tag{2.11}$$

and

$$\cos 3\varphi = \frac{2I_M^3 - 9I_M II_M + 27III_M}{\sqrt{2(I_M^2 - 3II_M)^3}}\,. \tag{2.12}$$

The basic invariants of **M** are given by

$$(2.13)\qquad \begin{gathered} I_M = \operatorname{tr}\mathbf{M}, \qquad II_M = \frac{1}{2}\left(\operatorname{tr}^2\mathbf{M} - \operatorname{tr}\mathbf{M}^2\right), \\ III_M = \det\mathbf{M} = \frac{1}{6}\left(\operatorname{tr}^3\mathbf{M} - 3\operatorname{tr}\mathbf{M}\operatorname{tr}\mathbf{M}^2 + 2\operatorname{tr}\mathbf{M}^3\right), \end{gathered}$$

where $\operatorname{tr}\mathbf{M}$ is the trace of **M**. In an orthonormal basis $\{\mathbf{e}_i\}$ $(i = 1, 2, 3)$ we have: $\mathbf{M} = M_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, $\operatorname{tr}\mathbf{M} = M_{ii}$, $(\mathbf{M}^2)_{ij} = (\mathbf{M}\mathbf{M})_{ij} = M_{ik}M_{kj}$, etc.

Note that if

$$(2.14)\qquad d = 4II_M^3 - I_M^2 II_M^2 + 4I_M^3 III_M - 18 I_M II_M II_M + 27 II_M^2 < 0,$$

then M_i in (2.11) are different; for $d = 0$ two of the eigenvalues are equal; in other words, the tensor **M** is then two-dimensional. Finally, for

$$(2.15)\qquad I_M^2 = 3II_M\,,$$

M is a spherical tensor.

In the case of three different eigenvalues, the eigentensors $\mathbf{i}_j \otimes \mathbf{i}_j$ (no summation over j) can be determined in a unique fashion:

$$(2.16)\qquad \mathbf{i}_j \otimes \mathbf{i}_j = \frac{1}{m_j}\left[\mathbf{M}^2 - (I_M - M_j)\mathbf{M} + III_M M_j^{-1}\mathbf{I}\right] \quad \text{(no summation over } j\text{)},$$

where

$$(2.17)\qquad m_j = 2M_j^2 - I_M M_j + III_M M_j^{-1}\,.$$

Consequently the fabric tensor (2.3) satisfying (2.7) can be represented in the following form

$$(2.18)\qquad \mathbf{H} = H_1 \mathbf{i}_1 \otimes \mathbf{i}_1 + H_2 \mathbf{i}_2 \otimes \mathbf{i}_2 + H_3 \mathbf{i}_3 \otimes \mathbf{i}_3\,,$$

where

$$(2.19)\qquad H_i = \frac{1}{\sqrt{M_i}}, \qquad i = 1, 2, 3.$$

In the case of multiple eigenvalues of **M**, the eigentensors (2.16) are not determined uniquely, cf. JEMIOŁO [17], MORMAN [23], TING [36]. As could be expected, for two (three) identical eigenvalues, (2.16) reduces to the representation of plane (spherical) tensors.

REMARK 1. If $d = 0$ and, for instance $M_1 \neq M_2 = M_3$, then instead of (2.11) one can calculate the eigenvalues similarly as for plane tensors $\overline{\mathbf{M}}$, i.e.:

$$(2.20)\qquad M_{1,2} = \overline{M}_{1,2} = \frac{1}{2}\left(\overline{I}_M \pm \sqrt{\overline{I}_M - 4\overline{II}_M}\right).$$

Consequently, (2.18) is to be replaced by the two-dimensional representation given by

$$\overline{\mathbf{H}} = \frac{1}{\sqrt{\overline{I}_M + 2\overline{II}_M}} \left[\left(1 + \frac{\overline{I}_M}{\sqrt{\overline{II}_M}} \right) \overline{\mathbf{I}} - \frac{1}{\sqrt{\overline{II}_M}} \overline{\mathbf{M}} \right], \tag{2.21}$$

where

$$\overline{I}_M = \operatorname{tr}\overline{\mathbf{M}}, \qquad \overline{II}_M = \det \overline{\mathbf{M}} = \frac{1}{2}\left(\operatorname{tr}^2\overline{\mathbf{M}} - \operatorname{tr}\overline{\mathbf{M}}^2 \right). \tag{2.22}$$

Finally, if (2.15) is satisfied, then $M_1 = M_2 = M_3 = M$ and

$$\overline{\mathbf{H}} = \frac{1}{\sqrt{M}} \overline{\mathbf{I}}. \tag{2.23}$$

3. Plastic or perfectly locking behaviour: common general structure of constitutive relationships

Constitutive relationships describing perfectly plastic and perfectly locking materials exhibit a common feature, as being rate-independent, cf. JEMIOŁO and TELEGA [18, 19]. Consequently, the general structure of constitutive relationships for both classes of materials is similar. MURAKAMI and SAWCZUK [24] extended the approach proposed in [29] to plastic materials with hardening, though softening is not precluded. It means that our subsequent developments can also be generalised to non-perfectly locking materials.

The considerations which follow are restricted to a class of materials obeying the constitutive relationship

$$\mathbf{A} = \mathbf{F}(\varrho, \dot{\mathbf{B}}, \mathbf{C}), \tag{3.1}$$

subject to

$$\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{B}}} \cdot \dot{\mathbf{B}} = \mathbf{0} \qquad \text{if} \qquad \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{B}}} \neq \tilde{\mathbf{0}}. \tag{3.2}$$

Physical interpretation of the tensors $\mathbf{A}$, $\mathbf{C}$ and $\dot{\mathbf{B}}$ will be given later on. It will also be assumed that

$$D = \phi(\varrho, \dot{\mathbf{B}}, \mathbf{C}) \equiv \mathbf{A} \cdot \dot{\mathbf{B}} = \operatorname{tr} \mathbf{A}\dot{\mathbf{B}} \geq 0. \tag{3.3}$$

Here $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are second-order symmetric tensors while $\dot{\mathbf{B}}$ is an objective time derivative of $\mathbf{B}$; ϱ is a scalar parameter, for instance the density. More generally, ϱ may be an internal scalar parameter, which is an isotropic function of $\mathbf{B}$. By $\mathbf{0}$

and $\widetilde{\mathbf{0}}$ we denote null tensors of the second and fourth order, respectively. In an orthonormal basis the relationship $(3.2)_1$ is obviously given by

$$\frac{\partial F_{ij}}{\partial \dot{B}_{kl}} \dot{B}_{kl} = 0. \tag{3.4}$$

From (3.2) and (3.3) we conclude that

$$\frac{\partial \phi}{\partial \dot{\mathbf{B}}} \cdot \dot{\mathbf{B}} = \phi. \tag{3.5}$$

According to the principle of isotropy of the physical space [28], the tensor function (3.1) has to be isotropic:

$$\forall\, \mathbf{Q} \in O(3) \qquad \mathbf{Q}\mathbf{F}(\varrho, \dot{\mathbf{B}}, \mathbf{C})\mathbf{Q}^T = \mathbf{F}(\varrho, \mathbf{Q}\dot{\mathbf{B}}\mathbf{Q}^T, \mathbf{Q}\mathbf{C}\mathbf{Q}^T), \tag{3.6}$$

where the full orthogonal group $O(3)$ is defined by (2.8). Note that the function $\mathbf{F}$ is an anisotropic function with respect to $\dot{\mathbf{B}}$. An anisotropy group S is given by

$$S \equiv \{\mathbf{Q} \in O(3): \quad \mathbf{Q}\mathbf{C}\mathbf{Q}^T = \mathbf{C}\}. \tag{3.7}$$

Consequently, the material anisotropy is determined by the structural (fabric) tensor $\mathbf{C} = \mathbf{H}$. We have

$$\forall\, \mathbf{Q} \in S \qquad \mathbf{Q}\mathbf{F}(\varrho, \dot{\mathbf{B}}, \mathbf{H})\mathbf{Q}^T = \mathbf{F}(\varrho, \mathbf{Q}\dot{\mathbf{B}}\mathbf{Q}^T, \mathbf{H}). \tag{3.8}$$

The tensor $\mathbf{H}$, being a symmetric second-order tensor, enables one to determine the following three cases of anisotropy:

(i) If $H_1 \neq H_2 \neq H_3 \neq H_1$ then S stands for the orthotropy group; more precisely, according to Schoenfliess' notation we then write $S = D_{4h}$, cf. [41]. In this case one has

$$S = S_1 \cap S_2 \cap S_3\,, \tag{3.9}$$

where

$$S_i \equiv \{\mathbf{Q} \in O(3): \quad \mathbf{Q}(\mathbf{i}_i \otimes \mathbf{i}_i)\mathbf{Q}^T = \mathbf{i}_i \otimes \mathbf{i}_i\} \qquad \text{(no summation over } i\text{)}. \tag{3.10}$$

(ii) If two eigenvalues coincide, say $H_1 \neq H_2 = H_3$, then S denotes the transverse isotropy group:

$$S = S_1 = D_{\infty h}\,. \tag{3.11}$$

(iii) If $H_1 = H_2 = H_3$ then $S = O(3)$ is the isotropy group.

The scalar function ϕ defined by (3.3) is an orthotropic function of $\dot{\mathbf{B}}$ provided that $\mathbf{C} = \mathbf{H}$:

$$\forall\, \mathbf{Q} \in S \qquad \phi(\varrho, \dot{\mathbf{B}}, \mathbf{H}) = \phi(\varrho, \mathbf{Q}\dot{\mathbf{B}}\mathbf{Q}^T, \mathbf{H}). \tag{3.12}$$

Let us investigate some important consequences implied by the homogeneity condition (3.2). Firstly, following SAWCZUK and STUTZ [28] we conclude that there exists a scalar function

$$f(\varrho, \mathbf{A}, \mathbf{C}) = 0, \tag{3.13}$$

satisfying the condition

$$\forall\, \mathbf{Q} \in S \qquad f(\varrho, \mathbf{A}, \mathbf{C}) = f(\varrho, \mathbf{Q}\mathbf{A}\mathbf{Q}^T, \mathbf{C}). \tag{3.14}$$

Secondly, one has

$$\det\left(\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{B}}}\right) = 0. \tag{3.15}$$

Thus the constitutive relationships (3.1) is not invertible. However, one can find a semi-invertible relation, now given by

$$\dot{\mathbf{B}} = \lambda \widetilde{\mathbf{F}}(\varrho, \mathbf{A}, \mathbf{C}), \qquad \lambda \geq 0, \tag{3.16}$$

where

$$\lambda = \eta(\varrho, \dot{\mathbf{B}}, \mathbf{C}). \tag{3.17}$$

Note that the notion of semi-invertibility was introduced by TRUESDELL and MOON [37], where a symmetric second-order tensor function of a symmetric, second order tensor was analysed.

REMARK 2. One can consider the case when ϱ and $\mathbf{C}$ are isotropic functions of $\mathbf{B}$: $\varrho = \widetilde{\varrho}(\mathbf{B})$ and $\mathbf{C} = \widetilde{\mathbf{C}}(\mathbf{B})$. The isotropy means that

$$\forall\, \mathbf{Q} \in O(3) \qquad \varrho = \widetilde{\varrho}(\mathbf{B}) = \widetilde{\varrho}(\mathbf{Q}\mathbf{B}\mathbf{Q}^T), \tag{3.18}$$

$$\forall\, \mathbf{Q} \in O(3) \qquad \mathbf{Q}\widetilde{\mathbf{C}}(\mathbf{B})\mathbf{Q}^T = \widetilde{\mathbf{C}}(\mathbf{Q}\mathbf{B}\mathbf{Q}^T). \tag{3.19}$$

To make such a theory complete, evolution laws $\dot{\varrho} = \widehat{\varrho}(\mathbf{B})$ and $\dot{\mathbf{C}} = \widehat{\mathbf{C}}(\mathbf{B})$ must additionally be specified. ◁

In general, the function ϕ is not a potential for $\mathbf{A}$. Below it will be shown that under the following condition, cf. [18, 19],

$$\dot{\mathbf{B}} \cdot \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{B}}} = \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{B}}} \cdot \dot{\mathbf{B}}, \tag{3.20}$$

one has

$$\mathbf{A} = \frac{\partial \phi}{\partial \dot{\mathbf{B}}}. \tag{3.21}$$

In general, the constitutive relationship (3.16) is also not associated with the scalar condition (3.13). The associated rule has the form

$$\dot{\mathbf{B}} = \lambda \frac{\partial f}{\partial \mathbf{A}}, \qquad \lambda \geq 0. \tag{3.22}$$

REMARK 3. Suppose that the set $C(\varrho, \mathbf{C})$ defined by

$$C(\varrho, \mathbf{C}) \equiv \{\mathbf{T} \in T^s \mid \ f(\varrho, \mathbf{T}, \mathbf{C}) \leq 0\}, \tag{3.23}$$

is convex and closed. Here T^s stands for the space of symmetric, second-order tensors. The indicator function of this set is given by, cf. ROCKAFELLAR [27]

$$\mathbf{I}_{\mathbf{C}(\varrho,\mathbf{C})}(\mathbf{T}) = \begin{cases} 0 & \text{if } \ \mathbf{T} \in C(\varrho, \mathbf{C}), \\ \infty & \text{otherwise.} \end{cases} \tag{3.24}$$

The subdifferential (associated) constitutive relationship has the following form:

$$\dot{\mathbf{B}} \in \partial I_{C(\varrho,\mathbf{C})}(\mathbf{A}). \tag{3.25}$$

In the case when f is differentiable with respect to the second argument, the last law is equivalent to (3.22).

The support function of $C(\varrho, \mathbf{C})$ is a particular case of (3.3) and is calculated as follows, cf. [27]:

$$\phi_1(\varrho, \dot{\mathbf{B}}, \mathbf{C}) = \sup_{\mathbf{T} \in T^s} \left\{ \dot{\mathbf{B}} \cdot \mathbf{T} - I_{C(\varrho,\mathbf{C})}(\mathbf{T}) \right\} = \sup \left\{ \dot{\mathbf{B}} \cdot \mathbf{T} \mid \ \mathbf{T} \in C(\varrho, \mathbf{C}) \right\}. \tag{3.26}$$

The function $\phi_1(\varrho, \cdot, \mathbf{C})$ is convex and subdifferentiable.

The constitutive relationship inverse to (3.25), and equivalent to it, is given by

$$\mathbf{A} \in \partial_2 \phi_1(\varrho, \dot{\mathbf{B}}, \mathbf{C}), \tag{3.27}$$

where $\partial_2 \phi_1(\varrho, \dot{\mathbf{B}}, \mathbf{C})$ stands for the subdifferential of the function $\phi_1(\varrho, \cdot, \mathbf{C})$ at a point $\dot{\mathbf{B}}$. If the function $\phi_1(\varrho, \cdot, \mathbf{C})$ is differentiable then the law (3.21) is recovered and ϕ_1 coincides with ϕ. $\lhd$

Let us now specify two particular classes of materials described by the introduced general relations within the theory of small deformations, cf. Fig. 2.

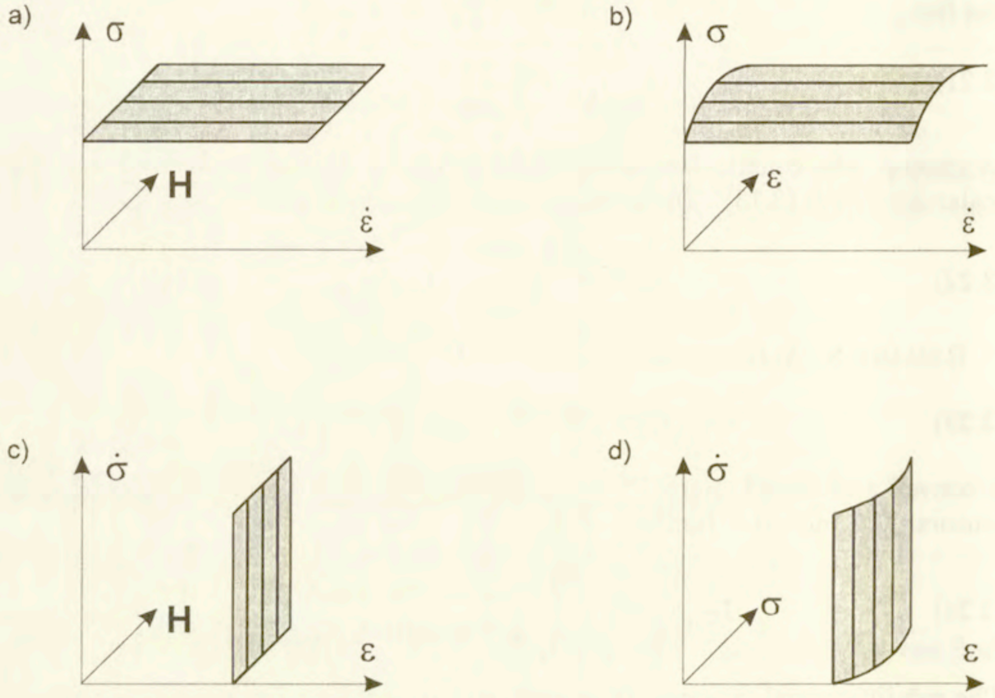

FIG. 2. Schematic representation of the constitutive relationships allowed by Eqs. (3.1) and (3.2): a) $\mathbf{C} = \mathbf{H}$, $\mathbf{A} = \boldsymbol{\sigma}$, $\dot{\mathbf{B}} = \dot{\boldsymbol{\varepsilon}}$, orthotropic perfect plasticity, b) $\mathbf{C} = \boldsymbol{\varepsilon}$, $\mathbf{A} = \boldsymbol{\sigma}$, $\dot{\mathbf{B}} = \dot{\boldsymbol{\varepsilon}}$, plasticity with hardening, c) $\mathbf{C} = \mathbf{H}$, $\mathbf{A} = \boldsymbol{\varepsilon}$, $\dot{\mathbf{B}} = \dot{\boldsymbol{\sigma}}$, orthotropic locking materials, d) $\mathbf{C} = \boldsymbol{\sigma}$, $\mathbf{A} = \boldsymbol{\varepsilon}$, $\dot{\mathbf{B}} = \dot{\boldsymbol{\sigma}}$, non-perfectly locking behaviour.

For

$$(\mathbf{A}, \dot{\mathbf{B}}) = (\boldsymbol{\sigma}, \dot{\boldsymbol{\varepsilon}}), \tag{3.28}$$

and $\mathbf{C} = \mathbf{H}$, perfect plasticity of at most orthotropic materials is recovered. Here $\boldsymbol{\sigma}$ denotes the stress tensor and $\dot{\boldsymbol{\varepsilon}}$ is the rate of plastic deformation. Assuming additionally (3.18), one obtains plasticity with isotropic hardening/softening. Incorporating (3.19), one describes an orthotropic hardening. For $\mathbf{C} = \boldsymbol{\varepsilon}$, the material behaviour is initially isotropic; $\boldsymbol{\varepsilon}$ denotes the strain tensor.

The second case (perfectly locking behaviour) is for $\mathbf{C} = \mathbf{H}$ and, cf. [18, 19]

$$(\mathbf{A}, \dot{\mathbf{B}}) = (\boldsymbol{\varepsilon}, \dot{\boldsymbol{\sigma}}). \tag{3.29}$$

For instance, if $\mathbf{C} = \boldsymbol{\sigma}$ then our approach describes non-perfectly locking behaviour; the orthotropy is then induced by the locking stress tensor $\boldsymbol{\sigma}$.

Summarizing, we conclude that (3.13) represents:

(a) The yield condition if $\mathbf{A} = \boldsymbol{\sigma}$,

(b) the locking condition if $\mathbf{A} = \boldsymbol{\varepsilon}$.

Similarly, (3.16) is:

(i) the flow rule if $(\mathbf{A}, \dot{\mathbf{B}}) = (\boldsymbol{\sigma}, \dot{\boldsymbol{\varepsilon}})$,

(ii) the locking rule if $(\mathbf{A}, \dot{\mathbf{B}}) = (\boldsymbol{\varepsilon}, \dot{\boldsymbol{\sigma}})$.

As we already know, in general both these rules are non-associated laws.

4. Representation of the tensor function (3.1)

To determine the general form of the tensor function (3.1) satisfying the isotropy condition (3.6) one can apply either the results due to SPENCER [32] on polynomial representation or those obtained by WANG [39] as well as BOEHLER and RACLIN [5], concerning nonpolynomial representations. For both representations the so-called canonical form of (3.1) is expressed by

$$\mathbf{A} = \widetilde{\alpha}_p \mathbf{G}_p \,, \tag{4.1}$$

where $p = 1, ..., 9$ for the polynomial representation, while $p = 1, ..., 8$ in the case of the nonpolynomial representation. The summation convention still applies, unless otherwise stated. Here $\widetilde{\alpha}_p$ are scalar functions of ϱ and of isotropic invariants of $\dot{\mathbf{B}}$ and $\mathbf{C}$:

$$\alpha_p = \widetilde{\alpha}_p(\varrho, I_r), \qquad r = 1, ..., 10. \tag{4.2}$$

The symmetric second-order tensors $\mathbf{G}_p$ are the so-called generators.

We have

$$\begin{aligned}\{I_r\} &= \{\operatorname{tr}\dot{\mathbf{B}}, \operatorname{tr}\dot{\mathbf{B}}^2, \operatorname{tr}\dot{\mathbf{B}}^3, \operatorname{tr}\dot{\mathbf{B}}\mathbf{C}, \operatorname{tr}\dot{\mathbf{B}}\mathbf{C}^2, \operatorname{tr}\dot{\mathbf{B}}^2\mathbf{C}, \operatorname{tr}\dot{\mathbf{B}}^2\mathbf{C}^2, \operatorname{tr}\mathbf{C}, \operatorname{tr}\mathbf{C}^2, \operatorname{tr}\mathbf{C}^3\} \\ &= \{J_s, \operatorname{tr}\mathbf{C}^i\}, \qquad s = 1, \ldots, 7, \quad i = 1, 2, 3,\end{aligned} \tag{4.3}$$

$$\{\mathbf{G}_p\} = \{\mathbf{I}, \dot{\mathbf{B}}, \dot{\mathbf{B}}^2, \mathbf{C}, \mathbf{C}^2, \mathbf{C}\dot{\mathbf{B}} + \dot{\mathbf{B}}\mathbf{C}, \mathbf{C}^2\dot{\mathbf{B}} + \dot{\mathbf{B}}\mathbf{C}^2, \mathbf{C}^2\dot{\mathbf{B}}^2 + \dot{\mathbf{B}}^2\mathbf{C}^2\}. \tag{4.4}$$

It can be easily shown that under the condition (3.20), the representation (4.1) simplifies since it contains only seven generators while $\widetilde{\alpha}_p$ satisfy additional relations, see below.

Since $\dot{\mathbf{B}}$ is a symmetric tensor, therefore one has

$$\begin{aligned}&\frac{\partial \operatorname{tr}\dot{\mathbf{B}}^i}{\partial \dot{\mathbf{B}}} = i\dot{\mathbf{B}}^{i-1}, \qquad i = 1, 2, 3, \quad \dot{\mathbf{B}}^0 = \mathbf{I}, \\ &\frac{\partial \operatorname{tr}\dot{\mathbf{B}}\mathbf{C}^\alpha}{\partial \dot{\mathbf{B}}} = \mathbf{C}^\alpha, \qquad \frac{\partial \operatorname{tr}\dot{\mathbf{B}}^2\mathbf{C}^\alpha}{\partial \dot{\mathbf{B}}} = \dot{\mathbf{B}}\mathbf{C}^\alpha + \mathbf{C}^\alpha\dot{\mathbf{B}}, \quad \alpha = 1, 2,\end{aligned} \tag{4.5}$$

and the representation (4.1) can be written in the following form:

$$\begin{aligned}\mathbf{A} = \gamma_1\mathbf{I} + 2\gamma_2\dot{\mathbf{B}} + 3\gamma_3\dot{\mathbf{B}}^2 + \gamma_4\mathbf{C} + \gamma_5\mathbf{C}^2 + \gamma_6(\dot{\mathbf{B}}\mathbf{C} + \mathbf{C}\dot{\mathbf{B}}) \\ + \gamma_7(\dot{\mathbf{B}}\mathbf{C}^2 + \mathbf{C}^2\dot{\mathbf{B}}) = \gamma_s\widetilde{\mathbf{G}}_s \,,\end{aligned} \tag{4.6}$$

provided that (3.20) is satisfied; $s = 1, ..., 7$. Here

$$\gamma_s = \frac{\partial \widetilde{g}}{\partial J_s}, \tag{4.7}$$

and $g = \widetilde{g}(\varrho, J_s, \operatorname{tr}\mathbf{C}^i)$, $i = 1, 2, 3$. The condition (3.20) implies

$$\frac{\partial \gamma_s}{\partial J_t} = \frac{\partial \gamma_t}{\partial J_s}, \qquad s, t = 1, ..., 7. \tag{4.8}$$

Note that (4.6) yields

$$\frac{\partial \mathbf{A}}{\partial \dot{\mathbf{B}}} = \frac{\partial \gamma_s}{\partial J_t} \widetilde{\mathbf{G}}_s \otimes \widetilde{\mathbf{G}}_t + \gamma_s \frac{\partial \widetilde{\mathbf{G}}_s}{\partial \dot{\mathbf{B}}} \tag{4.9}$$

and

$$\frac{\partial \widetilde{\mathbf{G}}_s}{\partial \dot{\mathbf{B}}} = \frac{\partial^2 J_s}{\partial \dot{\mathbf{B}} \otimes \partial \dot{\mathbf{B}}}. \tag{4.10}$$

Here the functions γ_s are interrelated by (4.8). Hence we conclude that (4.8) is equivalent to (3.20).

Comparing (4.1) with (4.6) and taking into account (4.7) and (4.8) one obtains additional conditions which have to be fulfilled by the scalar functions $\widetilde{\alpha}_p$. To satisfy (4.10) we conclude that the representation (4.6) has to be at least of class C^2 with respect to $\dot{\mathbf{B}}$.

The invariants (4.3) appearing in (4.2) constitute both polynomial and non-polynomial bases. A polynomial basis consisting of 10 invariants can be constructed from another set of independent invariants, provided, however, that they are polynomials in I_r. Obviously, we are discussing the general case of 3D tensors. In the remaining cases the relevant representations are simplified. The choice of a functional basis is also not unique. To satisfy the homogeneity condition (3.2) it is convenient to deal with the following functional basis consisting of invariants being functions of order one with respect to $\dot{\mathbf{B}}$ and including the invariants $\operatorname{tr}\mathbf{C}^i$ $(i = 1, 2, 3)$:

$$\{K_r\} = \left\{ \operatorname{tr}\dot{\mathbf{B}}, \sqrt{\operatorname{tr}\dot{\mathbf{B}}^2}, \sqrt[3]{\operatorname{tr}\dot{\mathbf{B}}^3}, \operatorname{tr}\dot{\mathbf{B}}\mathbf{C}, \operatorname{tr}\dot{\mathbf{B}}\mathbf{C}^2, \sqrt{\operatorname{tr}\dot{\mathbf{B}}^2\mathbf{C}}, \sqrt{\operatorname{tr}\dot{\mathbf{B}}^2\mathbf{C}^2}, \operatorname{tr}\mathbf{C}^i \right\} \tag{4.11}$$

$$r = 1, \ldots, 10, \qquad i = 1.2.3.$$

Note that $\sqrt{\operatorname{tr}\dot{\mathbf{B}}^2} = \|\dot{\mathbf{B}}\|$ is a norm of $\dot{\mathbf{B}}$.

In particular cases or in order to facilitate an experimental identification of material functions, one can choose the functional basis consisting of nine nonpolynomial invariants, say C_i and $\dot{B}_i$ $(i = 1,2,3)$ and, for instance, three Euler angles, which determine a mutual position of the eigenvectors of the tensors $\mathbf{C}$ and $\dot{\mathbf{B}}$; by C_i and $\dot{B}_i$ we have denoted the eigenvalues of $\mathbf{C}$ and $\dot{\mathbf{B}}$, respectively.

By using (2.11) and (2.16) one can construct a nine-element functional basis in the following way: six nonpolynomial invariants are just C_i and $\dot{B}_i$. To this end, M_i and $\mathbf{M}$ in (2.11) – (2.13) should be replaced first by C_i and $\mathbf{C}$ and next by $\dot{B}_i$ and $\dot{\mathbf{B}}$. To determine the remaining three invariants we apply (2.16) once again. For the tensor $\mathbf{C}$ we write

$$(4.12)\qquad \mathbf{i}_j \otimes \mathbf{i}_j = \frac{1}{2C_j^2 - I_C C_j + III_C C_j^{-1}}\left[\mathbf{C}^2 - (I_C - C_j)\mathbf{C} + III_C C_j^{-1}\mathbf{I}\right] = \frac{1}{a_0^{(j)}}\left(\mathbf{C}^2 - a_1^{(j)}\mathbf{C} + a_2^{(j)}\mathbf{I}\right) \qquad \text{(no summation over } j)$$

and similarly for the tensor $\dot{\mathbf{B}}$:

$$(4.13)\qquad \mathbf{j}_j \otimes \mathbf{j}_j = \frac{1}{b_0^{(j)}}\left(\dot{\mathbf{B}}^2 - b_1^{(j)}\dot{\mathbf{B}} + b_2^{(j)}\mathbf{I}\right),$$

where

$$(4.14)\qquad b_0^{(j)} = 2\dot{B}_j^2 - I_{\dot{B}}\dot{B}_j + III_{\dot{B}}\dot{B}_j^{-1}, \qquad b_1^{(j)} = I_{\dot{B}} - \dot{B}_j, \qquad b_2^{(j)} = III_{\dot{B}}\dot{B}_j^{-1}.$$

We recall that $\dot{\mathbf{B}}$ and $\mathbf{C}$ are symmetric, second-order tensors. The space T^s of symmetric second-order tensors is equipped with the scalar product defined by $\mathbf{T}\cdot\mathbf{Z} = \operatorname{tr}\mathbf{TZ}$, for each $\mathbf{T},\mathbf{Z} \in T^s$. Consequently, the natural norm of $\mathbf{T} \in T^s$ is given by

$$(4.15)\qquad \|\mathbf{T}\|^2 = \mathbf{T}\cdot\mathbf{T} = \operatorname{tr}\mathbf{T}^2.$$

The space $(T^s, \|\cdot\|)$ can be identified with the six-dimensional Euclidean space. Therefore one can calculate the scalar product of the eigenvectors (4.12) and (4.13)

$$(4.16)\qquad (\mathbf{i}_i \otimes \mathbf{i}_i)\cdot(\mathbf{j}_j \otimes \mathbf{j}_j) = \operatorname{tr}_{(1,4)}\operatorname{tr}_{(2,3)}\mathbf{i}_i \otimes \mathbf{i}_i \otimes \mathbf{j}_j \otimes \mathbf{j}_j = (\mathbf{i}_i\cdot\mathbf{j}_j)^2 = \cos^2\alpha_{ij} \geq 0 \qquad \text{(no summation over } i \text{ and } j),$$

where α_{ij} is the angle between the eigenvectors $\mathbf{i}_i$ and $\mathbf{j}_j$. We assume that the eigenvalues of C_i are different: $C_1 \neq C_2 \neq C_3 \neq C_1$ and similarly for $\dot{\mathbf{B}}$. The remaining cases are simpler since then a smaller number of invariants is involved.

We may write $[\cos\alpha_{ij}] = [Q_{ij}]$, obviously $\mathbf{Q} \in O(3)$. Further we find

$$(4.17)\qquad (\mathbf{j}_i \otimes \mathbf{j}_i)\cdot(\mathbf{i}_j \otimes \mathbf{i}_j) = \cos^2\beta_{ij} \qquad \text{(no summation over } i \text{ and } j),$$

and $[\cos\beta_{ij}]^T = [\cos\alpha_{ij}]$. Recalling that $Q_{ik}Q_{kj} = \delta_{ij}$, where δ_{ij} stands for Kronecker's delta, we deduce that only three of the angles α_{ij} are independent. From (4.12), (4.13) and (4.16) one obtains

$$\begin{aligned}(4.18)\qquad \cos^2\alpha_{ij} = \frac{1}{a_0^{(i)}b_0^{(j)}}\Big(&\operatorname{tr}\mathbf{C}^2\dot{\mathbf{B}}^2 - b_1^{(j)}\operatorname{tr}\mathbf{C}^2\dot{\mathbf{B}} + b_2^{(j)}\operatorname{tr}\mathbf{C}^2 - a_1^{(i)}\operatorname{tr}\mathbf{C}\dot{\mathbf{B}}^2\\ &+ a_1^{(i)}b_1^{(j)}\operatorname{tr}\mathbf{C}\dot{\mathbf{B}} - a_1^{(i)}b_2^{(j)}\operatorname{tr}\mathbf{C} + a_2^{(i)}\operatorname{tr}\dot{\mathbf{B}}^2 - a_2^{(i)}b_1^{(j)}\operatorname{tr}\dot{\mathbf{B}} + 3a_2^{(i)}b_2^{(j)}\Big)\\ &\text{(no summation over } i \text{ and } j).\end{aligned}$$

All in all, we conclude that it is possible to construct the functional basis consisting of nine invariants which in turn depend on ten polynomial invariants.

In the case of perfect plasticity or ideal locking, when $\mathbf{C} = \mathbf{H}$ is a fabric tensor, more convenient in applications seems to be a representation of the tensor function (3.1) different from (4.1). Suppose that $H_1 \neq H_2 \neq H_3 \neq H_1$ and assume that $\mathbf{i}_i$ $(i = 1,2,3)$ are the principal axes of orthotropy. We set

$$(4.19)\qquad \mathbf{M}_i = \mathbf{i}_i \otimes \mathbf{i}_i \qquad \text{(no summation over } i).$$

The representation (4.1) can then be written in the form

$$\begin{aligned}(4.20)\qquad \mathbf{A} = \beta_1\mathbf{M}_1 + \beta_2\mathbf{M}_2 + \beta_3\mathbf{M}_3 + \beta_4(\mathbf{M}_1\dot{\mathbf{B}} + \dot{\mathbf{B}}\mathbf{M}_1)&\\ + \beta_5(\mathbf{M}_2\dot{\mathbf{B}} + \dot{\mathbf{B}}\mathbf{M}_2) + \beta_6(\mathbf{M}_3\dot{\mathbf{B}} + \dot{\mathbf{B}}\mathbf{M}_3) + \beta_7(\mathbf{M}_1\dot{\mathbf{B}}^2 + \dot{\mathbf{B}}^2\mathbf{M}_1)&\\ + \beta_8(\mathbf{M}_2\dot{\mathbf{B}}^2 + \dot{\mathbf{B}}^2\mathbf{M}_2) + \beta_9(\mathbf{M}_3\dot{\mathbf{B}}^2 + \dot{\mathbf{B}}^2\mathbf{M}_3),&\end{aligned}$$

where

$$(4.21)\qquad \begin{aligned}\beta_i &= \alpha_1 + H_i\alpha_2 + H_i^2\alpha_6\,,\\ \beta_{i+3} &= \alpha_4 + H_i\alpha_6 + H_i^2\alpha_7\,,\\ \beta_{i+6} &= \alpha_5 + H_i\alpha_8 + H_i^2\alpha_9\,, \qquad i = 1,2,3.\end{aligned}$$

Since

$$\begin{aligned}(4.22)\qquad \operatorname{tr}\dot{\mathbf{B}}^\alpha\mathbf{H}^\beta = H_1^\beta\operatorname{tr}\mathbf{M}_1\dot{\mathbf{B}}^\alpha + H_2^\beta\operatorname{tr}\mathbf{M}_2\dot{\mathbf{B}}^\alpha + H_3^\beta\operatorname{tr}\mathbf{M}_3\dot{\mathbf{B}}^\alpha,&\\ \alpha = 1,2, \qquad \beta = 0,1,2,&\end{aligned}$$

therefore

$$(4.23)\qquad \beta_p = \tilde{\beta}_p(\varrho, \operatorname{tr}\mathbf{M}_i\dot{\mathbf{B}}^\alpha, \operatorname{tr}\dot{\mathbf{B}}^3, H_j).$$

The representation of the orthotropic symmetric tensor function has also been considered in the papers [2–5]. In those papers the same number of generators as in (4.20) describe such a representation. There is, however, a difference between our representation and the representation proposed in [2–5]. In the present paper the scalar functions associated with the generators depend on the invariants of the structural tensor **H**. In this manner we can take into account material inhomogeneities, because **H** and ϱ depend on a position of the material point considered. Allowance for the eigenvalues of **H** in the functions (4.23) delivers a possibility of determination of the material orthotropy. This is not the case with the structural tensors used in [2–5]. For instance, as a "measure" of the material orthotropy one can take the quantities H_2/H_1 and H_3/H_1 provided that $H_1 > H_2 > H_3$.

Applying the approach proposed by BOEHLER [2, 3], it can be shown that an alternative nonpolynomial representation of (4.20) contains only seven generators:

$$\mathbf{A} = \delta_1\mathbf{M}_1 + \delta_2\mathbf{M}_2 + \delta_3\mathbf{M}_3 + \delta_4(\mathbf{M}_1\dot{\mathbf{B}} + \dot{\mathbf{B}}\mathbf{M}_1) + \delta_5(\mathbf{M}_2\dot{\mathbf{B}} + \dot{\mathbf{B}}\mathbf{M}_2) + \delta_6(\mathbf{M}_3\dot{\mathbf{B}} + \dot{\mathbf{B}}\mathbf{M}_3) + \delta_7\dot{\mathbf{B}}^2, \tag{4.24}$$

where

$$\delta_p = \tilde{\delta}(\varrho, \operatorname{tr}\mathbf{M}_i\dot{\mathbf{B}}^\alpha, \operatorname{tr}\dot{\mathbf{B}}^3, H_j), \qquad p = 1, \ldots, 7. \tag{4.25}$$

Another form includes two arbitrary tensors from the set $\{\mathbf{M}_i\}$ $(i = 1, 2, 3)$, say $\mathbf{M}_1$ and $\mathbf{M}_2$,

$$\mathbf{A} = \kappa_1\mathbf{I} + \kappa_2\mathbf{M}_1 + \kappa_3\mathbf{M}_2 + \kappa_4\dot{\mathbf{B}} + \kappa_5(\mathbf{M}_1\dot{\mathbf{B}} + \dot{\mathbf{B}}\mathbf{M}_1) + \kappa_6(\mathbf{M}_2\dot{\mathbf{B}} + \dot{\mathbf{B}}\mathbf{M}_2) + \kappa_7\dot{\mathbf{B}}^2. \tag{4.26}$$

Here

$$\kappa_p = \tilde{\kappa}(\varrho, \operatorname{tr}\dot{\mathbf{B}}^i, \operatorname{tr}\mathbf{M}_\alpha\dot{\mathbf{B}}^\beta, H_j), \qquad \alpha, \beta = 1, 2, \quad p = 1, \ldots, 7. \tag{4.27}$$

Obviously, the representations (4.24) and (4.26) are equivalent. Consequently, for $\mathbf{C} = \mathbf{H}$ the nonpolynomial representation of the tensor function (4.1) has the form (4.6), with γ_p being functions of J_s and H_i where $p, s = 1, \ldots, 7$ and $i = 1, 2, 3$.

In the case of transverse isotropy, for instance when $H_1 = H_2 = H$ and $\mathbf{i}_3$ is a privileged direction, one obtains the following additional relations between the invariants involved in (4.25):

$$\begin{aligned} \operatorname{tr}\mathbf{H}^i &= 2H^i + H_3^i, \qquad i = 1, 2, 3, \\ \operatorname{tr}\mathbf{H}^\alpha\dot{\mathbf{B}}^\beta &= H^\alpha \operatorname{tr}\dot{\mathbf{B}}^\beta + (H_3^\alpha - H^\alpha)\operatorname{tr}\mathbf{M}_3\dot{\mathbf{B}}^\beta, \qquad \alpha, \beta = 1, 2, \end{aligned} \tag{4.28}$$

as well as between the generators appearing in (4.24):

$$\begin{aligned} \mathbf{H}^\alpha &= H^\alpha\mathbf{I} + (H_3^\alpha - H^\alpha)\mathbf{M}_3, \\ \mathbf{H}^\alpha\dot{\mathbf{B}}^\beta + \dot{\mathbf{B}}^\beta\mathbf{H}^\alpha &= 2H^\alpha\dot{\mathbf{B}}^\beta + (H_3^\alpha - H^\alpha)(\dot{\mathbf{B}}^\beta\mathbf{M}_3 + \dot{\mathbf{B}}^\beta\mathbf{M}_3). \end{aligned} \tag{4.29}$$

Thus we arrive at the transversely isotropic representation of the tensor function (3.1), cf. also BOEHLER [2, 3],

$$\mathbf{A} = \varrho_1\mathbf{I} + \varrho_2\mathbf{M}_3 + \varrho_3\dot{\mathbf{B}} + \varrho_4(\mathbf{M}_3\dot{\mathbf{B}} + \dot{\mathbf{B}}\mathbf{M}_3) + \varrho_5\dot{\mathbf{B}}^2 + \varrho_6(\mathbf{M}_3\dot{\mathbf{B}}^2 + \dot{\mathbf{B}}^2\mathbf{M}_3), \tag{4.30}$$

where

$$\varrho_p = \tilde{\varrho}(\varrho, \operatorname{tr}\dot{\mathbf{B}}^i, \operatorname{tr}\mathbf{M}_3\dot{\mathbf{B}}^\alpha, H, H_3), \qquad \alpha = 1,2, \qquad i = 1,2,3, \qquad p = 1,\ldots,6. \tag{4.31}$$

The simplest is the case of isotropy: $H_1 = H_2 = H_3 = H$. Then we easily obtain

$$\mathbf{A} = \gamma_1\mathbf{I} + \gamma_2\dot{\mathbf{B}} + \gamma_3\dot{\mathbf{B}}^2, \tag{4.32}$$

where

$$\gamma_i = \tilde{\gamma}_i(\varrho, \operatorname{tr}\dot{\mathbf{B}}^i, H). \tag{4.33}$$

REMARK 4. Assuming *a priori* that the tensor function (3.1) involves only symmetric two-dimensional tensors, (4.1) simplifies to

$$\mathbf{A} = \mathbf{F}(\varrho, \dot{\mathbf{B}}, \mathbf{C}) = \overline{\alpha}_1\mathbf{I} + \overline{\alpha}_2\mathbf{C} + \overline{\alpha}_3\dot{\mathbf{B}}, \tag{4.34}$$

where $\mathbf{I}$ denotes the two-dimensional unit tensor; moreover

$$\overline{a}_i = \tilde{a}_i(\operatorname{tr}\dot{\mathbf{B}}, \operatorname{tr}\dot{\mathbf{B}}^2, \operatorname{tr}\dot{\mathbf{B}}\mathbf{C}, \operatorname{tr}\mathbf{C}, \operatorname{tr}\mathbf{C}^2). \tag{4.35}$$

The representation (4.34) with (4.35) is formally the same in the case of both polynomial and nonpolynomial representation.

For $\mathbf{C} = \mathbf{H}$ and $H_1 = H_2 = H$, (4.34) reduces to

$$\mathbf{A} = \overline{\beta}_1\mathbf{I} + \overline{\beta}_2\dot{\mathbf{B}}, \tag{4.36}$$

where

$$\overline{\beta}_\alpha = \tilde{\beta}_\alpha(\operatorname{tr}\dot{\mathbf{B}}, \operatorname{tr}\dot{\mathbf{B}}^2, H), \qquad \alpha = 1,2. \quad \triangleleft \tag{4.37}$$

5. Some specific cases

The aim of this section is to examine the tensor function (4.34) by imposing suitable homogeneity requirements. Next, particular cases of plastic flow laws and locking rules as well as yield and locking conditions will be investigated.

5.1. Two-dimensional case

Deleting the bar over functions in (4.34) we write

$$\mathbf{A} = \alpha_1 \mathbf{I} + \alpha_2 \mathbf{C} + \alpha_3 \dot{\mathbf{B}} = \widetilde{\mathbf{F}}(\varrho, \dot{\mathbf{B}}, \mathbf{C}), \tag{5.1}$$

where

$$\alpha_i = \tilde{\alpha}_i(\varrho, \operatorname{tr}\dot{\mathbf{B}}, \|\dot{\mathbf{B}}\|, \operatorname{tr}\dot{\mathbf{B}}\mathbf{C}, \operatorname{tr}\mathbf{C}, \|\mathbf{C}\|), \quad i = 1, 2, 3. \tag{5.2}$$

Here, for the sake of convenience, a nonpolynomial basis has been chosen.

Since

$$\frac{\partial \widetilde{\mathbf{F}}}{\partial \dot{\mathbf{B}}} = \mathbf{I} \otimes \frac{\partial \tilde{\alpha}_1}{\partial \dot{\mathbf{B}}} + \mathbf{C} \otimes \frac{\partial \tilde{\alpha}_2}{\partial \dot{\mathbf{B}}} + \tilde{\alpha}_3 \mathbf{1} + \dot{\mathbf{B}} \otimes \frac{\partial \tilde{\alpha}_3}{\partial \dot{\mathbf{B}}}, \tag{5.3}$$

$$\frac{\partial \tilde{\alpha}_i}{\partial \dot{\mathbf{B}}} = \frac{\partial \tilde{\alpha}_i}{\partial \operatorname{tr}\dot{\mathbf{B}}} + \frac{\partial \tilde{\alpha}_i}{\partial \|\dot{\mathbf{B}}\|} \frac{\dot{\mathbf{B}}}{\|\dot{\mathbf{B}}\|} + \frac{\partial \tilde{\alpha}_3}{\partial \operatorname{tr}\mathbf{C}\dot{\mathbf{B}}} \mathbf{C}, \tag{5.4}$$

therefore (5.1) is a potential law provided that

$$\frac{1}{\|\dot{\mathbf{B}}\|} \frac{\partial \tilde{\alpha}_1}{\partial \|\dot{\mathbf{B}}\|} = \frac{\partial \tilde{\alpha}_3}{\partial \operatorname{tr}\dot{\mathbf{B}}}, \qquad \frac{\partial \tilde{\alpha}_1}{\partial \operatorname{tr}\mathbf{C}\dot{\mathbf{B}}} = \frac{\partial \tilde{\alpha}_2}{\partial \operatorname{tr}\dot{\mathbf{B}}}, \qquad \frac{1}{\|\dot{\mathbf{B}}\|} \frac{\partial \tilde{\alpha}_2}{\partial \|\dot{\mathbf{B}}\|} = \frac{\partial \tilde{\alpha}_3}{\partial \operatorname{tr}\mathbf{C}\dot{\mathbf{B}}}. \tag{5.5}$$

In (5.3), $\mathbf{1}$ denotes the 2D unit tensor of the fourth order.

The homogeneity condition of degree zero now gives

$$\frac{\partial \widetilde{\mathbf{F}}}{\partial \dot{\mathbf{B}}} \cdot \dot{\mathbf{B}} = \left(\frac{\partial \tilde{\alpha}_1}{\partial \dot{\mathbf{B}}} \cdot \dot{\mathbf{B}} \right) \mathbf{I} + \left(\frac{\partial \tilde{\alpha}_2}{\partial \dot{\mathbf{B}}} \cdot \dot{\mathbf{B}} \right) \mathbf{C} + \left(\frac{\partial \tilde{\alpha}_3}{\partial \dot{\mathbf{B}}} \cdot \dot{\mathbf{B}} + \tilde{\alpha}_3 \right) \dot{\mathbf{B}} = \mathbf{0}. \tag{5.6}$$

Equation (5.6) implies that $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are homogeneous functions of degree zero, while $\tilde{\alpha}_3$ is a homogeneous function of degree (-1) with respect to $\dot{\mathbf{B}}$.

Substituting (5.4) into (5.6) one obtains

$$\begin{aligned} \frac{\partial \tilde{\alpha}_\alpha}{\partial \dot{\mathbf{B}}} \cdot \dot{\mathbf{B}} &= \frac{\partial \tilde{\alpha}_\alpha}{\partial K_i} K_i = 0, \qquad \alpha = 1, 2, \quad i = 1, 2, 3, \\ \frac{\partial \tilde{\alpha}_3}{\partial \dot{\mathbf{B}}} \cdot \dot{\mathbf{B}} + \tilde{\alpha}_3 &= \frac{\partial \tilde{\alpha}_3}{\partial K_i} K_i + \tilde{\alpha}_3 = 0, \end{aligned} \tag{5.7}$$

where

$$\{K_i\} = \left\{ \operatorname{tr}\dot{\mathbf{B}}, \operatorname{tr}\dot{\mathbf{B}}\mathbf{C}, \|\dot{\mathbf{B}}\| \right\}. \tag{5.8}$$

Introducing new variables

$$x = \ln \|\dot{\mathbf{B}}\|, \qquad p_1 = \frac{\operatorname{tr}\dot{\mathbf{B}}}{\|\dot{\mathbf{B}}\|}, \qquad p_2 = \frac{\operatorname{tr}\mathbf{C}\dot{\mathbf{B}}}{\|\dot{\mathbf{B}}\|}, \tag{5.9}$$

from (5.7) one gets the equations

$$\frac{\partial \tilde{\alpha}_\alpha}{\partial x} = 0, \qquad \frac{\partial \tilde{\alpha}_3}{\partial x} + \alpha_3 = 0. \tag{5.10}$$

Their solutions are

$$\tilde{\alpha}_\alpha = a_\alpha(\varrho, p_1, p_2, \operatorname{tr}\mathbf{C}, \|\mathbf{C}\|), \qquad \tilde{\alpha}_3 = \frac{1}{\|\dot{\mathbf{B}}\|} a_3(\varrho, p_1, p_2, \operatorname{tr}\mathbf{C}, \|\mathbf{C}\|). \tag{5.11}$$

Taking into account (5.11) in (5.1) one obtains

$$\mathbf{A} = a_1 \mathbf{I} + a_2 \mathbf{C} + \frac{a_3}{\|\dot{\mathbf{B}}\|} \dot{\mathbf{B}}. \tag{5.12}$$

The last constitutive law is of a potential type provided that

$$\frac{\partial a_1}{\partial p_\alpha} p_\alpha = -\frac{\partial a_3}{\partial p_1}, \qquad \frac{\partial a_2}{\partial p_\alpha} p_\alpha = -\frac{\partial a_3}{\partial p_2}, \qquad \frac{\partial a_1}{\partial p_2} = \frac{\partial a_2}{\partial p_1}, \quad \alpha = 1, 2. \tag{5.13}$$

These relations are obtained by substitution of (5.11) into (5.5).

From (5.12) one gets

$$\begin{aligned} \operatorname{tr}\mathbf{A} &= 2a_1 + a_2 \operatorname{tr}\mathbf{C} + a_3 p_1, \\ \operatorname{tr}\mathbf{AC} &= a_1 \operatorname{tr}\mathbf{C} + a_2 \|\mathbf{C}\|^2 + a_3 p_2, \\ \|\mathbf{A}\|^2 &= 2a_1^2 + a_3^2 + a_2 \left(2a_1 \operatorname{tr}\mathbf{C} + a_2 \|\mathbf{C}\|^2\right) + 2a_3(a_1 p_1 + a_2 p_2). \end{aligned} \tag{5.14}$$

In the case when $\mathbf{A} = \boldsymbol{\sigma}$, the set (5.14) is called a parametric yield condition; similarly if $\mathbf{A} = \boldsymbol{\varepsilon}$, a parametric locking condition is obtained.

If the parameters p_α can be eliminated from (5.14), then the invariants of $\mathbf{A}$ are interrelated by a scalar relation:

$$f(\varrho, \operatorname{tr}\mathbf{A}, \|\mathbf{A}\|, \operatorname{tr}\mathbf{AC}, \operatorname{tr}\mathbf{C}, \|\mathbf{C}\|) = 0. \tag{5.15}$$

From (5.12) one can easily derive the semi-inverse relation

$$\frac{\dot{\mathbf{B}}}{\|\dot{\mathbf{B}}\|} = \frac{1}{a_3}(\mathbf{A} - a_1 \mathbf{I} - a_2 \mathbf{C}), \tag{5.16}$$

which represents either a flow rule ($\mathbf{A} = \boldsymbol{\sigma}$), or a locking law ($\mathbf{A} = \boldsymbol{\varepsilon}$).

To derive specific forms of the constitutive relationships, one has to postulate concrete forms of the functions a_i or b_i ($i = 1, 2, 3$). For instance, simple polynomial forms were used by the second author in the case of perfectly plastic isotropic materials [34].

As we already know, the spherical tensor $\mathbf{C} = \mathbf{H}$ characterises an isotropic material. The functions appearing in (4.36) are then given by

$$\beta_1 = \tilde{\beta}_1\left(\frac{\mathrm{tr}\dot{\mathbf{B}}}{\|\dot{\mathbf{B}}\|},\ H\right), \qquad \beta_2 = \frac{1}{\|\dot{\mathbf{B}}\|}\tilde{\beta}_2\left(\frac{\mathrm{tr}\dot{\mathbf{B}}}{\|\dot{\mathbf{B}}\|},\ H\right). \tag{5.17}$$

We pass now to providing simple examples.

EXAMPLE 1

Consider the two-dimensional case when the perfectly locking constraint affects only the volumetric part of the strain tensor of an isotropic material, cf. [11, 33]. We set

$$\mathbf{A} = \frac{1}{2}(\mathrm{tr}\boldsymbol{\varepsilon})\mathbf{I}, \qquad \dot{\mathbf{B}} = \frac{1}{2}(\mathrm{tr}\dot{\boldsymbol{\sigma}})\mathbf{I}.$$

Now we have

$$\|\dot{\mathbf{B}}\| = \frac{1}{\sqrt{2}}|\mathrm{tr}\dot{\boldsymbol{\sigma}}|. \tag{5.18}$$

Taking into account (5.17), the constitutive relationship (4.36) reduces to

$$(\mathrm{tr}\boldsymbol{\varepsilon})\mathbf{I} = \sqrt{2}c\frac{\mathrm{tr}\dot{\boldsymbol{\sigma}}}{|\mathrm{tr}\dot{\boldsymbol{\sigma}}\|}\mathbf{I}, \tag{5.19}$$

where c is a material coefficient. Hence the locking condition is given by

$$|\mathrm{tr}\boldsymbol{\varepsilon}| = \sqrt{2}c. \tag{5.20}$$

EXAMPLE 2 (Perfect plasticity, isotropic material)

For the case of plane stresses, we set $\mathbf{A} = \boldsymbol{\sigma}$, $\dot{\mathbf{B}} = \dot{\boldsymbol{\varepsilon}}$ while the functions appearing in (5.17) are assumed in the form:

$$\tilde{\beta}_1 = \frac{\sqrt{2}kp}{\sqrt{1+p^2}}, \qquad \tilde{\beta}_2 = \frac{\sqrt{2}k}{\sqrt{1+p^2}}, \qquad p = \frac{\mathrm{tr}\dot{\boldsymbol{\varepsilon}}}{\|\dot{\boldsymbol{\varepsilon}}\|}, \quad k = \mathrm{const}. \tag{5.21}$$

Then (4.26) gives

$$\boldsymbol{\sigma} = \frac{\sqrt{2}k}{\sqrt{1+p^2}}\left(p\mathbf{I} + \frac{\dot{\boldsymbol{\varepsilon}}}{\|\dot{\boldsymbol{\varepsilon}}\|}\right), \tag{5.22}$$

and the yield condition has the following form

$$\frac{\|\mathrm{s}\|^2}{2k^2} + \frac{\mathrm{tr}^2\boldsymbol{\sigma}}{12k^2} = 1, \tag{5.23}$$

where $\mathbf{s} = \boldsymbol{\sigma} - \frac{1}{2}(\mathrm{tr}\boldsymbol{\sigma})\mathbf{I}$; k is the yield limit in shear. Obviously, (5.23) represents the Huber–Mises yield condition. The flow rule (5.22) is associated with this condition. The plastic dissipation density is given by

$$D(\dot{\boldsymbol{\varepsilon}}) = \mathrm{tr}\boldsymbol{\sigma}\dot{\boldsymbol{\varepsilon}} = \sqrt{2}k\|\dot{\boldsymbol{\varepsilon}}\|\sqrt{1+p^2}. \tag{5.24}$$

EXAMPLE 3

We shall now consider a perfectly plastic, incompressible and orthotropic material in the case of plane deformations, $\mathbf{C} = \mathbf{H}$. In this case one obviously has $\mathrm{tr}\dot{\boldsymbol{\varepsilon}} = 0$ and (5.12) gives

$$\mathrm{tr}\boldsymbol{\sigma} = 2c_1\,, \qquad \mathbf{s} = c_2\mathbf{H}_d + \frac{c_3}{\|\dot{\mathbf{d}}\|}\dot{\mathbf{d}}, \tag{5.25}$$

where

$$\begin{gathered} \mathbf{H}_d = \mathbf{H} - \frac{1}{2}(\mathrm{tr}\mathbf{H})\mathbf{I}, \qquad \dot{\mathbf{d}} = \dot{\boldsymbol{\varepsilon}} - \frac{1}{2}(\mathrm{tr}\dot{\boldsymbol{\varepsilon}})\mathbf{I} = \dot{\boldsymbol{\varepsilon}}, \\ c_\alpha = \tilde{c}_\alpha\left(\varrho, \frac{\mathrm{tr}\dot{\mathbf{d}}\mathbf{H}_d}{\|\dot{\mathbf{d}}\|}, \mathrm{tr}\mathbf{H}, \|\mathbf{H}_d\|\right). \end{gathered} \tag{5.26}$$

Note that for $c_1 = 0$ the yield condition and flow rule do not depend upon $\mathrm{tr}\boldsymbol{\sigma}$. The parametric yield condition resulting from (5.25) has then the form

$$\mathrm{tr}\mathbf{s}\mathbf{H}_d = c_2\|\mathbf{H}_d\|^2 + c_3 q, \qquad \|\mathbf{s}\|^2 = c_2^2\|\mathbf{H}_d\|^2 + 2c_2c_3q + c_3^2\,, \tag{5.27}$$

where

$$q = \frac{\mathrm{tr}\dot{\mathbf{d}}\mathbf{H}_d}{\|\dot{\mathbf{d}}\|}. \tag{5.28}$$

For $q = 0$ the yield condition is given by

$$(1 - \cos^2\alpha)\|\mathbf{s}\|^2 = c_3^2, \tag{5.29}$$

where

$$\cos\alpha = \frac{\mathrm{tr}\mathbf{s}\mathbf{H}_d}{\|\mathbf{s}\|\|\mathbf{H}_d\|}, \qquad \text{if} \quad \mathbf{s} \neq \mathbf{0}. \tag{5.30}$$

Obviously, for an isotropic material one has the well known flow rule

$$\mathbf{s} = \frac{\sqrt{2}k}{\|\dot{\mathbf{d}}\|}\dot{\mathbf{d}}, \qquad k = \mathrm{const}, \tag{5.31}$$

and the yield condition

$$\|\mathbf{s}\|^2 = 2k^2. \tag{5.32}$$

EXAMPLE 4

Consider now a plastic incompressible material, still in the case of plane deformation. Let now

$$\mathbf{A}_d = \mathbf{s}, \quad \dot{\mathbf{B}}_d = \dot{\mathbf{d}}, \quad \mathbf{C}_d = \mathbf{d}, \quad \mathbf{d} = \boldsymbol{\varepsilon} - \frac{1}{2}(\mathrm{tr}\boldsymbol{\varepsilon})\mathbf{I}, \quad \mathrm{tr}\dot{\boldsymbol{\varepsilon}} = 0.$$

Equation $(5.25)_2$ is written in the form

$$\overset{*}{\mathbf{s}} = \mathbf{s} - c_2\mathbf{d} = \frac{c_3}{\|\dot{\mathbf{d}}\|}\dot{\mathbf{d}}. \tag{5.33}$$

Here we assume that

$$c_\alpha = \tilde{c}_\alpha(\|\mathbf{d}\|), \qquad \alpha = 2, 3. \tag{5.34}$$

The yield condition following from (5.33) is given by

$$\mathrm{tr}\left(\overset{*}{\mathbf{s}}\right)^2 = c_3^2. \tag{5.35}$$

It takes into account the simple kinematic hardening and the isotropic hardening. Under the influence of plastic deformations, initially isotropic material becomes an orthotropic one. We note that specific cases studied in [30, 31] fall within our general framework.

So far only simple cases of constitutive relationships have been discussed, provided that deformations are plane. Here we shall not study more complex and essentially new constitutive relationships, which naturally result from our rather general approach. Note that there still remains the problem of identification of material functions, particularly for locking materials.

6. Three-dimensional case

Proceeding similarly to the two-dimensional case, one obtains the following constitutive relationships satisfying (3.2):

$$\begin{aligned}\mathbf{A} = a_1\mathbf{I} + a_2\mathbf{C} + a_3\mathbf{C}^2 + \frac{a_4}{\|\dot{\mathbf{B}}\|}\dot{\mathbf{B}} + \frac{a_5}{\|\dot{\mathbf{B}}\|^2}\dot{\mathbf{B}}^2 + \frac{a_6}{\|\dot{\mathbf{B}}\|}\left(\mathbf{C}\dot{\mathbf{B}} + \dot{\mathbf{B}}\mathbf{C}\right)\\ + \frac{a_7}{\|\dot{\mathbf{B}}\|}\left(\mathbf{C}^2\dot{\mathbf{B}} + \dot{\mathbf{B}}\mathbf{C}^2\right) + \frac{a_8}{\|\dot{\mathbf{B}}\|^2}\left(\mathbf{C}\dot{\mathbf{B}}^2 + \dot{\mathbf{B}}^2\mathbf{C}\right) + \frac{a_9}{\|\dot{\mathbf{B}}\|^2}\left(\mathbf{C}^2\dot{\mathbf{B}}^2 + \dot{\mathbf{B}}^2\mathbf{C}^2\right),\end{aligned} \tag{6.1}$$

where

$$a_p = \widetilde{a}_p(\varrho, p_m, \operatorname{tr}\mathbf{C}^i), \qquad p = 1, \dots, 9, \quad m = 1, \dots, 6, \quad i = 1, 2, 3. \tag{6.2}$$

Here

$$\begin{gathered} p_m = \frac{r_m}{\|\dot{\mathbf{B}}\|}, \qquad r_1 = \operatorname{tr}\dot{\mathbf{B}}, \qquad r_2 = \operatorname{tr}\dot{\mathbf{B}}\mathbf{C}, \qquad r_3 = \operatorname{tr}\dot{\mathbf{B}}\mathbf{C}^2, \\ r_4 = \sqrt{\operatorname{tr}\dot{\mathbf{B}}^2\mathbf{C}}, \qquad r_5 = \sqrt{\operatorname{tr}\dot{\mathbf{B}}^2\mathbf{C}^2}, \qquad r_6 = \sqrt[3]{\operatorname{tr}\dot{\mathbf{B}}^3}. \end{gathered} \tag{6.3}$$

Note that the functions $\widetilde{\alpha}_p$ cannot be polynomials constructed from the elements of the integrity basis (4.3). Moreover, $\widetilde{\alpha}_1$, $\widetilde{\alpha}_4$ and $\widetilde{\alpha}_5$ are homogeneous functions of degree zero; $\widetilde{\alpha}_2$, $\widetilde{\alpha}_6$, $\widetilde{\alpha}_7$ are homogeneous of degree (-1) while $\widetilde{\alpha}_3$, $\widetilde{\alpha}_8$, $\widetilde{\alpha}_9$ are homogeneous of degree (-2). Obviously, the homogeneity holds with respect to $\dot{\mathbf{B}}$.

Applying the generalized Cayley–Hamilton theorem due to RIVLIN [26], from (6.1) one can construct 7 invariants K_s $(s = 1, \dots, 7)$ as follows,

$$\{K_s\} = \{\operatorname{tr}\mathbf{A}^i, \operatorname{tr}\mathbf{A}^\alpha\mathbf{C}^\beta\}, \qquad i = 1, 2, 3, \quad \alpha, \beta = 1, 2. \tag{6.4}$$

Taking into account (6.2) we write

$$K_s = g_s(\varrho, p_m, \operatorname{tr}\mathbf{C}^i). \tag{6.5}$$

For $\mathbf{A} = \boldsymbol{\sigma}$ the last relation represents the yield condition in a parametric form while for $\mathbf{A} = \boldsymbol{\varepsilon}$ a parametric form of the locking condition is obtained.

Suppose that the parameters p_m $(m = 1, \dots, 6)$ can be eliminated from (6.5). Then

$$f(\varrho, K_s, \operatorname{tr}\mathbf{C}^i) = 0, \tag{6.6}$$

is the yield condition when $\mathbf{A} = \boldsymbol{\sigma}$ or locking condition for $\mathbf{A} = \boldsymbol{\varepsilon}$.

To derive the semi-inverse form of (6.1) one has to find the following generators, cf. BOEHLER [2]:

$$\mathbf{A}^i, \ \mathbf{A}^\alpha\mathbf{C}^\beta + \mathbf{C}^\beta\mathbf{A}^\alpha, \qquad i = 1, 2, 3, \quad \alpha, \beta = 1, 2. \tag{6.7}$$

After some algebraic manipulations we finally obtain

$$\begin{aligned} \frac{\dot{\mathbf{B}}}{\sqrt{\operatorname{tr}\dot{\mathbf{B}}^2}} = \tilde{\lambda} \Big[& b_1\mathbf{I} + b_2\mathbf{C} + b_3\mathbf{C}^2 + b_4\mathbf{A} + b_5\mathbf{A}^2 + b_6(\mathbf{CA} + \mathbf{AC}) \\ & + b_7(\mathbf{C}^2 A + \mathbf{AC}^2) + b_8(\mathbf{CA}^2 + \mathbf{A}^2\mathbf{C}) + b_9(\mathbf{C}^2\mathbf{A}^2 + \mathbf{A}^2\mathbf{C}^2) \Big], \qquad \tilde{\lambda} \geq 0, \end{aligned} \tag{6.8}$$

where $\tilde{\lambda}$ and b_p $(p = 1, \dots, 9)$ are functions of ϱ, p_m $(m = 1, \dots, 6)$ and $\operatorname{tr}\mathbf{C}^i$.

The general constitutive relationships just derived provide a rational basis for more specific equations, which model orthotropic, perfectly plastic or perfectly locking materials for $\mathbf{C} = \mathbf{H}$. For instance, associated laws follow by assuming (6.6) as a yield or locking condition, which is equivalent to the condition, see (6.11) below,

$$\frac{\partial \tilde{b}_r}{\partial K_s} = \frac{\partial \tilde{b}_s}{\partial K_r}, \qquad s, r = 1, \ldots, 7. \tag{6.9}$$

If the function f is sufficiently regular, then (6.8) takes the form

$$\dot{\mathbf{B}} = \lambda\Big[\tilde{b}_1\mathbf{I} + 2\tilde{b}_2\mathbf{A} + 3\tilde{b}_3\,\mathrm{tr}\,\mathbf{A}^2 + \tilde{b}_4\,\mathrm{tr}\,\mathbf{H} + \tilde{b}_5\,\mathrm{tr}\,\mathbf{H}^2 + \tilde{b}_6(\mathbf{HA} + \mathbf{AH}) + \tilde{b}_7(\mathbf{H}^2\mathbf{A} + \mathbf{AH}^2)\Big], \qquad \lambda \geq 0, \tag{6.10}$$

where

$$\tilde{b}_s = \frac{\partial f}{\partial K_s}, \qquad s = 1, \ldots, 7; \tag{6.11}$$

and $\{K_s\} = \{\mathrm{tr}\,\mathbf{A}, \mathrm{tr}\,\mathbf{A}^2, \mathrm{tr}\,\mathbf{A}^3, \mathrm{tr}\,\mathbf{HA}, \mathrm{tr}\,\mathbf{H}^2\mathbf{A}, \mathrm{tr}\,\mathbf{HA}^2, \mathrm{tr}\,\mathbf{H}^2\mathbf{A}^2\}$. Note that the plastic or locking multiplier λ can be calculated from (3.3) as follows:

$$\lambda = \frac{D}{D'}, \tag{6.12}$$

where

$$D' = \tilde{b}_1\,\mathrm{tr}\,\mathbf{A} + 2\tilde{b}_2\,\mathrm{tr}\,\mathbf{A}^2 + 3\tilde{b}_3\,\mathrm{tr}\,\mathbf{A}^3 + \tilde{b}_4\,\mathrm{tr}\,\mathbf{HA} + \tilde{b}_5\,\mathrm{tr}\,\mathbf{H}^2\mathbf{A} + 2\tilde{b}_6\,\mathrm{tr}\,\mathbf{HA}^2 + 2\tilde{b}_7\,\mathrm{tr}\,\mathbf{H}^2\mathbf{A}^2.$$

The last form of λ is important for numerical computations.

REMARK 5. Suppose that (6.8) has been derived from a potential $g(\varrho, K_s, \mathrm{tr}\,\mathbf{C}^i)$ and $\mathbf{C} = \mathbf{H}$. One readily verifies that in such case b_8 and b_9 vanish while the remaining material functions satisfy (6.9) and (6.11), with f being replaced by g.

REMARK 6. Following BOEHLER [4] and BOEHLER and SAWCZUK [6, 7], a simpler version of (3.1) is obtained when

$$\mathbf{A}^* = \mathbf{F}^*(\varrho, \dot{\mathbf{B}}), \tag{6.13}$$

where

$$\mathbf{A}^* = \mathbb{C}(\mathbf{H}) \cdot \mathbf{A}. \tag{6.14}$$

Here $\mathbb{C}$ is a fourth order tensor function of $\mathbf{H}$, satisfying $\mathbb{C}_{ijkl} = \mathbb{C}_{jikl} = \mathbb{C}_{klij}$. The representation of such tensor function was considered by the second author in [35]. The tensor function appearing on the r.h.s. of (6.13) and satisfying the homogeneity condition

$$\frac{\partial \mathbf{F}^*}{\partial \dot{\mathbf{B}}} \cdot \dot{\mathbf{B}} = \mathbf{0} \qquad \text{if} \qquad \frac{\partial \mathbf{F}^*}{\partial \dot{\mathbf{B}}} \neq \tilde{\mathbf{0}}, \tag{6.15}$$

was discussed in detail in our previous papers [18, 19]. To use the results presented in those two papers, $\mathbf{A}$ has to be replaced with $\mathbf{A}^*$.

EXAMPLE 5. Of practical interest is the following specific case of (6.6) for $\mathbf{C} = \mathbf{H}$, independent of $\operatorname{tr}\mathbf{A}^3$:

$$\begin{aligned} f(\varrho, K_s \operatorname{tr}\mathbf{H}^i) = {} & c_1 \operatorname{tr}\mathbf{A} + c_2 \operatorname{tr}\mathbf{AH} + c_3 \operatorname{tr}\mathbf{AH}^2 + d_1 \operatorname{tr}^2\mathbf{A} + d_2 \operatorname{tr}\mathbf{A}^2 \\ & + d_3 \operatorname{tr}^2\mathbf{AH} + d_4 \operatorname{tr}\mathbf{A} \operatorname{tr}\mathbf{AH} + d_5 \operatorname{tr}^2\mathbf{AH}^2 + d_6 \operatorname{tr}\mathbf{A} \operatorname{tr}\mathbf{AH}^2 \\ & + d_7 \operatorname{tr}\mathbf{AH} \operatorname{tr}\mathbf{AH}^2 + d_8 \operatorname{tr}\mathbf{A}^2\mathbf{H} + d_9 \operatorname{tr}\mathbf{A}^2\mathbf{H}^2 - 1, \end{aligned} \tag{6.16}$$

where c_i $(i = 1, 2, 3)$ and d_p $(p = 1, \ldots, 9)$ are scalar functions of ϱ and $\operatorname{tr}\mathbf{H}^i$. Note that for a given material with prescribed ϱ, one has $c_i = \text{const}$, $d_p = \text{const}$; obviously c_i and d_p may also depend on the position of a material point.

The condition (6.16) represents an orthotropic yield condition for $\mathbf{A} = \boldsymbol{\sigma}$; similarly, if $\mathbf{A} = \boldsymbol{\varepsilon}$ then it describes an orthotropic locking condition. The yield condition of the form (6.16) is an invariant form of the condition proposed by TSAI and WU [37]. For more details including the description of experimental tests for the determination of the constants c_i and d_p we refer the reader to the paper by COWIN [9].

The anisotropic yield conditions due to HOFFMAN [16] and Mises-Hill [15] are particular cases of (6.16), cf. also [35].

The functions $\tilde{b}_r$ appearing in the associated law (6.10) are now given by

$$\begin{aligned} \tilde{b}_1 &= \frac{\partial f}{\partial \operatorname{tr}\mathbf{A}} = c_1 + 2d_1 \operatorname{tr}\mathbf{A} + d_4 \operatorname{tr}\mathbf{AH} + d_6 \operatorname{tr}\mathbf{AH}^2, \\ \tilde{b}_2 &= \frac{\partial f}{\partial \operatorname{tr}\mathbf{A}^2} = d_2\,, \qquad \tilde{b}_3 = \frac{\partial f}{\partial \operatorname{tr}\mathbf{A}^3} = 0, \\ \tilde{b}_4 &= \frac{\partial f}{\partial \operatorname{tr}\mathbf{AH}} = c_2 + 2d_3 \operatorname{tr}\mathbf{AH} + d_4 \operatorname{tr}\mathbf{A} + d_7 \operatorname{tr}\mathbf{AH}^2, \\ \tilde{b}_5 &= \frac{\partial f}{\partial \operatorname{tr}\mathbf{AH}^2} = c_3 + 2d_5 \operatorname{tr}\mathbf{AH}^2 + d_6 \operatorname{tr}\mathbf{A} + d_7 \operatorname{tr}\mathbf{AH}, \\ \tilde{b}_6 &= \frac{\partial f}{\partial \operatorname{tr}\mathbf{A}^2\mathbf{H}} = d_8\,, \qquad \tilde{b}_7 = \frac{\partial f}{\partial \operatorname{tr}\mathbf{A}^2\mathbf{H}^2} = d_9\,, \end{aligned} \tag{6.17}$$

where f is obviously specified by (6.16). It can easily be verified that (6.9) is satisfied.

Let us set $d_p = 0$ in (6.16). Then the resulting locking condition, i.e. for $\mathbf{A} = \boldsymbol{\varepsilon}$, represents a generalization of the isotropic condition (5.25) to orthotropic materials.

7. Final remarks

Real materials may exhibit elastic, plastic and locking behaviour. In the case of small deformations the tensor of elastic deformations can easily be included into our scheme. In a separate paper we shall be concerned with modelling the elastic-plastic-locking behaviour. It seems that locking behaviour does not affect all components of the strain tensor, i.e. the locking condition should probably be imposed on particular modes of deformations. Unfortunately, in the relevant literature one cannot find reliable experimental data. Our considerations clearly reveal the role of nonpolynomial representations for modelling plastic and locking behaviour of materials. For instance, such representations involve quotients of polynomials, square roots, etc.

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Adiabatic shear band localization in single crystals under dynamic loading processes

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THE MAIN OBJECTIVE of the paper is the investigation of adiabatic shear band localization phenomena in inelastic single crystals under dynamic loading processes. In the first part, a rate-dependent plastic model of single crystals is developed within the thermodynamic framework of the rate-type covariance constitutive structure. This model takes account of the effects as follows: (i) influence of covariance terms, lattice rotations and plastic spin; (ii) thermomechanical coupling; (iii) evolution of the dislocation substructure. An adiabatic process is formulated and examined. The relaxation time is used as a regularization parameter. The viscoplastic regularization assures the stable integration algorithm by using the finite element method. It has been shown that the evolution problem (the initial-boundary value problem) for rate-dependent plastic model of single crystals is well posed. The second part is devoted to the investigation of criteria of localization of plastic deformation in both single slip and symmetric double slip processes. The adiabatic shear band formation in elastic-plastic rate-independent single crystals during dynamic loading processes is investigated. The critical value of the strain hardening rate and the misalignment of the shear band from the active slip systems in the crystal's matrix have been determined. Particular attention is focused on the investigation of synergetic effects. Calculations have been obtained for aluminum single crystals. The results obtained are compared with available experimental observations.

1. Introduction

RECENT EXPERIMENTAL observations and theoretical investigations have shown that the synergetic effects have great influence on the behaviour of inelastic single crystals. Particularly the adiabatic shear band localization in single crystals is affected very much by cooperative phenomena.

Experimental observations of the macroscopic adiabatic shear band localization in single crystals performed by CHANG and ASARO [6, 7], SPITZIG [35] and LISIECKI *et al.* [18] showed that the strain-hardening rate h_{crit} at the inception of shear band localization is positive and the direction of the localized shear band is misaligned by some small angle δ from the active slip system.

On the other hand, the investigations presented by MECKING and KOCKS [21], FOLLANSBEE [12] and FOLLANSBEE and KOCKS [13] showed the great influence of the strain rate sensitivity on the behaviour of inelastic metallic single crystals in dynamic loading processes. To describe the strain rate sensitivity effects, FOLLANSBEE [12] suggested to take into consideration the evolution of the dislocation substructure.

Experimental study of highly heterogeneous deformations in copper single crystals performed by RASHID *et al.* [33] showed that the strain rate history dependence of the substructure evolution plays an important role, particularly in adiabatic shear band formation phenomena.

ASARO and RICE [3] have clearly shown that the classical theory of crystals based on the Schmid constitutive law does not seem to be appropriate to explain the shear band localization phenomenon in ductile metallic single crystals.

ASARO and RICE [3] have focused attention on the localization criteria for "an assumed class of materials that essentially obey Schmid's rule but display modest departure from it". They proved that the plastic hardening rate h_{crit} at the inception of localization may be positive when there are deviations from the Schmid law, cf. also PEIRCE, ASARO and NEEDLEMAN [21, 22], QIN and BASSANI [32, 33] and BASSANI [5].

To describe the main experimentally observed facts connected with the macroscopic shear band formation, DUSZEK–PERZYNA and PERZYNA [9] have considered the synergetic effects resulting from taking into account spatial covariance effects and thermomechanical couplings. PERZYNA and KORBEL [29] have investigated the influence of the evolution of substructure on the shear band localization phenomena in single crystals for single slip process. DUSZEK–PERZYNA and PERZYNA [10] have examined the influence of thermal expansion, thermal plastic softening and spatial covariance effects on shear band localization criteria for a planar model of an f.c.c. crystal undergoing symmetric primary-conjugate double slip process, cf. also PERZYNA and DUSZEK–PERZYNA [27]. In the paper by PERZYNA and KORBEL [30] the attention is focused on the discussion of the cooperative influence of various effects on criteria for shear band localization in both the symmetric double slip and single slip processes.

It has been proved by previously mentioned theoretical investigations that the main cooperative phenomena which affect the behaviour of metallic single crystals are generated by thermomechanical couplings and the evolution of the dislocation substructure.

To describe the influence of main cooperative phenomena on the behaviour of metallic single crystals, we intend to start from the development of the thermodynamic theory of single crystals with special emphasis on the investigations of thermomechanical couplings and internal dissipative effects. Then this theory is used for the investigations of the adiabatic shear band formation in single crystals under dynamic loading processes.

In Sec. 2 the constitutive rate-dependent model and the formulation of the initial-boundary value problem (evolution problem) are presented. Section 3 is devoted to the rate-independent model of inelastic single crystals. In Secs. 4 and 5 the investigation of the adiabatic shear band localization is given. Single slip and symmetric double slip processes are considered. The identification procedure for the material functions and constants is presented in Sec. 6.

In Sec. 7 the numerical investigation for the localization criteria is given. Discussion of the results and final comments are presented in Sec. 8.

2. Formulation of the initial-boundary value problem

2.1. General postulates

To formulate the initial-boundary value problem which describes an adiabatic plastic deformation process for an elastic-viscoplastic model of single crystals, we take advantage of the results obtained previously in the following papers: DUSZEK–PERZYNA and PERZYNA [9, 10], PERZYNA [25, 28].

Let us assume that the regular motion of a body $\mathcal{B}$ is described by the mapping

$$\mathbf{x} = \phi(\mathbf{X}, t), \tag{2.1}$$

where points in $\mathcal{B}$ are denoted by $\mathbf{X}$ and those in the actual configuration $\mathcal{S}$ of a body $\mathcal{B}$ at time t by $\mathbf{x}$. Then *the kinematic equation* has the form

$$\boldsymbol{\upsilon} = \dot{\phi}(\mathbf{X}, t) \mid_{\mathbf{X}=\phi^{-1}(\mathbf{x},t)}, \tag{2.2}$$

where $\boldsymbol{\upsilon}$ denotes the spatial velocity and the dot – the material differentiation with respect to time t.

Let $\varrho(\mathbf{x}, t)$ be the mass density of the deformed body $\mathcal{B}$ at time t. Then *the conservation of mass* states that the equation of continuity

$$\dot{\varrho} = -\varrho \operatorname{div} \boldsymbol{\upsilon} \tag{2.3}$$

is satisfied.

Assume that conservation of mass (2.3) holds, then *the balance of momentum* is equivalent to (provided there is no body force field)

$$\dot{\boldsymbol{\upsilon}} = \frac{1}{\varrho} \operatorname{div} \left(\frac{1}{J} \boldsymbol{\tau} \right), \tag{2.4}$$

where J is the Jacobian of a mapping ϕ and $\boldsymbol{\tau}$ is the Kirchhoff stress tensor.

It is postulated that the free energy function is given by

$$\psi = \widehat{\psi}(\mathbf{e}, \mathbf{F}, \vartheta; \boldsymbol{\mu}), \tag{2.5}$$

where $\mathbf{e}$ denotes the Eulerian strain tensor, $\mathbf{F}$ is the deformation gradient, ϑ temperature and $\boldsymbol{\mu}$ is the matrix of the internal state variables.

The form of the free energy function $\psi = \widehat{\psi}(\mathbf{e}, \mathbf{F}, \vartheta)$ is suggested for spatial description in thermoelasticity. To extend the domain of description of the material properties of a single crystal and particularly, to take into consideration the plastic flow effects and the dislocation substructure, we have to introduce a set of the internal state variables. In our case the matrix $\boldsymbol{\mu}$ is postulated in the form

$$\boldsymbol{\mu} = (\boldsymbol{\gamma}, \boldsymbol{\alpha}_M, \boldsymbol{\alpha}_D), \tag{2.6}$$

where

$$\boldsymbol{\gamma} = \begin{bmatrix} \gamma^{(1)} \\ \vdots \\ \gamma^{(n)} \end{bmatrix}, \qquad \boldsymbol{\alpha}_M = \begin{bmatrix} \alpha_M^{(1)} \\ \vdots \\ \alpha_M^{(n)} \end{bmatrix}, \qquad \boldsymbol{\alpha}_D = \begin{bmatrix} \alpha_D^{(1)} \\ \vdots \\ \alpha_D^{(n)} \end{bmatrix}, \tag{2.7}$$

$\boldsymbol{\gamma}$ defines the shearing, $\boldsymbol{\alpha}_M$ the density of mobile dislocations and $\boldsymbol{\alpha}_D$ the density of obstacle dislocations in the slip system of a crystal.

Let us postulate evolution equations for the internal state variables $\boldsymbol{\gamma}$, $\boldsymbol{\alpha}_M$ and $\boldsymbol{\alpha}_D$ as follows

$$\dot{\boldsymbol{\gamma}} = \begin{bmatrix} \dot{\gamma}^{(1)} \\ \vdots \\ \dot{\gamma}^{(n)} \end{bmatrix} \equiv \mathbf{G}, \tag{2.8}$$

where $$\dot{\gamma}^{(\beta)} = \frac{1}{T^{(\beta)}} \left\langle \varPhi \left[\frac{\tau^{(\beta)}}{\tau_c^{(\beta)}(\boldsymbol{\gamma}, \boldsymbol{\alpha}_D, \vartheta) + \boldsymbol{\kappa}^{(\beta)} : \boldsymbol{\tau}} - 1 \right] \right\rangle \operatorname{sgn}\tau^{(\beta)},$$

$$\begin{aligned} \dot{\boldsymbol{\alpha}}_M &= \mathbf{a}_1 \cdot \dot{\gamma} + \mathbf{a}_2 \cdot \dot{\vartheta} + \mathbf{a}_3 \cdot \dot{\boldsymbol{\alpha}}_D , \\ \dot{\boldsymbol{\alpha}}_D &= \mathbf{b}_1 \cdot \dot{\gamma} + \mathbf{b}_2 \cdot \dot{\vartheta} + \mathbf{b}_3 \cdot \dot{\boldsymbol{\alpha}}_M , \end{aligned} \tag{2.9}$$

where $T^{(\beta)}$ denotes the relaxation time, $\varPhi$ is the overstress empirical viscoplastic function, $\tau^{(\beta)}$ defines the Schmid resolved stress on the slip system β ($\beta = 1, 2, \ldots, n$), $\tau_c^{(\beta)}$ is the critical Schmid resolved stress on the slip system β, $\boldsymbol{\kappa}^{(\beta)}$ denotes the symmetric tensor of the non-Schmid effects, and the matrices $\mathbf{a}_i$, $\mathbf{b}_i$ ($i = 1, 2, 3$) are the material functions of $\boldsymbol{\gamma}$, ϑ, $\boldsymbol{\alpha}_D$ and $\boldsymbol{\alpha}_M$.

Consistency condition for the evolution equations (2.9) needs the assertions

$$\det[\mathbf{1} - \mathbf{a}_3 \cdot \mathbf{b}_3] \neq 0 \qquad \text{and} \qquad \det[\mathbf{1} - \mathbf{b}_3 \cdot \mathbf{a}_3] \neq 0. \tag{2.10}$$

We postulate that the balance of energy and the entropy production inequality hold. Then we obtain two fundamental evolution equations for the Kirchhoff stress tensor $\boldsymbol{\tau}$ and for temperature ϑ in the form

$$\begin{aligned} \mathrm{L}_{\boldsymbol{\upsilon}} \boldsymbol{\tau} &= \mathcal{L}^e : \mathbf{d} - \mathcal{L}^{\mathrm{th}} \dot{\vartheta} - (\mathcal{L}^e : \mathbf{N} + \mathbf{b}) \cdot \dot{\boldsymbol{\gamma}}, \\ \varrho c_p \dot{\vartheta} &= -\operatorname{div}\mathbf{q} + \vartheta \frac{\varrho}{\varrho_{\mathrm{Ref}}} \frac{\partial \boldsymbol{\tau}}{\partial \vartheta} : \mathbf{d} + \chi \sum_{\beta=1}^{n} \tau^{(\beta)} \dot{\gamma}^{(\beta)} \\ &\quad + \chi^* \sum_{\beta=1}^{n} \sum_{\delta=1}^{n} \left(a_1^{-1}\right)^{(\beta\delta)} \tau^{(\delta)} \dot{\alpha}_M^{(\beta)} + \chi^{**} \sum_{\beta=1}^{n} \sum_{\delta=1}^{n} \left(b_1^{-1}\right)^{(\beta\delta)} \tau^{(\delta)} \dot{\alpha}_D^{(\beta)}, \end{aligned} \tag{2.11}$$

where $\mathrm{L}_{\boldsymbol{\upsilon}}$ denotes the Lie derivative of $\boldsymbol{\tau}$ with respect to $\boldsymbol{\upsilon}$, $(a_1^{-1})^{(\beta\delta)}$ and $(b_1^{-1})^{(\beta\delta)}$ are components of $\mathbf{a}_1^{-1}$ and $\mathbf{b}_1^{-1}$, and

$$\mathcal{L}^e = \varrho_{\mathrm{Ref}} \frac{\partial^2 \widehat{\psi}}{\partial \mathbf{e}^2}, \qquad \mathcal{L}^{\mathrm{th}} = -\varrho_{\mathrm{Ref}} \frac{\partial^2 \widehat{\psi}}{\partial \mathbf{e} \partial \vartheta},$$

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}^{(1)} \\ \vdots \\ \mathbf{N}^{(n)} \end{bmatrix}, \qquad \mathbf{N}^{(\beta)} = \frac{1}{2}\left[\mathbf{s}^{(\beta)}\mathbf{m}^{(\beta)} + \mathbf{m}^{(\beta)}\mathbf{s}^{(\beta)}\right],$$

$$\mathbf{W} = \begin{bmatrix} \mathbf{W}^{(1)} \\ \vdots \\ \mathbf{W}^{(n)} \end{bmatrix}, \qquad \mathbf{W}^{(\beta)} = \frac{1}{2}\left[\mathbf{s}^{(\beta)}\mathbf{m}^{(\beta)} - \mathbf{m}^{(\beta)}\mathbf{s}^{(\beta)}\right],$$

$$\mathbf{b} = \begin{bmatrix} \mathbf{b}^{(1)} \\ \vdots \\ \mathbf{b}^{(n)} \end{bmatrix}, \qquad \mathbf{b}^{(\beta)} = (\mathbf{N}^{(\beta)} + \mathbf{W}^{(\beta)})\cdot\boldsymbol{\tau} + \boldsymbol{\tau}(\mathbf{N}^{(\beta)} - \mathbf{W}^{(\beta)}), \tag{2.12}$$

$$c_p = -\vartheta \frac{\partial^2 \widehat{\psi}}{\partial \vartheta^2}, \qquad \chi\tau^{(\beta)} = -\varrho \frac{\partial \widehat{\psi}}{\partial \gamma^{(\beta)}},$$

$$\chi^*\tau^{(\beta)} = -\varrho \sum_{\delta=1}^{n} a_1^{(\beta\delta)} \frac{\partial \widehat{\psi}}{\partial \alpha_M^{(\delta)}},$$

$$\chi^{**}\tau^{(\beta)} = -\varrho \sum_{\delta=1}^{n} b_1^{(\beta\delta)} \frac{\partial \widehat{\psi}}{\partial \alpha_D^{(\delta)}}.$$

2.2. Adiabatic process

For an adiabatic process ($\mathbf{q} = 0$) from $(2.11)_2$ we obtain

$$\dot{\vartheta} = \mathcal{F} : \mathbf{d} + K\mathbf{G}, \tag{2.13}$$

where

$$\begin{aligned} \mathcal{F} &= \frac{\varrho}{\varrho_{\mathrm{Ref}}} \frac{\vartheta}{\varrho c_p - D} \frac{\partial \boldsymbol{\tau}}{\partial \vartheta}, \\ K &= \frac{\boldsymbol{\tau} : \mathbf{N}}{\varrho c_p - D}\left(\chi + \chi^* \mathbf{a}_1^{-1}\cdot\mathbf{A}_1 + \chi^{**}\mathbf{b}_1^{-1}\cdot\mathbf{B}_1\right), \\ D &= \boldsymbol{\tau} : \mathbf{N}\cdot\left(\chi^*\mathbf{a}_1^{-1}\cdot\mathbf{A}_2 + \chi^{**}\mathbf{b}_1^{-1}\cdot\mathbf{B}_2\right), \\ \mathbf{A}_1 &= (\mathbf{a}_1 + \mathbf{a}_3\cdot\mathbf{b}_1)(\mathbf{1} - \mathbf{a}_3\cdot\mathbf{b}_3)^{-1}, \\ \mathbf{B}_1 &= (\mathbf{b}_1 + \mathbf{b}_3\cdot\mathbf{a}_1)(\mathbf{1} - \mathbf{b}_3\cdot\mathbf{a}_3)^{-1}. \end{aligned} \tag{2.14}$$

To guarantee the existence of $\mathcal{F}$ and K we have to assume $D \neq \varrho c_p$.

Taking into account (2.13) we can write the evolution equations (2.9) in the form

$$\begin{aligned} \dot{\boldsymbol{\alpha}}_M &= (\mathbf{A}_1 + \mathbf{A}_2 K)\cdot\mathbf{G} + \mathbf{A}_2\mathcal{F} : \mathbf{d}, \\ \dot{\boldsymbol{\alpha}}_D &= (\mathbf{B}_1 + \mathbf{B}_2 K)\cdot\mathbf{G} + \mathbf{B}_2\mathcal{F} : \mathbf{d}, \end{aligned} \tag{2.15}$$

and the rate-type constitutive relation for the Kirchhoff stress tensor $\boldsymbol{\tau}$ as follows,

$$\mathrm{L}_{\boldsymbol{\upsilon}}\boldsymbol{\tau} = \left(\mathcal{L}^e - \mathcal{L}^{\mathrm{th}}\mathcal{F}\right) : \mathbf{d} - \left(\mathcal{L}^e : \mathbf{N} + \mathbf{b} + \mathcal{L}^{\mathrm{th}} K\right)\cdot\mathbf{G}. \tag{2.16}$$

2.3. An abstract form of the evolution problem

In an abstract form Eqs. (2.2), (2.3), (2.4), (2.16), (2.8), (2.15) and (2.13) can be written as follows:

$$\dot{\boldsymbol{\varphi}} = \mathcal{A}(t, \boldsymbol{\varphi})\boldsymbol{\varphi} + \mathbf{f}(t, \boldsymbol{\varphi}) \tag{2.17}$$

where

$$\boldsymbol{\varphi} = \begin{bmatrix} \phi \\ \boldsymbol{\upsilon} \\ \varrho \\ \boldsymbol{\tau} \\ \boldsymbol{\gamma} \\ \boldsymbol{\alpha}_M \\ \boldsymbol{\alpha}_D \\ \vartheta \end{bmatrix}, \qquad \mathbf{f} = \begin{bmatrix} \boldsymbol{\upsilon} \\ 0 \\ 0 \\ (\mathcal{L}^e : \mathbf{N} + \mathbf{b} + \mathcal{L}^{\mathrm{th}} K)\cdot\mathbf{G} \\ \mathbf{G} \\ (\mathbf{A}_1 + \mathbf{A}_2 K)\cdot\mathbf{G} \\ (\mathbf{B}_1 + \mathbf{B}_2 K)\cdot\mathbf{G} \\ K\mathbf{G} \end{bmatrix},$$

$$\mathcal{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0\;0\;0\;0 \\ 0 & 0 & \dfrac{\tau}{\varrho_{\mathrm{Ref}}}\mathrm{grad} & \dfrac{\varrho}{\varrho_{\mathrm{Ref}}}\mathrm{div} & 0\;0\;0\;0 \\ 0 & \varrho\,\mathrm{div} & 0 & 0 & 0\;0\;0\;0 \\ 0 & (\mathcal{L}^e - \mathcal{L}^{\mathrm{th}}\mathcal{F}) : \mathrm{sym}\left[\dfrac{\partial}{\partial\mathbf{x}}\right] - 2\,\mathrm{sym}\left[\boldsymbol{\tau}\cdot\dfrac{\partial}{\partial\mathbf{x}}\right] & 0 & 0 & 0\;0\;0\;0 \\ 0 & 0 & 0 & 0 & 0\;0\;0\;0 \\ 0 & \mathbf{A}_2\mathcal{F} : \mathrm{sym}\left[\dfrac{\partial}{\partial\mathbf{x}}\right] & 0 & 0 & 0\;0\;0\;0 \\ 0 & \mathbf{B}_2\mathcal{F} : \mathrm{sym}\left[\dfrac{\partial}{\partial\mathbf{x}}\right] & 0 & 0 & 0\;0\;0\;0 \\ 0 & \mathcal{F} : \mathrm{sym}\left[\dfrac{\partial}{\partial\mathbf{x}}\right] & 0 & 0 & 0\;0\;0\;0 \end{bmatrix}. \tag{2.18}$$

To investigate the behaviour of rate-dependent elasto-plastic single crystal during an adiabatic process and particularly, to examine the shear band formation, let us formulate the initial-boundary value problem (the evolution problem) as follows.

Find $\boldsymbol{\varphi}$ as a function of t and $\mathbf{x}$ such that the following assertions are satisfied:

(i) the differential equations in an abstract form (2.17) are fulfilled;

(ii) boundary conditions:

(a) displacement ϕ is prescribed on a part ∂_ϕ of $\partial\phi(\mathcal{B})$ and tractions $(\boldsymbol{\tau}\cdot\mathbf{n})^a$ are prescribed on a part $\partial_{\boldsymbol{\tau}}$ of $\partial\phi(\mathcal{B})$, where $\partial_\phi \cap \partial_{\boldsymbol{\tau}} = 0$ and $\overline{\partial_\phi \cup \partial_{\boldsymbol{\tau}}} = \partial\phi(\mathcal{B})$;

(b) heat flux $(\mathbf{q}\cdot\mathbf{n}) = 0$ is prescribed on $\partial\phi(\mathcal{B})$;

(iii) the initial conditions: $\boldsymbol{\varphi}$ is given at $X \in \mathcal{B}$ at $t = 0$.

3. Rate-independent response of single crystals

Assuming in (2.8) $T^{(\beta)} = 0$ we obtain

$$\tau^{(\beta)} = \tau_c^{(\beta)}(\gamma, \alpha_D, \vartheta) + \boldsymbol{\kappa}^{(\beta)} : \boldsymbol{\tau}. \tag{3.1}$$

Material differentiation with respect to time of (3.1) yields

$$\dot{\gamma}^{(\beta)} = \sum_{\delta=1}^{n} \hat{h}_{\beta\delta}^{-1}\left(\dot{\tau}^{(\delta)} - \boldsymbol{\kappa}^{(\delta)} : \dot{\boldsymbol{\tau}}\right) + \pi^{(\beta)}\dot{\vartheta} - \sum_{\delta=1}^{n} g_{\beta\delta}\dot{\alpha}_D^{(\delta)}, \tag{3.2}$$

where

$$\hat{h}_{\beta\delta} = \frac{\partial\tau_c^{(\beta)}}{\partial\gamma^{(\delta)}}, \qquad \pi^{(\beta)} = -\sum_{\delta=1}^{n}\frac{\partial\tau_c^{(\delta)}}{\partial\vartheta}\hat{h}_{\beta\delta}^{-1}, \qquad g_{\beta\delta} = \sum_{\nu=1}^{n}\frac{\partial\tau_c^{(\nu)}}{\partial\alpha_D^{(\delta)}}\hat{h}_{\beta\nu}^{-1}. \tag{3.3}$$

For rate-independent process the evolution equation for shearing $\boldsymbol{\gamma}$ has to be replaced by

$$\dot{\gamma} = \mathbf{M}^{-1}\Big[\mathbf{N} : \mathcal{L}^e : \mathbf{N} + \mathbf{b} : \mathbf{N} - \mathcal{F}\left(\mathcal{L}^{\mathrm{th}} : \mathbf{N} + \frac{\partial\tau_c}{\partial\vartheta} + \frac{\partial\tau_c}{\partial\boldsymbol{\alpha}_D}\cdot\mathbf{B}_2\right) - \boldsymbol{\kappa} : \mathcal{L}^e\Big] : \mathbf{d}, \tag{3.4}$$

where

$$\mathbf{M} = \mathbf{h} + (\mathbf{N} - \boldsymbol{\kappa}) : \mathcal{L}^e : \mathbf{N} + \mathbf{b} : \mathbf{N} + \left(\mathcal{L}^{\mathrm{th}} : \mathbf{N} + \frac{\partial\tau_c}{\partial\vartheta}\right)K. \tag{3.5}$$

4. Analysis of acceleration waves

4.1. General consideration

To investigate the intrinsic mathematical structure of the set of the field equations which determine the adiabatic inelastic flow processes, let us analyse the problem of propagation of acceleration waves. We shall show that the theory of acceleration waves in materials considered can be based on the notion of an instantaneous adiabatic acoustic tensor.

Let $\sum(t)$ denote a smooth surface with outward normal $\mathbf{n}$ which is moving through the solid body with velocity $\mathbf{w}(t,\mathbf{x})$. Some field quantities or their derivatives may be discontinuous across $\sum(t)$ which is then called a singular surface. If the surface $\sum(t)$ is composed of the same material points at all times, one then refers to $\sum(t)$ as a stationary discontinuity. Otherwise, the surface $\sum(t)$ is called a propagating singular surface or wave, cf. Hill [15].

Let c denote the normal speed of propagation of $\sum(t)$ with respect to the material in its current configuration. It is related to the spatial velocity $\boldsymbol{\upsilon}(t,\mathbf{x})$ and to the normal wave speed $w = \mathbf{w}\cdot\mathbf{n}$, by the following equation: $c = w - \boldsymbol{\upsilon}\cdot\mathbf{n}$.

It is said that $\sum(t)$ is an acceleration wave if the fields ϕ, $\boldsymbol{\upsilon}$, $\mathbf{F}$, $\boldsymbol{\mu}$ and ϑ are continuous functions of t and $\mathbf{x}$ while $\dot{\boldsymbol{\upsilon}}$, $\nabla\boldsymbol{\upsilon}$, $\dot{\mathbf{F}}$, $\nabla\mathbf{F}$, $\dot{\boldsymbol{\mu}}$, $\nabla\boldsymbol{\mu}$, $\dot{\vartheta}$, $\nabla\vartheta$ have (at most) jump discontinuities across $\sum(t)$ but are continuous in t and $\mathbf{x}$ jointly everywhere else ($\boldsymbol{\mu}$ denotes a set of the internal state variables).

An acceleration wave in which $\dot{\vartheta}$ and $\nabla\vartheta$ are continuous functions of t and $\mathbf{x}$ is called homothermal.

From the definition of an acceleration wave and the constitutive assumption

$$\psi = \widehat{\psi}(\mathbf{e},\mathbf{F},\vartheta,\boldsymbol{\mu}) \tag{4.1}$$

we have

$$[\![\psi]\!] = [\![\boldsymbol{\sigma}]\!] = 0, \tag{4.2}$$

where $[\![\cdot]\!]$ denotes the jump of a quantity across $\sum(t)$ in the direction of its local normal $\mathbf{n}(t,\mathbf{x})$.

Hadamard's compatibility conditions require that jumps in velocity and stress derivatives be related as follows (cf. Hadamard [14] and Hill [15]):

$$\begin{aligned} [\![\nabla\boldsymbol{\upsilon}]\!] &= -\frac{1}{c}[\![\mathbf{a}]\!]\mathbf{n}, \\ [\![\nabla\boldsymbol{\sigma}]\!] &= -\frac{1}{c}[\![\dot{\boldsymbol{\sigma}}]\!]\mathbf{n}, \end{aligned} \tag{4.3}$$

where ∇ denotes the spatial gradient, $\boldsymbol{\sigma} = (1/J)\boldsymbol{\tau}$ is the Cauchy tensor, and $\mathbf{a} = \dot{\boldsymbol{\upsilon}}$.

Balance of momentum requires that

$$\operatorname{div}\boldsymbol{\sigma} = \varrho \mathbf{a}. \tag{4.4}$$

Finally we have

$$\mathbf{n}\cdot[\![\dot{\boldsymbol{\sigma}}]\!] = -\varrho c[\![\mathbf{a}]\!]. \tag{4.5}$$

From the last result it becomes clear that the existence and propagation speed of acceleration waves in solids is directly related to the assumed constitutive structure of the material.

Since ϑ is continuous across $\sum(t)$, we have

$$[\![\dot{\vartheta}]\!] = -c[\![\nabla\vartheta]\!]\cdot\mathbf{n}. \tag{4.6}$$

For an acceleration wave in an adiabatic process we have (cf. PERZYNA [26])

$$[\![\mathbf{q}]\!] = 0, \qquad [\![\dot{\mathbf{q}}]\!] = 0, \qquad [\![\dot{\vartheta}]\!] \neq 0, \qquad [\![\nabla\vartheta]\!] \neq 0, \tag{4.7}$$

where $\mathbf{q}$ denotes the heat flux vector field.

Hence an acceleration wave in inelastic solids for an adiabatic process is not homothermal.

This conclusion will play an important role in an analysis of acceleration waves in particular material models for adiabatic process of a crystal.

4.2. Rate-dependent adiabatic process

For rate-dependent adiabatic process we have

$$[\![\mathrm{L}_{\boldsymbol{\upsilon}}\boldsymbol{\tau}]\!] = \mathbb{E} : [\![\mathbf{d}]\!]. \tag{4.8}$$

where

$$\mathbb{E} = \mathcal{L}^{e} - \mathcal{L}^{\mathrm{th}}\mathcal{F}. \tag{4.9}$$

THEOREM 1. *For an adiabatic rate-dependent plastic flow process of single crystal described by* Eq. (2.17), *the acceleration discontinuity* $[\![\mathbf{a}]\!]$ *is the solution of the eigenvalue problem*

$$\mathbf{A}\cdot[\![\mathbf{a}]\!] = \varrho_{\mathrm{Ref}}c^{2}[\![\mathbf{a}]\!], \tag{4.10}$$

where

$$\mathbf{A} = \mathbf{n}\cdot(\mathbb{E}\cdot\mathbf{n} + \boldsymbol{\tau}\cdot\mathbf{n}\mathbf{g}) \tag{4.11}$$

denotes the instantaneous adiabatic acoustic tensor and $\mathbf{g}$ *is the metric tensor.*

4.3. Rate-independent adiabatic process

For rate-independent adiabatic process we obtain

$$[\![\mathrm{L}_{\boldsymbol{\upsilon}}\boldsymbol{\tau}]\!] = \mathbb{L} : [\![\mathbf{d}]\!], \tag{4.12}$$

where

$$\begin{aligned}\mathbb{L} = \mathcal{L}^e - \mathcal{L}^{\mathrm{th}}\mathcal{F} - [\mathcal{L}^e : \mathbf{N} + \mathbf{b} + \mathcal{L}^{\mathrm{th}}K]\cdot\mathbf{M}^{-1}\cdot\Big[&\mathbf{N} : \mathcal{L}^e : \mathbf{N} + \mathbf{b} : \mathbf{N} \\ &- \mathcal{F}\left(\mathcal{L}^{\mathrm{th}} : \mathbf{N} + \frac{\partial\tau_c}{\partial\vartheta} + \frac{\partial\tau_c}{\partial\boldsymbol{\alpha}_D}\cdot\mathbf{B}_2\right) - \boldsymbol{\kappa} : \mathcal{L}^e\Big],\end{aligned} \tag{4.13}$$

$$h^{(\alpha\beta)} = \frac{\partial\tau_c^{(\alpha)}}{\partial\gamma^{(\beta)}}, \qquad \left(\frac{\partial\tau_c}{\partial\boldsymbol{\alpha}_D}\right)^{(\beta\delta)} = \frac{\partial\tau_c^{(\beta)}}{\partial\alpha_D^{(\delta)}}, \qquad \frac{\partial\tau_c}{\partial\vartheta} = \begin{bmatrix}\dfrac{\partial\tau_c^{(1)}}{\partial\vartheta}\\ \vdots \\ \dfrac{\partial\tau_c^{(n)}}{\partial\vartheta}\end{bmatrix}.$$

THEOREM 2. *For an adiabatic rate-independent plastic flow process of single crystal, the acceleration discontinuity* $[\![\mathbf{a}]\!]$ *is the solution of the eigenvalue problem*

$$\widehat{\mathbf{A}}\cdot[\![\mathbf{a}]\!] = \varrho_{\mathrm{Ref}}c^2[\![\mathbf{a}]\!], \tag{4.14}$$

where

$$\widehat{\mathbf{A}} = \mathbf{n}\cdot(\mathbb{L}\cdot\mathbf{n} + \boldsymbol{\tau}\cdot\mathbf{n}\mathbf{g}) \tag{4.15}$$

denotes the instantaneous adiabatic acoustic tensor.

4.4. Analysis of eigenvalues of acoustic tensors

In the case of a rate-dependent process, all eigenvalues of the instantaneous adiabatic acoustic tensor **A** are positive. This of course is implied by hyperbolicity of the initial-boundary value problem.

For simplicity let us introduce rectangular Cartesian coordinates $\{x^i\}$ in such a way that **n** is in x^2-direction. We can assume, without loss of generality, $a_3 = 0$, and consider the reduced problem

$$\det[\mathbf{A}^{jk} - \zeta\delta^{jk}] = 0 \qquad \text{for} \quad j,k = 1,2, \tag{4.16}$$

where

$$\zeta = \varrho_{\mathrm{Ref}}c^2. \tag{4.17}$$

This leads to the result

$$\zeta_{1,2} = \frac{1}{2}(A^{11} + A^{22}) \pm \frac{1}{2}\sqrt{(A^{11} - A^{22})^2 + 4A^{12}A^{21}}\,. \tag{4.18}$$

Assuming the linear behaviour of the crystal and linear thermal expansion, i.e.

$$
\begin{aligned}
(\mathcal{L}^e)^{ijkl} &= \tau^{jl}\delta^{ik} + \mu\left(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\right) + \lambda\delta^{ij}\delta^{kl},\\
(\mathcal{L}^e)^{-1} : \mathcal{L}^{\mathrm{th}} &= \theta\mathbf{I},
\end{aligned}
\tag{4.19}
$$

where μ and λ are Lamé moduli and θ denotes the thermal expansion coefficient, we can find

$$
\begin{aligned}
\zeta_{1,2} = {}& \frac{1}{2}(3\mu + \lambda + 3\tau^{22})\\
& - \frac{1}{2}\theta\frac{\varrho}{\varrho_{\mathrm{Ref}}}\frac{\vartheta}{\varrho c_p - D}\left[\tau^{12}\cdot\frac{\partial\tau^{12}}{\partial\vartheta} + (2\mu + 2\lambda + \tau^{22})\frac{\partial\tau^{22}}{\partial\vartheta}\right]\\
& \pm \frac{1}{2}\left\{\left[\mu + \lambda + \tau^{22} - \frac{\varrho}{\varrho_{\mathrm{Ref}}}\frac{\theta\vartheta}{\varrho c_p - D}\left((2\mu + 2\lambda + \tau^{22})\frac{\partial\tau^{22}}{\partial\vartheta} - \tau^{12}\frac{\partial\tau^{12}}{\partial\vartheta}\right)\right]^2\right.\\
& \left. - 4\frac{\varrho}{\varrho_{\mathrm{Ref}}}\frac{\theta\vartheta}{\varrho c_p - D}\left[\frac{4}{3}\mu + \lambda + \tau^{22}\right]\cdot\left[1 - \frac{\varrho}{\varrho_{\mathrm{Ref}}}\frac{\theta\vartheta}{\varrho c_p - D}\frac{\partial\tau^{22}}{\partial\vartheta}\right]\tau^{12}\frac{\partial\tau^{12}}{\partial\vartheta}\right\}^{1/2}.
\end{aligned}
\tag{4.20}
$$

Neglecting thermal effects and the evolution of the dislocation substructure we have

$$
\zeta_1 = 2\mu + \lambda + 2\tau^{22}, \qquad \zeta_2 = \mu + \tau^{22}.
\tag{4.21}
$$

In the case of a rate-independent process it may happen that some eigenvalues of the instantaneous adiabatic acoustic tensor $\widehat{\mathbf{A}}$ are equal zero. Then the associated discontinuity does not propagate ($c = 0$). This situation is referred to as the strain localization condition, i.e.

$$
\det\widehat{\mathbf{A}} = 0.
\tag{4.22}
$$

5. Shear band formation

5.1. Single slip process ($n = 1$)

Mathematical analysis of the governing equations and the perturbation method give the hardening modulus rate h and the direction of the shear band $\mathbf{n}$ at the initiation of localization (cf. DUSZEK–PERZYNA and PERZYNA [9, 10], PERZYNA and DUSZEK–PERZYNA [28] and PERZYNA and KORBEL [29, 30])

$$
\begin{aligned}
\left(\frac{h}{\tau}\right)_{\mathrm{crit}} &= \Pi + \frac{\Gamma}{\tau} - \Omega + \frac{\tau}{\mu}\left(\Theta^2\nu + \Theta + \frac{1}{4\nu}\right) + \frac{1}{4}\kappa^2\frac{\mu}{\tau},\\
\mathbf{n} &= \mathbf{m} + \left(\frac{\Theta\tau}{2\mu} + \frac{\tau}{4\mu\nu}\right)\mathbf{s} - \frac{1}{2}\kappa\mathbf{z},
\end{aligned}
\tag{5.1}
$$

where **z** is a unit vector perpendicular to **s** and **m**, so that **s**, **m**, **z** form a right-handed triad

$$\Theta = \Delta\theta\mu, \qquad \Pi = \Delta\pi h, \qquad \Gamma = b_1 \frac{\partial \tau_c}{\partial \alpha_D},$$

$$\Omega = b_2 \Delta \frac{\partial \tau_c}{\partial \alpha_D}, \qquad \Delta = \frac{\chi + \chi^* + \chi^{**}}{\varrho c_p}, \qquad \nu = \frac{\lambda + \mu}{\lambda + 2\mu},$$

μ and λ are Lamé moduli, and

(5.2)
$$\boldsymbol{\kappa}^{(1)} = \boldsymbol{\kappa}^{(2)} = \begin{bmatrix} 0 & \frac{1}{2}\kappa \\ \frac{1}{2}\kappa & 0 \end{bmatrix}.$$

It is assumed that $\kappa = 0.0017$, $a_1 = 6.5 \cdot 10^{16}$, $b_1 = -6.85 \cdot 10^{13}$ and $b_2 = 0$. All other material parameters are given in Table 1.

Table 1. Material parameters.

	Parameter	Unit	Aluminium
ϱ	density	Kg m^{-3}	2702
c_p	specific heat	J Kg^{-1} K^{-1}	896
μ	shear modulus	G Pa	26.0
E	Young's modulus	G Pa	71.0
K	bulk modulus	G Pa	73.2
θ	coefficient of thermal expansion	K^{-1}	$23.8 \cdot 10^{-6}$
χ	irreversibility coefficient	–	0.65 – 0.85

5.2. Symmetric double slip process ($n = 2$)

It is assumed that the crystal has two active slip systems, symmetrically oriented with respect to the maximum principal stress τ^{22} (the tensile axis is x^2) at the angle φ.

The condition for localization is reduced to the equation

(5.3)
$$A\left(\frac{n_1}{n_2}\right)^4 + B\left(\frac{n_1}{n_2}\right)^3 + C\left(\frac{n_1}{n_2}\right)^2 + D\left(\frac{n_1}{n_2}\right) + E = 0,$$

where

$$
(5.4)\qquad
\begin{aligned}
A &= (\mathbb{L}^{2222}+\tau^{22})(\mathbb{L}^{2112}+\tau^{22}),\\
B &= (\mathbb{L}^{2222}+\tau^{22})(\mathbb{L}^{1112}+\mathbb{L}^{2111})+(\mathbb{L}^{2112}+\tau^{22})(\mathbb{L}^{1222}+\mathbb{L}^{2221})\\
&\qquad -\mathbb{L}^{2212}(\mathbb{L}^{1122}+\mathbb{L}^{2121}),\\
C &= (\mathbb{L}^{1111}+\tau^{11})(\mathbb{L}^{2222}+\tau^{22})+(\mathbb{L}^{1222}+\mathbb{L}^{2221})(\mathbb{L}^{1112}+\mathbb{L}^{2111})\\
&\qquad +(\mathbb{L}^{1221}+\tau^{11})(\mathbb{L}^{2112}+\tau^{22})-(\mathbb{L}^{1122}+\mathbb{L}^{2121})(\mathbb{L}^{1212}+\mathbb{L}^{2211})\\
&\qquad -\mathbb{L}^{1121}\mathbb{L}^{2212},\\
D &= (\mathbb{L}^{1111}+\tau^{11})(\mathbb{L}^{1222}+\mathbb{L}^{2221})+(\mathbb{L}^{1221}+\tau^{11})(\mathbb{L}^{1112}+\mathbb{L}^{2111})\\
&\qquad -\mathbb{L}^{1121}(\mathbb{L}^{1212}+\mathbb{L}^{2211}),\\
E &= (\mathbb{L}^{1111}+\tau^{11})(\mathbb{L}^{1221}+\tau^{11}).
\end{aligned}
$$

The ratio $(n_2/n_1)=\tan\beta$ gives us the direction of the macroscopic shear band. Solving the fourth order algebraic equation (5.3) we have to choose the real and positive solutions for (n_2/n_1), and next to take such value of the angle β that maximizes the value of the hardening modulus rate h/τ^{22}. Then the misalignment angle $\delta=\beta-\varphi$. We shell denote

$$
(5.5)\qquad \mathbf{h}=\begin{bmatrix} h & h_1\\ h_1 & h\end{bmatrix},
$$

and assume $h_1/h=q=1.2$.

In such a way we obtain the critical value of the hardening modulus rate and the misalignment angle between the shear band and the slip plane in point of the inception of the shear band formation.

6. Physical identification of introduced coefficients

For better presentation of theoretical results we have performed numerical estimations of the considered quantities for two particular cases of uniaxial tensile test.

The paper of CHANG and ASARO [7] for Al–Cu single crystals tested at 298 K was taken as the experimental base for calculations.

For simplicity we have made some additional assumptions about the process and we have to write the specific evolution equations for density of dislocation parameters. We used the evolution equations proposed by BALKE and ESTRIN [4] in the form

$$
(6.1)\qquad
\begin{aligned}
&\dot{\alpha}_M = A_1\dot{\gamma}, \qquad A_1^{(\beta\beta)}=\frac{c_1}{b^2}\frac{\alpha_D^{(\beta)}}{\alpha_M^{(\beta)}}-c_2\alpha_M^{(\beta)}-\frac{c_3}{b}\sqrt{\alpha_D^{(\beta)}}, \qquad A_1^{(\beta\delta)}=-\xi\alpha_M^{(\beta)},\\
&\dot{\alpha}_D = B_1\dot{\gamma}, \qquad B_1^{(\beta\beta)}=c_2\alpha_M^{(\beta)}+\frac{c_3}{b^2}\sqrt{\alpha_D^{(\beta)}}-c_4\alpha_D^{(\beta)}, \qquad B_1^{(\beta\delta)}=\xi\alpha_M^{(\beta)},\\
&\qquad\qquad \beta,\delta=1,2.
\end{aligned}
$$

Preserving proportional dependence of generated and annihilated dislocations proposed by the authors we have been able to estimate coordinates of matrices $\mathbf{A}_1$, $\mathbf{B}_1$ in the point of inception of the shear band formation.

For the considered process we have obtained

$$A_1^{(\beta\beta)} \sim 4.2 \times 10^{16}\ \mathrm{m}^{-2}, \qquad A_1^{(\beta\delta)} \sim -1.6 \times 10^{14}\ \mathrm{m}^{-2},$$
$$B_1^{(\beta\beta)} \sim -2.45 \times 10^{14}\ \mathrm{m}^{-2}, \qquad B_1^{(\beta\delta)} \sim 1.6 \times 10^{14}\ \mathrm{m}^{-2}.$$

From the experimental data we have $\partial\tau_2/\partial\vartheta = -0.06$ MPa/K, $\tau_c = 195$ MPa, $\gamma_c = 0.4$ and $\xi = 0.01$.

When we assume the linear elastic behaviour of the crystal and linear thermal expansion cf. (4.19), we can estimate influence of various effects on the value of the fundamental matrix $\mathbb{L}$.

For example the non-dissipative thermal effects are introduced by $\mathcal{F}$, the non-Schmid effects by the matrix $\boldsymbol{\kappa}$, and the substructure evolution effects by coefficient K.

7. Numerical results

Analysing the localization conditions we can observe that these various effects cooperate and then some synergetic effects are generated (cf. Figs. 4–7). The synergetic effect is defined as the difference between the result obtained for two cooperative phenomena and the sum of the results obtained when these two phenomena are active separately. For the considered process in single slip case we may say that most significant are the evolution of the dislocation substructure and the non-Schmid synergetic effects, while in double slip case the non-Schmid effects can be neglected. However, both cases indicate that the evolution of the dislocation substructure effects should be taken into consideration in entire analysis of simple crystal deformation process.

Besides the synergetic effects we obtain also the values of critical hardening modulus and misalignment angle between the shear band and the slip planes for processes in which different effects were considered (Figs. 1–3).

We can observe that in the processes, in which non-dissipative thermal effect is included, the value of hardening modulus is only slightly higher than in the process without this effect.

Next, the two processes in which one of them has the non-Schmid effect but the other not, show more significant difference in value of hardening modulus, but this difference is not so high as that introduced by thermal couplings and the evolution of substructure.

Very important observation is that for both cases: single slip and double slip processes, the value of misalignment angle does not depend on the fact whether the various effects are considered or not. Of course, the value of this angle is more reasonable for double slip process (Fig. 3).

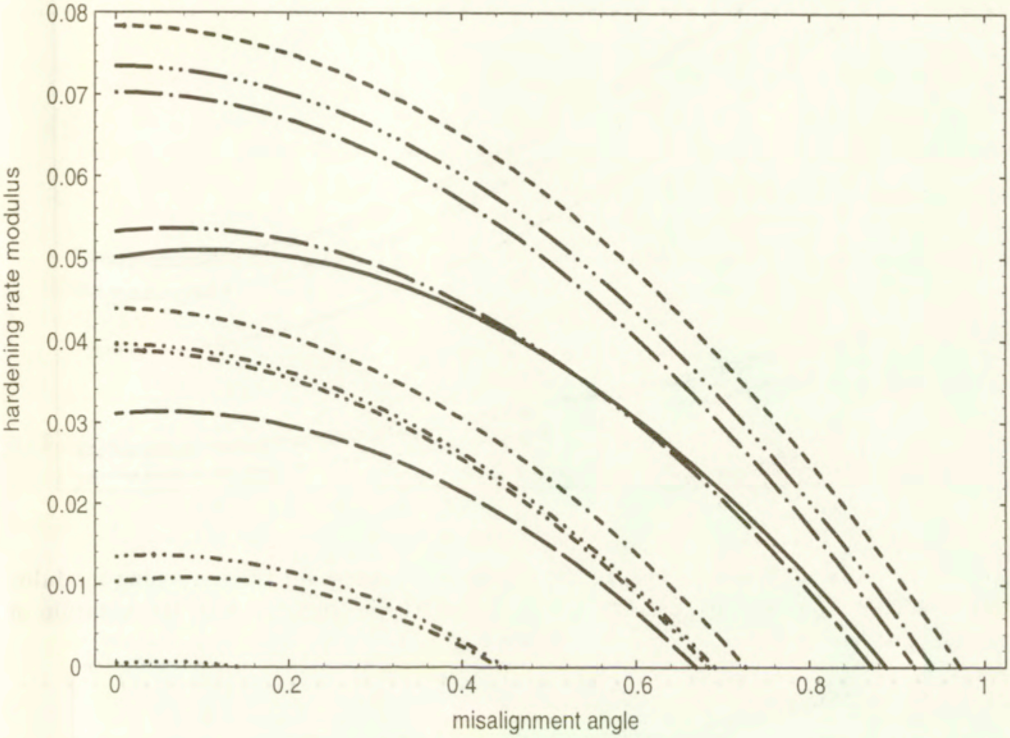

FIG. 1. Numerical results for single slip process for the hardening modulus rate h/τ^{22} as a function of the misalignment angle δ for Al–Cu single crystal.

– – – – – – – –	adiabatic process with substructure and non-Schmid effects,
— · · · — · · ·	adiabatic process with substructure, non-dissipative thermal term and non-Schmid effects,
— · — · — ·	isothermal process with substructure and non-Schmid effects,
— — · · — —	adiabatic process with substructure and non-dissipative thermal term,
———————	adiabatic process with substructure,
– – · – – – · – –	adiabatic process with non-Schmid effects,
– · · · – – · · · – ·	isothermal process with non-Schmid effects,
· – – – · · – – – · ·	adiabatic process with non-dissipative thermal term and non-Schmid effects,
— — — — —	isothermal process with substructure,
– · · – – · · – · · –	adiabatic process with non-dissipative thermal term,
– · – · – · – · – ·	adiabatic process,
· · · · · · · · · · · · · ·	isothermal process.

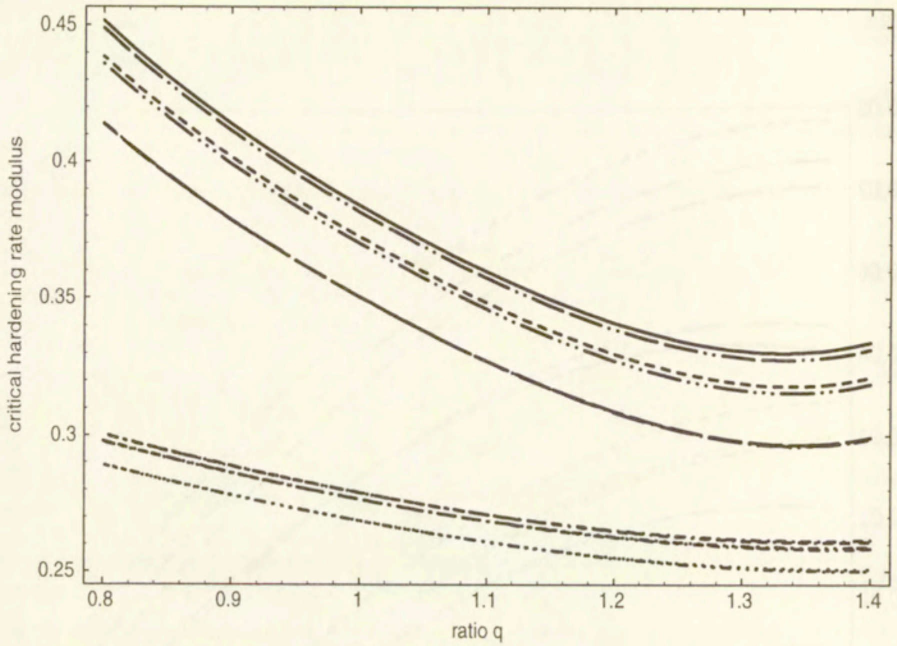

FIG. 2. Numerical results for symmetric double slip process for the hardening modulus rate h/τ^{22} as a function of the ratio $q = h_1/h$ for Al–Cu single crystal. For notation of lines see Fig. 1.

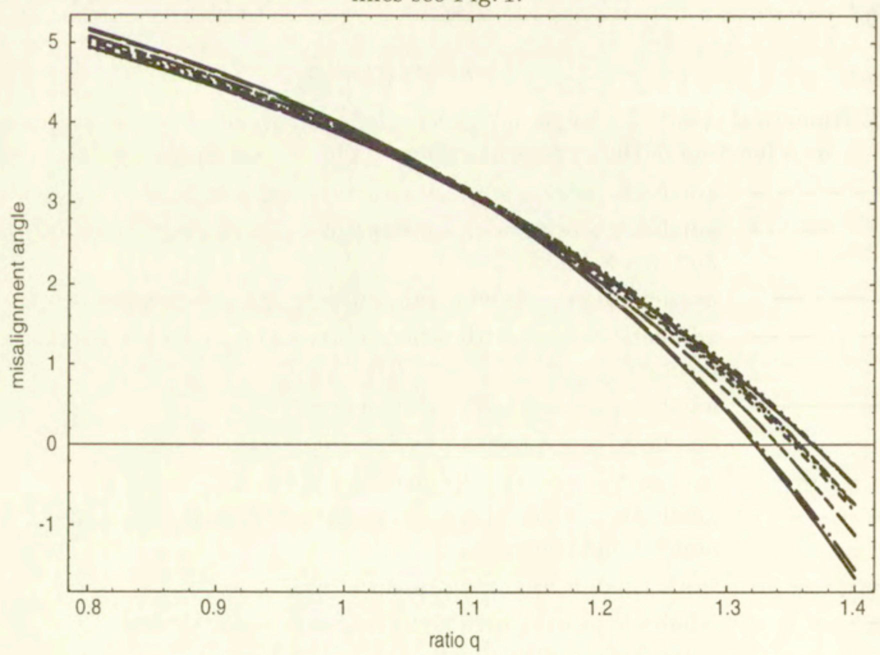

FIG. 3. Numerical results for symmetric double slip process for the misalignment angle δ as a function of the ratio $q = h_1/h$ for Al–Cu single crystal. For notation of lines see Fig. 1.

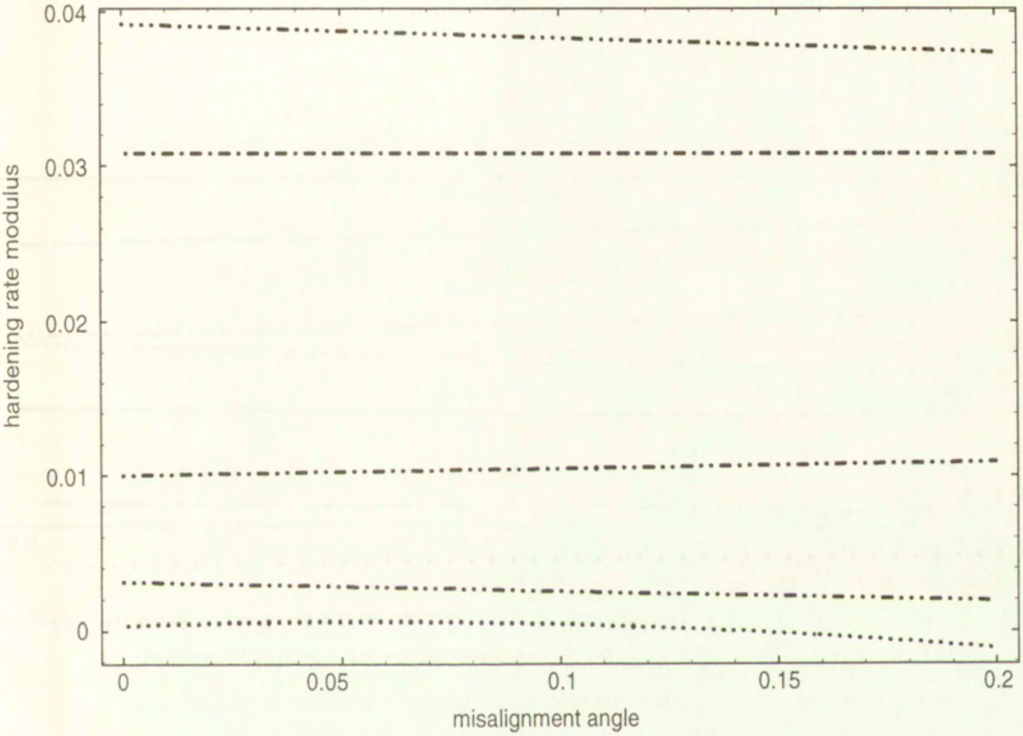

FIG. 4. Synergetic effects for single slip process for the hardening modulus rate h/τ^{22} as a function of the misalignment angle δ for Al–Cu single crystal.

- – · · · · – · · · · – spatial covariance and non-Schmid effects,
- – · – · – · – · – spatial covariance and substructure effects,
- – · · – · · – · · – spatial covariance and thermomechanical effects,
- – · · · – · · · – · spatial covariance and non-dissipative thermal effects,
- · · · · · · · · · · · · · · isothermal process (spatial covariance effect).

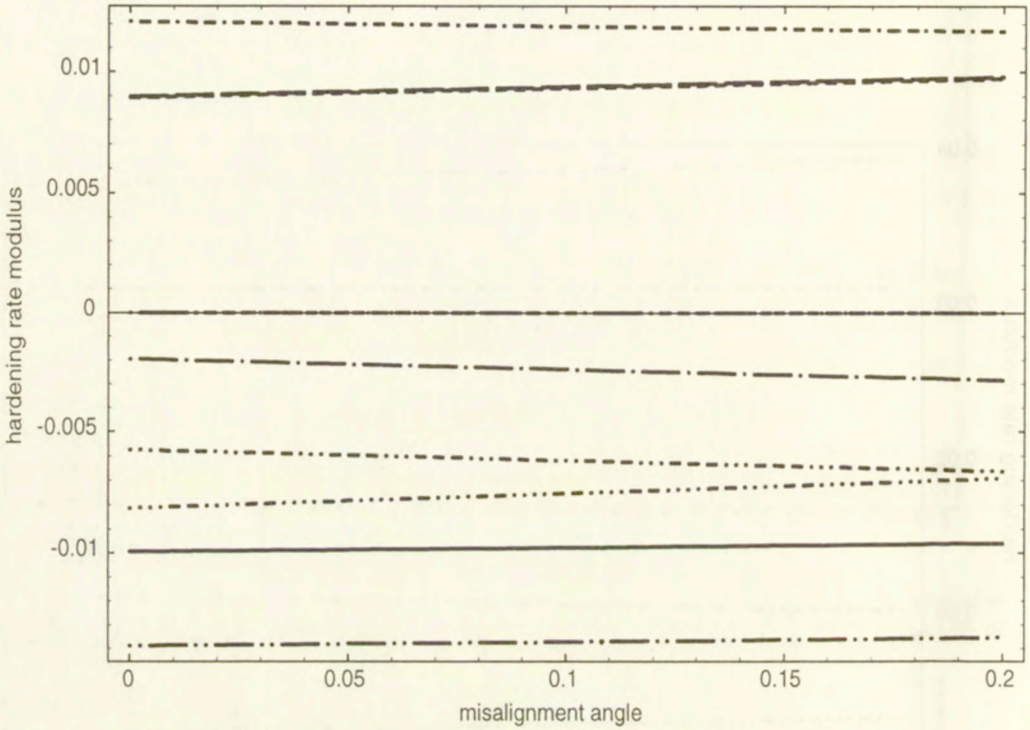

FIG. 5. Synergetic effects for single slip process for the hardening modulus rate h/τ^{22} as a function of the misalignment angle δ for Al–Cu single crystal.

— — · — — · — —	substructure and non-dissipative thermal effects,
— — — — — — — —	substructure and thermomechanical effects,
—— —— —— —— ——	substructure, thermomechanical and non-dissipative thermal effects,
— — — · — — — · —	substructure and non-Schmid effects,
— — · · — — · · —	thermomechanical and non-dissipative thermal effects,
—— · —— · —— ·	substructure, thermomechanical and non-Schmid effects,
— — — · · · — — — · · ·	thermomechanical and non-Schmid effects,
— — · · · · — — · · · ·	non-dissipative thermal and non-Schmid effects,
————————	substructure, thermomechanical, non-dissipative thermal and non-Schmid effects,
—— · · —— · · ——	thermomechanical, non-dissipative thermal and non-Schmid effects.

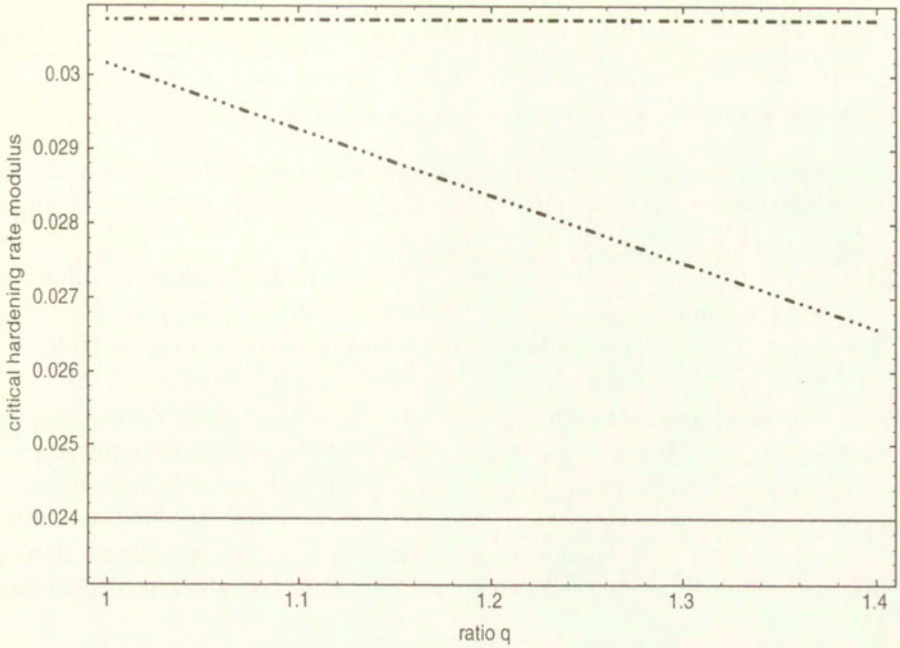

FIG. 6. Synergetic effects for symmetric double slip process for the hardening modulus rate h/τ^{22} as a function of the ratio $q = h_1/h$ for Al–Cu single crystal. For notation of lines see Fig. 4.

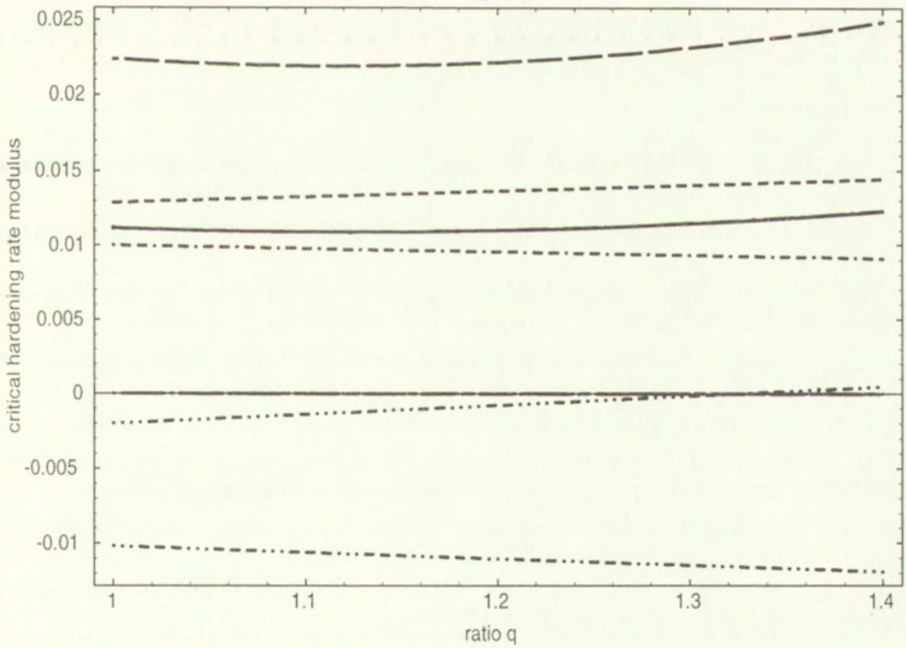

FIG. 7. Synergetic effects for symmetric double slip process for the hardening modulus rate h/τ^{22} as a function of the ratio $q = h_1/h$ for Al–Cu single crystal. For notation of lines see Fig. 5.

8. Final comments

The evolution of the dislocation substructure affects very much the results for the critical modulus rate $(h/\tau^{22})_{\text{crit}}$. It has no influence on the misalignment angle δ.

Two cooperative phenomena, namely the thermomechanical coupling and the dislocation substructure give distinct synergetic effect (mostly on $(h/\tau^{22})_{\text{crit}}$).

The change of the irreversibility coefficient χ in the range of $(0.65 - 0.85)$ does not give important influence on the localization results.

Comparison of the theoretical results for $(h/\tau^{22})_{\text{crit}}$ with those obtained experimentally by CHANG and ASARO [7] and SPITZIG [35] shows good agreement.

Comparison of the theoretical results for the misalignment angle δ with those obtained experimentally by CHANG and ASARO [7], LISIECKI, NELSON and ASARO [18] and SPITZIG [35] shows good agreement only for symmetric double slip process when the geometry of the deformed specimen is taken into consideration.

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Treating a singular case for a motion of rigid body in a Newtonian field of force

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THIS PAPER presents a rotational motion of a rigid body about a fixed point in a Newtonian force field for a singular value of the natural frequency $\omega = 3$. Such a singularity appears in [3] and has been never studied in full generality. Poincaré's small parameter method [4] is applied to investigate analytical periodic solutions, with non-zero basic amplitudes, for equations of motion of the body. A geometric interpretation of motion is given using Euler's angles to describe the orientation of the body at any instant of time.

1. Introduction

CONSIDER A RIGID BODY of mass (M), with one fixed point; its ellipsoid of inertia is arbitrary, and its center of mass does not necessarily coincide with the fixed point. Assume that $0x$, $0y$ and $0z$ represent the principal axes of the ellipsoid of inertia (fixed frame of the body), and $0X$, $0Y$ and $0Z$ represent the fixed frame in space. Assume A, B and C to be the principal moments of inertia, x_0, y_0 and z_0 to be the coordinates of the center of mass in the moving coordinate system, γ, γ' and γ'' to be the direction cosines of the vertical Z-axis, directed downwards and p, q and r to be the projections of the angular velocity vector of the body on the principal axes of inertia. It is taken into consideration that at the initial instant of time, the body rotates about z-axis with a high angular velocity r_0 and that this axis makes an angle $\theta_0 \neq m\pi/2$ $(m = 0, 1, 2, \ldots)$ with Z-axis. The six nonlinear differential equations of motion and their three first integrals are reduced to the following system of two degrees of freedom and one first integral [3]

$$(1.1)\qquad \begin{aligned} \ddot{p}_2 + 9p_2 &= \mu^2 F(p_2, \dot{p}_2, \gamma_2, \dot{\gamma}_2, \mu), \\ \ddot{\gamma}_2 + \gamma_2 &= \mu^2 \Phi(p_2, \dot{p}_2, \gamma_2, \dot{\gamma}_2, \mu); \end{aligned}$$

$$(1.2)\qquad \begin{aligned} \gamma_0''^{-2} - 1 = \gamma_2^2 + \dot{\gamma}_2^2 + 2\mu(\nu p_2\gamma_2 + \nu_2\dot{p}_2\dot{\gamma}_2 + s_{21}) \\ + \mu^2\left[\nu_2^2\dot{p}_2^2 - 2\dot{\gamma}_2\left(e_2A_1^{-1}\dot{\gamma}_2 + A_1^{-1}\dot{p}_2 s_{21} + \frac{1}{2}\dot{\gamma}_2 s_{11} - y_0' a^{-1}A_1^{-1}\right)\right. \\ \left. + \nu^2 p_2^2 + s_{21}^2 + 2\left(s_{22} - \frac{1}{2}s_{11}\right)\right] + \mu^3(\ldots), \end{aligned}$$

where

$$
\begin{aligned}
F &= F_2 + \mu F_3 + \ldots, & \Phi &= \Phi_2 + \mu\Phi_3 + \ldots,\\
F_2 &= f_2 + 8\nu e_1 p_2\,, & \Phi_2 &= \phi_2 - 8\nu(e + e_1\gamma_2),\\
F_3 &= f_3 - e_1\phi_2 + 8\nu e_1(e + e_1\gamma_2), & \Phi_3 &= \phi_3 - \nu f_2 - 8\nu^2 e_1 p_2\,,
\end{aligned}
$$

$$
\begin{aligned}
f_2 = {} & A_1 b^{-1} x_0' s_{21} - 9p_2 s_{11} + C_1 A_1^{-1} p_2 \dot{p}_2^2 - y_0' a^{-1} p_2 \dot{\gamma}_2\\
& - y_0' A_1^{-1}(A_1 + a^{-1})\gamma_2 \dot{p}_2 + x_0' \dot{p}_2 \dot{\gamma}_2 - z_0' a^{-1} p_2\\
& - k\left[(1 - C_1)\gamma_2 \dot{p}_2 \dot{\gamma}_2 + A_1(1 + B_1)\gamma_2 s_{21} - A_1 p_2 (1 - \dot{\gamma}_2^2)\right],\\
\phi_2 = {} & -\gamma_2 s_{11} + (1 + B_1) p_2 s_{21} - (1 - C_1) A_1^{-1} p_2 \dot{p}_2 \dot{\gamma}_2 + x_0' \dot{\gamma}_2^2 - y_0' \gamma_2 \dot{\gamma}_2\\
& - z_0' b^{-1}\gamma_2 + x_0' b^{-1} - A_1^{-2}\gamma_2 \dot{p}_2^2 + k(C_1 \dot{\gamma}_2^2 - B_1)\gamma_2\,,\\
f_3 = {} & C_1 A_1^{-1} \dot{p}_2 \left[e\dot{p}_2 + e_1 \gamma_2 \dot{p}_2 - 2p_2(y_0' a^{-1} - e_2 \dot{\gamma}_2)\right]\\
& - 9(e s_{11} + e_1 \gamma_2 s_{11} + 2p_2 s_{12}) + A_1 b^{-1} x_0' s_{22}\\
& + x_0'\left[\nu_2 \dot{p}_2^2 - \dot{\gamma}_2(y_0' a^{-1} - e_2 \dot{\gamma}_2)\right] - y_0' a^{-1}\left[\dot{\gamma}_2(e + e_1\gamma_2) + \nu_2 p_2 \dot{p}_2\right]\\
& + y_0'(1 + A_1^{-1} a^{-1})\left[\gamma_2(y_0' a^{-1} - e_2\dot{\gamma}_2) - \nu p_2 \dot{p}_2\right]\\
& + \frac{1}{2} z_0'(a^{-1} - A_1 b^{-1})\gamma_2 s_{11} - z_0' a^{-1}(e + e_1\gamma_2 + p_2 s_{21})\\
& + k\Big[(1 - C_1)(y_0' a^{-1} - e_2\dot{\gamma}_2)\gamma_2\dot{\gamma}_2 - \nu(1 - C_1)p_2 \dot{p}_2 \dot{\gamma}_2\\
& - 2\nu_2 A_1 p_2 \dot{p}_2 \dot{\gamma}_2 - \nu_2(1 - C_1)\gamma_2 \dot{p}_2^2 - \nu A_1(1 + B_1) p_2 s_{21} + 2A_1 p_2 s_{21}\\
& + (9 - A_1)\gamma_2 s_{22} + A_1(e + e_1\gamma_2)(1 - \dot{\gamma}_2^2)\Big],\\
\phi_3 = {} & 2x_0'\nu_2 \dot{p}_2 \dot{\gamma}_2 - 2\gamma_2 s_{12} - \nu p_2 s_{11} + (1 + B_1)[p_2 s_{22} + (e + e_1\gamma_2)s_{21}]\\
& + (1 - C_1)A_1^{-1}\left[p_2\dot{\gamma}_2(y_0' a^{-1} - e_2\dot{\gamma}_2) - \nu_2 p_2 \dot{p}_2^2 - (e + e_1\gamma_2)\dot{p}_2\dot{\gamma}_2\right]\\
& - z_0' b^{-1}(\nu p_2 + \gamma_2 s_{21}) + 2x_0' b^{-1} s_{21} + A_1^{-2}\left[2\gamma_2 \dot{p}_2(y_0' a^{-1} - e_2\dot{\gamma}_2) - \nu p_2 \dot{p}_2^2\right]\\
& - y_0'(\nu p_2 \dot{\gamma}_2 + \nu_2 \gamma_2 \dot{p}_2) + k\left[\nu p_2(C_1\dot{\gamma}_2^2 - B_1) + 2\gamma_2(\nu_2 C_1 \dot{p}_2 \dot{\gamma}_2 - B_1 s_{21})\right];
\end{aligned}
\tag{1.3}
$$

$$
\begin{aligned}
p_2 = {} & p_1 - \mu e - \mu e_1 \gamma_2\,, \qquad \gamma_2 = \gamma_1 - \mu\nu p_2\,,\\
q_1 = {} & -A_1^{-1}\dot{p}_2 + \mu A_1^{-1}(y_0' a^{-1} - e_2\dot{\gamma}_2) + \mu^2\Big[(aA_1)^{-1} y_0' s_{21} + \frac{1}{2} A_1^{-1}\dot{p}_2 s_{11}\\
& + k\dot{\gamma}_2 s_{21} - \nu_2 \dot{p}_2(a^{-1}A_1^{-1} z_0' - k)\Big] + \mu^3\Big[(aA_1)^{-1} y_0' s_{22} + \frac{1}{2} A_1^{-1} e_1 \dot{\gamma}_2 s_{11}\\
& + A_1^{-1}\dot{p}_2 s_{12} + a^{-1}A_1^{-2} e_1 z_0' \dot{\gamma}_2 + a^{-1}A_1^{-1} s_{11}(z_0'\dot{\gamma}_2 - y_0')\\
& + (k - a^{-1}A_1^{-1} z_0')(a^{-1}A_1^{-1} y_0' - a^{-1}A_1^{-1} z_0'\dot{\gamma}_2 + k\dot{\gamma}_2) + a^{-1}A_1^{-2} z_0' \dot{p}_2 s_{21}\\
& + k(\nu\dot{p}_2 s_{21} + \dot{\gamma}_2 s_{22} - A_1^{-1} e_1\dot{\gamma}_2 - \frac{3}{2}\dot{\gamma}_2 s_{11} - 2A_1^{-1}\dot{p}_2 s_{21})\Big] + \ldots,
\end{aligned}
\tag{1.4}
$$

(1.4) [cont.]

$$r_1 = 1 + \frac{1}{2}\mu^2 s_{11} + \mu^3 s_{12} + \ldots,$$

$$\gamma_1' = \dot{\gamma}_2 + \mu\nu_2\dot{p}_2 + \mu^2\left[(aA_1)^{-1}y_0' - A_1^{-1}(e_2\dot{\gamma}_2 + \dot{p}_2 s_{21}) - \frac{1}{2}\dot{\gamma}_2 s_{11}\right]$$
$$+ \mu^3\Big[-A_1^{-1}(e_1\dot{\gamma}_2 s_{21} + \dot{p}_2 s_{22}) + \frac{1}{2}(3A_1^{-1} - \nu)\dot{p}_2 s_{11} - \dot{\gamma}_2 s_{12}$$
$$+ \nu_2(k - a^{-1}A_1^{-1}z_0')\dot{p}_2 + 2a^{-1}A_1^{-1}y_0' s_{21} + (2k - a^{-1}A_1^{-1}z_0')\dot{\gamma}_2 s_{21}\Big] + \ldots,$$

$$\gamma_1'' = 1 + \mu s_{21} + \mu^2\left(s_{22} - \frac{1}{2}s_{11}\right) - \mu^3\left(s_{12} + \frac{1}{2}s_{11}s_{21}\right) + \ldots;$$

(1.5)

$$p_1 = p/c\sqrt{\gamma_0''}, \quad q_1 = q/c\sqrt{\gamma_0''}, \quad r_1 = r/r_0, \quad \gamma_1 = \gamma/\gamma_0'',$$
$$\gamma_1' = \gamma'/\gamma_0'', \quad \gamma_1'' = \gamma''/\gamma_0'', \quad r = r_0 t, \quad (\cdot \equiv d/d\tau);$$

(1.6)

$$s_{11} = a(p_{20}^2 - p_2^2) + b(\dot{p}_{20}^2 - \dot{p}_2^2)/A_1^2 - 2\left[x_0'(\gamma_{20} - \gamma_2) + y_0'(\dot{\gamma}_{20} - \dot{\gamma}_2)\right]$$
$$+ k\left[a(\gamma_{20}^2 - \gamma_2^2) + b(\dot{\gamma}_{20}^2 - \dot{\gamma}_2^2)\right],$$

$$s_{12} = a\left[e(p_{20} - p_2) + e_1(p_{20}\gamma_{20} - p_2\gamma_2)\right]$$
$$- bA_1^{-2}\left[y_0'a^{-1}(\dot{p}_{20} - \dot{p}_2) - e_2(\dot{p}_{20}\dot{\gamma}_{20} - \dot{p}_2\dot{\gamma}_2)\right]$$
$$- vx_0'(p_{20} - p_2) - \nu_2 y_0'(\dot{p}_{20} - \dot{p}_2) + (z_0' - k)s_{21}$$
$$+ k\left[\nu a(p_{20}\gamma_{20} - p_2\gamma_2) + \nu_2 b(\dot{p}_{20}\dot{\gamma}_{20} - \dot{p}_2\dot{\gamma}_2)\right],$$

$$s_{21} = a(p_{20}\gamma_{20} - p_2\gamma_2) - bA_1^{-1}(\dot{p}_{20}\dot{\gamma}_{20} - \dot{p}_2\dot{\gamma}_2),$$

$$s_{22} = a\left[\nu(p_{20}^2 - p_2^2) + e(\gamma_{20} - \gamma_2) + e_1(\gamma_{20}^2 - \gamma_2^2)\right]$$
$$+ bA_1^{-1}\left[-\nu_2(\dot{p}_{20}^2 - \dot{p}_2^2) + a^{-1}y_0'(\dot{\gamma}_{20} - \dot{\gamma}_2) - e_2(\dot{\gamma}_{20}^2 - \dot{\gamma}_2^2)\right];$$

(1.7)

$$A_1 = \frac{C - B}{A}, \quad B_1 = \frac{A - C}{B}, \quad C_1 = \frac{B - A}{C}, \quad \gamma_0 \geq 0, \quad 0 < \gamma_0'' < 1,$$

$$a = \frac{A}{C}, \quad b = \frac{B}{C}, \quad c^2 = \frac{Mg\ell}{C},$$

$$\mu = \frac{c\sqrt{\gamma_0''}}{r_0}, \quad x_0 = \ell x_0', \quad y_0 = \ell y_0',$$

$$z_0 = \ell z_0', \quad \ell^2 = x_0^2 + y_0^2 + z_0^2, \quad A_1 B_1 = -9, \quad e = \frac{1}{9}x_0' A_1 b^{-1},$$

$$e_1 = \frac{1}{8}\left[k(9 - A_1) + z_0'(a^{-1} - A_1 b^{-1})\right],$$

$$\nu = -\frac{1}{8}(1 + B_1), \quad e_2 = e_1 + a^{-1}z_0' - kA_1,$$

$$\nu_2 = \nu - A_1^{-1}, \quad k = N\gamma_0''/c^2, \quad N = 3g/R, \quad g = \lambda/R^2.$$

Here R is the distance from the fixed point to the attracting center; λ is the coefficient of attraction of such a center; p_0, q_0, r_0, γ_0, γ_0' and γ_0'' are the initial values of the corresponding variables. Since r_0 is very large, then μ is considered as a small parameter.

2. Proposed method

In this section, Poincaré's small parameter method is applied to satisfy periodic solutions, with non-zero basic amplitudes, of system (1.1). For such a considered system, the following generating system ($\mu = 0$) is obtained

$$\ddot{p}_2^{(0)} + 9p_2^{(0)} = 0, \qquad \ddot{\gamma}_2^{(0)} + \gamma_2^{(0)} = 0, \tag{2.1}$$

which gives periodic solutions in the forms

$$p_2^{(0)} = M_1 \cos 3\tau + M_2 \sin 3\tau, \qquad \gamma_2^{(0)} = M_3 \cos \tau, \tag{2.2}$$

with the period $T_0 = 2\pi$, and M_1, M_2 and M_3 which are constants. Consider the following initial condition

$$\dot{\gamma}_2(0, \mu) = 0, \tag{2.3}$$

which does not affect the generality of the required solutions [4].

The periodic solutions for system (1.1) are expressed by the following forms [5]

$$\begin{aligned} p_2(\tau, \mu) &= \widetilde{M}_1 \cos 3\tau + \widetilde{M}_2 \sin 3\tau + \sum_{k=2}^{\infty} \mu^k G_k(\tau), \\ \gamma_2(\tau, \mu) &= \widetilde{M}_3 \cos \tau + \sum_{k=2}^{\infty} \mu^k H_k(\tau), \end{aligned} \tag{2.4}$$

where

$$\widetilde{M}_i = M_i + \beta_i \qquad (i = 1, 2, 3), \tag{2.5}$$

$$\begin{aligned} U = u &+ \frac{\partial u}{\partial M_1}\beta_1 + \frac{\partial u}{\partial M_2}\beta_2 + \frac{\partial u}{\partial M_3}\beta_3 \\ &+ \frac{1}{2}\frac{\partial^2 u}{\partial M_1^2}\beta_1^2 + \ldots, \qquad \begin{Bmatrix} U = G_k, & H_k, \\ u = g_k, & h_k \end{Bmatrix}, \end{aligned} \tag{2.6}$$

the quantities β_1, $3\beta_2$ and β_3 representing the deviations of the initial values of p_2, $\dot{p}_2$ and γ_2 of system (1.1) from their initial values of system (2.1); these deviations are functions of μ and satisfy the condition $\beta_i(0) = 0$. These functions $g_k(\tau)$ and $h_k(\tau)$ take the forms [1]

(2.7)
$$g_k(\tau) = \frac{1}{3}\int_0^\tau F_k^{(0)}(t_1)\sin 3(\tau - t_1)\,dt_1\,,$$
$$h_k(\tau) = \int_0^\tau \Phi_k^{(0)}(t_1)\sin(\tau - t_1)\,dt_1 \qquad (k = 2, 3).$$

The solutions (2.4) have the period $T = T_0 + \alpha(\mu)$ which reduces to T_0 at $\mu = 0$, that is $\alpha(0) = 0$. The initial condition (2.3) can be rewritten using the following relations:

(2.8)
$$p_2(0,\mu) = \widetilde{M}_1, \qquad \dot{p}_2(0,\mu) = 3\widetilde{M}_2,$$
$$\gamma_2(0,\mu) = \widetilde{M}_3, \qquad \dot{\gamma}_2(0,\mu) = 0.$$

The solutions (2.2) are rewritten in the following forms

(2.9)
$$p_2^{(0)} = E\cos(3\tau - \varepsilon), \qquad \gamma_2^{(0)} = M_3\cos\tau,$$

where $E = \sqrt{M_1^2 + M_2^2}$ and $\varepsilon = \tan^{-1} M_2/M_1$. Making use of (2.9) and (1.4), one gets

(2.10)
$$s_{ij}^{(0)} = s_{ij}^{(0)}\left(p_2^{(0)}, \dot{p}_2^{(0)}, \gamma_2^{(0)}, \dot{\gamma}_2^{(0)}\right) \qquad (i, j = 1, 2).$$

The functions $F_k^{(0)}$ and $\Phi_k^{(0)}$ are obtained from (2.9), (2.10) and (1.3). Then making use of (2.7) one obtains $g_k(2\pi)$, $h_k(2\pi)$, $\dot{g}_k(2\pi)$ and $\dot{h}_k(2\pi)$. The quantity $\widetilde{M}_3$ is determined by means of (2.8) into the first integral (1.2), with $\tau = 0$, and can be written in the form

(2.11)
$$\widetilde{M}_3 = \sqrt{1 - \gamma_0''/\gamma_0''} - \mu\nu\widetilde{M}_1 - 9\mu^2\nu_2^2\widetilde{M}_2^2/2M_3 - 3\mu^3 y_0'\nu_2\widetilde{M}_2/aA_1M_3 + \ldots\,.$$

The independent periodicity conditions [2] of the solutions $p_2(\tau,\mu)$, $\dot{p}_2(\tau,\mu)$, $\gamma_2(\tau,\mu)$ and $\dot{\gamma}_2(\tau,\mu)$ take the following forms:

(2.12)
$$L_{310} + \mu(\ldots) = 0, \qquad (\widetilde{L}_{21} - 9\widetilde{N}_{21}) + \mu(\ldots) = 0;$$

(2.13)
$$\alpha(\mu) = \mu^2\widetilde{M}_3^{-1}\left[\dot{H}_2(2\pi) + \mu\dot{H}_3(2\pi) + \ldots\right],$$

where

$$L_{310} = \frac{1}{4}k\Big[8e_1(a-b) - \frac{1}{2}z_0'(a^{-1} - A_1 b^{-1})(a-b) + e_2(1 - C_1) + e_1 A_1 + (A_1 - 9)(ae_1 + be_2 A_1^{-1})\Big],$$

(2.14)
$$\widetilde{L}_{21} - 9\widetilde{N}_{21} = a_1(\widetilde{M}_1^2 + \widetilde{M}_2^2) - [a_2 + 9kb(2M_3\beta_3 + \beta_3^2)],$$
$$a_1 = (a-1)(a+b-2)/2b,$$
$$a_2 = z_0'(ab)^{-1}[3(a+b) - 2(2ab+1)] + 18k\left[1 - (a+b) + \frac{1}{2}bM_3^2\right].$$

Equations of the basic amplitudes of (2.12) give

$$M_i = \pm\left[a_2 a_1^{-1} - M_j^2\right]^{1/2} \qquad (i = 1,2, \quad j = 2,1). \tag{2.15}$$

The functions β_1 and β_2 are assumed in the forms

$$\beta_1 = \sum_{k=1}^{3}\mu^k \ell_k + O(\mu^4), \qquad \beta_2 = \sum_{k=1}^{3}\mu^k m_k + O(\mu^4). \tag{2.16}$$

Making use of (2.16), (2.12) and (2.5), one obtains

(2.17)
$$\ell_1 = -a_1^{-1}M_1^{-1}[a_1 M_2 m_1 + 9bk\nu M_1 M_3],$$
$$\ell_2 = \frac{1}{2}a_1^{-1}M_1^{-1}[9bk\nu^2 M_1^2 - 18bkM_3(\nu\ell_1 + 9\nu_2^2 M_2^2/2M_3) - a_1(m_1^2 + \ell_1^2 + 2M_2 m_2)],$$
$$\ell_3 = \frac{1}{2}a_1^{-1}M_1^{-1}[9b\nu k M_1(2\nu\ell_1 + 9\nu_2^2 M_2^2 M_3^{-1}) - 54kb\nu_2 M_2 y_0' a^{-1}A_1^{-1} - 2a_1(m_1 m_2 + M_2 m_3 + \ell_1\ell_2)].$$

Having Eqs. (2.15) and (2.17), we get a family of arbitrary solutions for the constants M_1 and M_2, and the quantities β_1 and β_2. Equations (2.6) and (2.7) give the functions $G_k(\tau)$ and $H_k(\tau)$; then, the periodic solutions (2.4) are constructed up to the third power of μ. Making use of (1.5) and (1.6), the following periodic solutions are obtained:

(2.18)
$$p = c\sqrt{\gamma_0''}\Big\{M_1\cos 3\tau + M_2\sin 3\tau + \mu(e + \ell_1\cos 3\tau + m_1\sin 3\tau + e_1 M_3\cos\tau) + \mu^2\sum_{i=0}^{9}(Q_{1i}\cos i\tau + Q_{1i}'\sin i\tau) + \mu^3\sum_{j=0}^{9}(Q_{2j}\cos j\tau + Q_{2j}'\sin j\tau)\Big\} + \ldots, \qquad i \neq 6,7,8, \quad j \neq 8,$$

(2.18) [cont.]

$$q = c\sqrt{\gamma_0''}\Big\{A_1^{-1}(3M_1\sin 3\tau - 3M_2\cos 3\tau) + \mu A_1^{-1}(y_0' a^{-1} + e_2 M_3 \sin\tau + 3\ell_1 \sin 3\tau - 3m_1\cos 3\tau) + \mu^2\sum_{i=0}^{9}(Q_{10i}\cos i\tau + Q_{10i}'\sin i\tau) + \mu^3\sum_{j=0}^{9}(Q_{11j}\cos j\tau + Q_{11j}'\sin j\tau)\Big\} + \ldots, \qquad i\neq 6,7,8, \quad j\neq 8,$$

$$r = r_0\Big\{1 + \frac{1}{2}\mu^2\Big\{E^2\left[a\cos^2\varepsilon - \frac{1}{2} + 9bA_1^{-2}\left(\sin^2\varepsilon - \frac{1}{2}\right)\right] - 2M_3[x_0'(1-\cos\tau) + y_0'\sin\tau] - \frac{1}{2}kC_1M_3^2(1-\cos 2\tau) + \frac{1}{2}E^2(9bA_1^{-2} - a)(\sin 2\varepsilon\sin 6\tau + \cos 2\varepsilon\cos 6\tau)\Big\} + \mu^3\sum_{i=0}^{6}(Q_{5i}\cos i\tau + Q_{5i}'\sin i\tau\Big\} + \ldots, \qquad i\neq 5,$$

$$\gamma = \gamma_0''\Big\{M_3\cos\tau + \mu\nu\,[M_1(\cos 3\tau - \cos\tau) + M_2\sin 3\tau] + \mu^2\sum_{i=0}^{7}(Q_{3i}\cos i\tau + Q_{3i}'\sin i\tau) + \mu^3\sum_{j=0}^{9}(Q_{4j}\cos j\tau + Q_{4j}'\sin j\tau)\Big\} + \ldots, \qquad i\neq 4,6, \quad j\neq 6,8,$$

$$\gamma' = \gamma_0''\Big\{-M_3\sin\tau + \mu\,[\nu M_1\sin\tau + 3\nu_2(M_2\cos 3\tau - M_1\sin 3\tau)] + \mu^2\sum_{i=0}^{7}(Q_{8i}\cos i\tau + Q_{8i}'\sin i\tau) + \mu^3\sum_{j=0}^{9}(Q_{9j}\cos j\tau + Q_{9j}'\sin j\tau)\Big\} + \ldots, \qquad i\neq 4,6, \quad j\neq 6,8,$$

$$\gamma'' = \gamma_0''\Big\{1 + \mu M_3 E\left[a\cos\varepsilon + \frac{1}{2}(3bA_1^{-1} - a)(\cos\varepsilon\cos 2\tau + \sin\varepsilon\sin 2\tau) - \frac{1}{2}(3bA_1^{-1} + a)(\cos\varepsilon\cos 4\tau + \sin\varepsilon\sin 4\tau)\right] + \mu^2\sum_{i=0}^{6}(Q_{6i}\cos i\tau + Q_{6i}'\sin i\tau) + \mu^3\sum_{j=0}^{10}(Q_{7j}\cos 7\tau + Q_{7j}'\sin j\tau)\Big\} + \ldots, \qquad i\neq 3,5, \quad j\neq 7,9,$$

and the correction of the period $\alpha(\mu)$ becomes

$$\alpha(\mu) = \pi\mu^2 N_{21} + \mu^3\pi(N_{21}^* + N_{31} + N_{38}M_1M_3) + \ldots, \tag{2.19}$$

where the constants N, Q and Q' are determined in terms of the rigid body motion parameters and occupy about twenty pages. The symbols $(\ldots)$ mean terms of order higher than $O(\mu^3)$.

3. Geometric interpretation of motion

Analyzing the obtained motion of the rigid body about a fixed point, using the Eulerian angles θ, ψ and ϕ, the following relations are obtained [6],

$$\begin{aligned} \cos\theta &= \gamma'', & \frac{d\psi}{dt} &= \frac{p\gamma + q\gamma'}{1-\gamma''^2}, \\ \tan\phi_0 &= \frac{\gamma_0}{\gamma_0'}, & \frac{d\phi}{dt} &= r - \frac{d\psi}{dt}\cos\theta. \end{aligned} \tag{3.1}$$

Since the initial system (1.1) is autonomous then the periodic solutions remain periodic if t is replaced by $t+t_0$, where t_0 is an arbitrary constant. Taking into consideration that the initial instant of time corresponds to the instant $t=t_0$, substituting the solutions (2.18) into Eqs. (3.1), the following angles are deduced:

$$\begin{aligned} &\phi_0 = \frac{\pi}{2} + r_0 t_0 + \ldots, \qquad \theta_0 = \tan^{-1} M_3, \\ &\theta = \theta_0 - \mu\cot\theta_0\{\theta_1(t+t_0) - \theta_1(t_0) + \mu[\theta_2(t+t_0) - \theta_2(t_0)] \\ &\qquad + \mu^2[\theta_3(t+t_0) - \theta_3(t_0)]\} + \ldots, \\ &\psi = \psi_0 + (Mg\ell\alpha_{10}C^{-1}r_0^{-}\cot^2\theta_0)t + \frac{1}{4}\mu\operatorname{cosec}\theta_0[\psi_1(t+t_0) - \psi_1(t_0)] \\ &\qquad + \mu^2\cot\theta_0\operatorname{cosec}\theta_0[\psi_2(t+t_0) - \psi_2(t_0)] + \ldots, \\ &\phi = \phi_0 + A_1^* t - \frac{1}{2}\mu\cot\theta_0[\phi_1(t+t_0) - \phi_1(t_0)] \\ &\qquad + \mu^2[\phi_2(t+t_0) - \phi_2(t_0)] + \ldots, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} &\theta_1(t) = \frac{1}{2}M_3E[(3bA_1^{-1} - a)(\cos\varepsilon\cos 2r_0t + \sin\varepsilon\sin 2r_0t) \\ &\qquad - (3bA_1^{-1} + a)(\cos\varepsilon\cos 4r_0t + (\sin\varepsilon\sin 4r_0t)], \\ &\theta_2(t) = \sum_{i=1}^{6}(Q_{6i}\cos ir_0t + Q'_{6i}\sin ir_0t), \qquad i \neq 3,5, \\ &\theta_3(t) = \sum_{j=1}^{10}(Q_{7j}\cos jr_0t + Q'_{7j}\sin jr_0t), \qquad j \neq 7,9, \\ &\psi_1(t) = (1-3A_1^{-1})(M_1\sin 2r_0t - M_2\cos 2r_0t) \\ &\qquad + \frac{1}{2}(1+3A_1^{-1})(M_1\sin 4r_0t - M_2\cos 4r_0t), \end{aligned} \tag{3.3}$$

(3.3) [cont.]

$$\psi_2(t) = \sum_{i=1}^{8} \frac{1}{i}(\alpha_{1i} \sin ir_0 t - \alpha'_{1i} \cos ir_0 t), \qquad i \neq 3, 5, 7,$$

$$\phi_1(t) = \frac{1}{2}(1 - 3A_1^{-1})(M_1 \sin 2r_0 t - M_2 \cos 2r_0 t) + \frac{1}{4}(1 + 3A_1^{-1})(M_1 \sin 4r_0 t - M_2 \cos 4r_0 t),$$

$$\phi_2(t) = \sum_{j=1}^{8}(\beta_{1j} \cos jr_0 t + \beta'_{1j} \sin jr_0 t), \qquad j \neq 3, 5, 7,$$

the formulae for constants A_1^*, α, α', β and β' occupying about three pages. We note that the expressions for the Eulerian angles θ, ψ and ϕ depend on four arbitrary constants θ_0, ψ_0, ϕ_0 and r_0 (r_0 is large).

4. Discussion of the solutions

In this section, we give a qualitative analysis of the results obtained, and several diagrams, explanations and examples.

The motion considered in this paper is investigated by introducing Euler's angles of nutation θ, precession ψ and pure rotation ϕ, see expressions (3.2). We note that θ is the angle between $0Z$ and $0z$; ψ is the angle between $0X$ and the line $0j$ of intersection of the fixed plane $0XY$ and the moving one $0xy$; and ϕ is the angle between the line $0j$ and the moving axis $0x$, see Fig. 1.

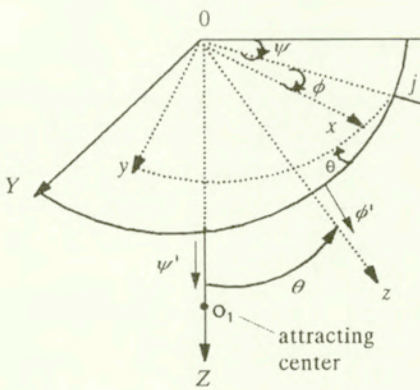

FIG. 1. Representation of Euler's angles.

The zero order approximation of the expressions (3.2) can be formulated as:

$$\theta = \theta_0, \quad \psi = \psi_0 + A^* t, \quad \phi = \phi_0 + A_1^* t, \tag{4.1}$$
$$\theta' = 0, \quad \psi' = A^* = \text{const}, \quad \phi' = A_1^* = \text{const}, \qquad ' = \frac{d}{dt},$$

which represents regular precession with spin A_1^* about $0z$-axis and precession A^* about the fixed axis $0Z$, see for example Fig. 2.

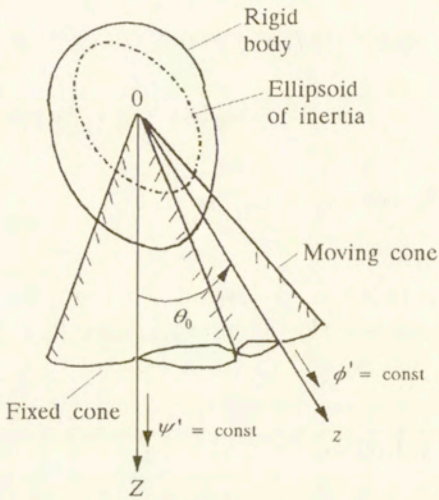

FIG. 2. Representation of zero-order approximation; regular precession.

The first approximation is formulated by

$$\begin{aligned}\theta &= \theta_0 + \mu f_1^*(t), \qquad \psi = \psi_0 + A^* t + \mu f_2^*(t),\\ \phi &= \phi_0 + A_1^* t + \mu f_3^*(t),\\ \theta' &= \mu f_1^{*'}(t), \qquad \psi' = A^* + \mu f_2^{*'}(t),\\ \phi' &= A_1^* + \mu f_3^{*'}(t),\end{aligned} \tag{4.2}$$

where f_i^* and $f_i^{*'}$ are periodic functions with periods proportional to $1/r_0$ which means that they are of a fast character. The formulas (4.2) indicate, up to the first approximation, a perturbed pseudo-regular precession, see for example Fig. 3.

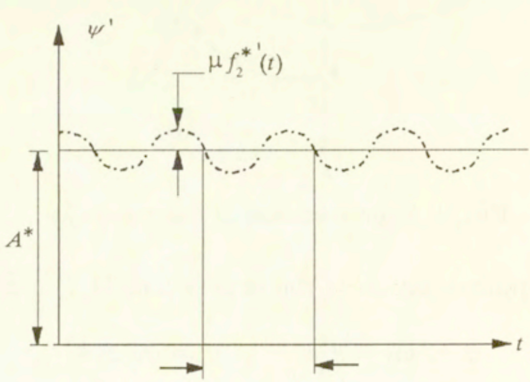

FIG. 3. A perturbed pseudo-regular precession for the first-order approximation.

The second and the third approximations represent the perturbations of the pseudo-regular precession and improves, qualitatively, the geometric interpretation of motion.

5. Conclusions

The periodic solutions with non-zero basic amplitudes for the system of the equations of motion, of the singular case $\omega = 3$, are investigated using the small parameter method of Poincaré. This problem deals with rigid bodies being classified according to the moments of inertia as follows:

1. $C > A > B, \quad B < \frac{1}{4}C, \quad A > \frac{1}{4}C,$
2. $C > B > A, \quad A < \frac{1}{4}C, \quad B > \frac{1}{4}C.$

The obtained solutions are considered as a generalization of the corresponding ones in the uniform gravity field ($k = 0$). Such solutions contain the solutions for the special cases of the basic amplitudes ($M_1 = M_2 = 0$; $M_1 \neq 0$, $M_2 = 0$ and $M_1 = 0$, $M_2 \neq 0$). The geometric interpretation of motion (using Euler's angles) is obtained to show the orientation of the body at any instant of time.

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2-D boundary value problems of thermoelasticity in a multi-wedge – multi-layered region
Part 1. Sweep method

G.S. MISHURIS (RZESZÓW)

A METHOD PROPOSED earlier to solve the BVP for Poisson's equation in a domain consisting of wedges and plane layers, is discussed and applied to 2D thermoelasticity problems. Linear conditions of general form are prescribed along the exterior boundaries as well as at all the interfaces. The essence of the method consists in combining the Fourier and Mellin transforms along the common interface. This allows to reduce the boundary value problems to special systems of singular equations. The analysis is significantly simplified by incorporating the fact that layers and wedges represent chain-like systems. In the paper, relations between the Fourier (Mellin) transformations of solutions for the layered (wedge-shaped) part of the domain are found by using the sweep method of LINKOV and FILIPPOV (1991, Mecchanica, 26, 195–209). All matrix-functions in the relations are slowly increasing ones. Their asymptotic behaviour is analyzed depending on the types of the exterior boundary conditions.

1. Introduction

ELASTICITY PROBLEMS for inhomogeneous bodies of regular structures (for example layered media) were intensively investigated in [2, 4, 5, 6, 7, 11, 12, 27, 32, 34, 39, 40]. We do not discuss here the problems and methods of their solution for composite laminates or periodic composite plates (see for this purpose [13, 31]). The important point of the mentioned bodies is the fact that they have chain-like structures. Independently of the applied technique (FEM, BEM, Fourier transform, etc.) in each layer, this made it possible to use the methods for chain-like systems of rods and beams to solve the problems under consideration. One of the most commonly encountered methods is the so-called "transfer-matrix" method having various modifications (see for example [1, 9, 17, 19, 30, 33]).

However, in the process of adaptation of this method, an intrinsic defect often occurs: the square matrices are ill-conditioned, and this defect is redoubled for products of matrices. Shortages of the "transfer-matrix" method are discussed in details in [16, 17, 19].

To eliminate such difficulties, many authors used special modifications of the method [18, 19, 38]. The modifications were based on the explicit forms of the boundary conditions between the layers.

As it was noted in [16], all the mentioned modifications were particular cases of a "sweep method" in fact. "General sweep method" for layered medium with arbitrary boundary conditions along the interfaces was proposed and investigated in [16]. It consists in reduction of the problems to three-points difference equations. Corresponding results are then based on the theory of difference equations (see for example [10]). The stability of the proposed method is investigated, and conclusions of its efficiency and interconnections with the other methods are presented in [16], depending on the number of the layers and some types of the intermediate conditions.

For the case of multi-wedge bodies analogous results are obtained in [3]. Besides, the last work allows us to build stable algorithm to calculate parameters determining singularity of the gradient of solutions near the common corner tip for an arbitrary number of the wedges.

In [23, 24] classical two-dimensional boundary value problems for Poisson's equation in multi-layered – multi-wedge regions are investigated. Then the Fourier and Mellin transforms are applied to based domains (layer and wedge), respectively. As the simplest example of such geometry we can note elasticity problems for a crack normally terminating at the layered media, which were investigated by different techniques in [14, 15, 21, 36] and others.

Previously the idea of using the Fourier and Mellin transforms simultaneously to solve some plane and Mode III problems of linear elasticity for layered media with a notch or, in particular case, a crack was presented in [21, 22]. The notch (crack) was symmetric with respect to the normal to the interfaces, but intermediate boundary conditions were of the "ideal type" (defined by given discontinuities of displacements and tractions along the interfaces). At that time, explicit form of the interconnection formulae for the arising matrices (which takes into account "ideal" type of interfacial conditions) was very important. This made it possible to reduce the problems to a special class of systems of singular integral equations with fixed point singularities, and to investigate symbols of the corresponding systems in some Banach spaces with a relevant weight. The justification of the method [22] in a relevant space of distributions is presented from [25].

In the papers [23, 24] arbitrary numbers of the layers and wedges as well as the types of intermediate and external boundary conditions are considered. At that time, the "sweep method" proposed in [16] plays an important role. Moreover, in Appendix [23] exact asymptotic formulas (which are absent from [16]) for arising functions are obtained for all possible types of exterior and interior boundary conditions. This allows us to reduce the problems by the method [22] to the mentioned class of systems of integral equations and to investigate its symbols. Besides, in Appendix [24] it is shown that general partial differential equations of the divergent form (not only Poisson's equation) in the mentioned regions can be analogously solved.

In this paper, two-dimensional boundary value problems of thermoelasticity (see [28, 29]) in the multi-layered – multi-wedge region are considered. In the first part, necessary formulas derived in the process of solution of different two-dimensional boundary value problems of thermoelasticity in layered (wedge-shaped) media by the "sweep method" are presented. They are generalizations of the formulas obtained in [3, 16]. Besides, asymptotic expansions of resultant matrix-functions near zero and infinity points, similar to those derived in [23], are found and justified for all types of linear interior and exterior boundary conditions. Then in the next part of the paper we shall make it possible to reduce all problems under consideration to systems of integral equations by the method of integral transforms.

General formulations of the problems are presented exactly in the second section of the paper. In the next two sections, the "sweep method" proposed in [16] is consequently applied in the layered and wedge-shaped parts of the domain. The main results consist in the Lemma 1 and Lemma 2.

2. Problem formulation

Let us consider the infinite domain presented in Fig. 1 consisting of a layered part $\Omega_L = \bigcup_{i=1}^{n} \Omega_i$ and two wedge parts $\Omega^+ = \bigcup_{j=1}^{m_+} \Omega_j^+$, $\Omega^- = \bigcup_{k=1}^{m_-} \Omega_k^-$.

$$\begin{aligned} \Omega_i &= \{(x_1,x_2):\ x_1 \in \mathbb{R},\ x_2 \in (y_{i-1}, y_i)\}, && i = 1,2,\dots,n,\\ \Omega_j^+ &= \{(r,\theta):\ r \in \mathbb{R}_+,\ \theta \in (\theta_{j-1}^+, \theta_j^+)\}, && j = 1,2,\dots,m_+,\\ \Omega_k^- &= \{(r,\theta):\ r \in \mathbb{R}_+,\ \theta \in (\theta_{k-1}^-, \theta_k^-)\}, && k = 1,2,\dots,m_-. \end{aligned}$$

By Γ_i $(i = 1,2,...,n-1)$ we denote an interior boundary between the regions Ω_i and Ω_{i+1}. Similarly, Γ_j^+ $(j = 1,2,...,m_+ - 1)$ and Γ_k^- $(k = 1,2,...,m_- - 1)$ be the interior boundaries between the corresponding wedges.

$$0 = y_0 < \ldots < y_i < y_{i+1} < \ldots < y_n \le \infty, \qquad h_i = y_i - y_{i-1},$$

$$\pi = \theta_0^- < \ldots < \theta_k^- < \theta_{k+1}^- < \ldots < \theta_{m_-}^- = -\frac{\pi}{2} + \phi_*^-, \qquad \phi_k^- = \theta_k^- - \theta_{k-1}^-,$$

$$-\frac{\pi}{2} + \phi_*^+ = \theta_0^+, \ldots, < \theta_j^+ < \theta_{j+1}^+ < \ldots < \theta_{m_+}^+ = 0, \qquad \phi_j^+ = \theta_j^+ - \theta_{j-1}^+.$$

Thus, by Γ_n, Γ_0^+ and $\Gamma_{m_-}^-$ we denote the exterior boundaries of the layered region (Ω_L), or the wedge-shaped regions $(\Omega^{\pm})$, respectively. Besides, let $\Gamma_0 = \Gamma_{m_+}^+ \cup \Gamma_0^-$ be the interior boundary between the different parts of domain Ω.

We shall seek for the vector of displacements $\mathbf{u}(x_1,x_2)$ and the tensor of stresses $\underline{\sigma}(x_1,x_2)$ with components satisfying the equilibrium equations:

$$\sigma_{\beta\alpha,\beta} + X_\alpha = 0, \qquad \alpha,\beta = 1,2, \tag{2.1}$$

and the Duhamel–Neumann relations (see [28]):

$$\sigma_{\alpha\beta} = \mu(u_{\alpha,\beta} + u_{\beta,\alpha}) + (\lambda\varepsilon - \gamma\Theta)\delta_{\alpha\beta}, \qquad \varepsilon = u_{\alpha,\alpha}, \tag{2.2}$$

where X_α are the defined internal forces, μ, λ – Lamé constants, $\gamma = (2\mu + 3\lambda)\alpha_t$ – thermoelasticity constant (α_t is the coefficient of linear thermal expansion). Besides, if ν is the Poisson coefficient then $\lambda = 2\mu\nu^*/(1 - 2\nu^*)$, where $\nu^* = \nu$ under plane stress conditions, $\nu^* = \nu/(1 - \nu)$ under plane strain conditions. Further on we omit the upper index $*$ in the symbol ν^*. All constants are different inside the regions Ω_i, Ω_j^+, Ω_k^-, in general. We assume here that the temperature $\Theta(x_1, x_2) = T(x_1, x_2) - T(0, 0)$ is a known function. Corresponding boundary value problems for Poisson's equation for the function $\Theta(x_1, x_2)$ in a similar domain with different boundary and interfacial conditions have been solved in [23, 24].

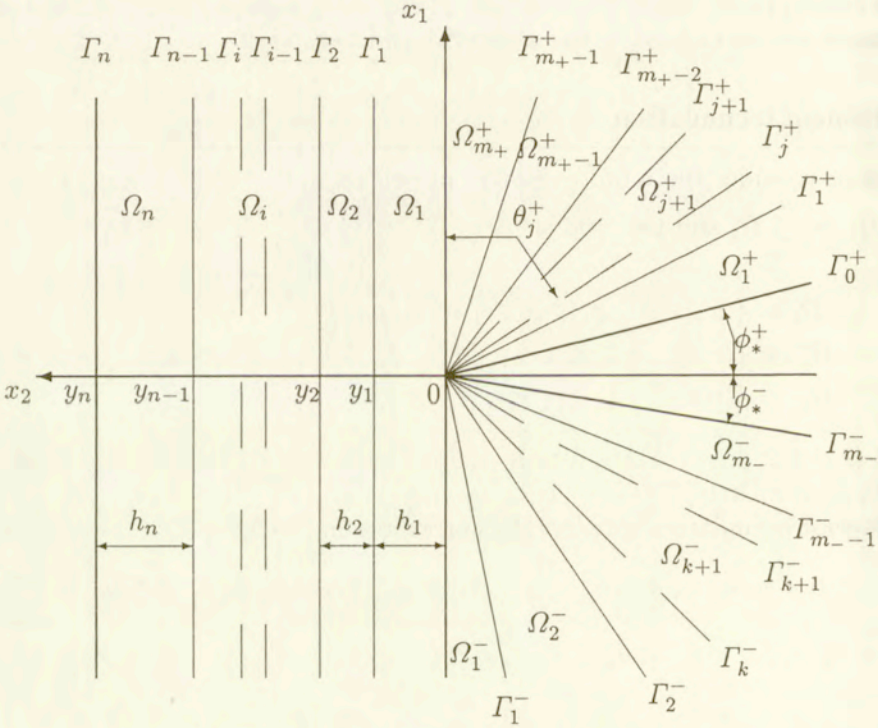

FIG. 1. The domain Ω under consideration.

Let us introduce the following symbols:

$$\mathbf{u} = \begin{cases} \mathbf{u}^{(i)} \\ \mathbf{v}^{(j)} \\ \mathbf{w}^{(k)} \end{cases}, \quad \underline{\boldsymbol{\sigma}} = \begin{cases} \underline{\boldsymbol{\sigma}}^{(i)} \\ \underline{\mathbf{p}}^{(j)} \\ \underline{\mathbf{q}}^{(k)} \end{cases}, \qquad \mu = \begin{cases} \mu_i \\ \mu_j^+ \\ \mu_k^- \end{cases},$$

$$\lambda = \begin{cases} \lambda_i \\ \lambda_j^+ \\ \lambda_k^- \end{cases}, \quad \beta = \begin{cases} \beta_i \\ \beta_j^+ \\ \beta_k^- \end{cases}, \quad (x_1, x_2) \in \begin{cases} \Omega_i \\ \Omega_j^+ \\ \Omega_k^- \end{cases}. \tag{2.3}$$

Besides, we shall use Cartesian coordinates in the layered part and polar coordinates in the wedge parts of the domain.

Along the interior boundaries of layered domain Ω_L the conditions hold:

$$(2.4)\qquad \begin{aligned} (\mathbf{u}^{(i+1)} - \mathbf{u}^{(i)} - \boldsymbol{\tau}_i\, \boldsymbol{\sigma}^{(i)})_{|\Gamma_i} &= \delta \mathbf{u}_i(x_1), \quad x_1 \in \mathbb{R}, \quad i = 1,2,...,n-1, \\ (\boldsymbol{\sigma}^{(i+1)} - \boldsymbol{\sigma}^{(i)})_{|\Gamma_i} &= \delta\, \boldsymbol{\sigma}_i(x_1), \quad x_1 \in \mathbb{R}, \end{aligned}$$

where $\mathbf{u}^{(i)} = (u^{(i)}_{x_2}, u^{(i)}_{x_1})^\top$, but $\boldsymbol{\sigma}^{(i)}$ is vector of stresses along boundary Γ_i; $\boldsymbol{\tau}_i$ is a diagonal matrix with positive constant components, but $\delta\boldsymbol{\sigma}_i$, $\delta\mathbf{u}_i$ are some known vector-functions. Analogous relations for the interior boundaries of wedge domains $\Omega^{\pm}$ are given in the form:

$$(2.5)\qquad \begin{aligned} (\mathbf{v}^{(j+1)} - \mathbf{v}^{(j)} - r\boldsymbol{\tau}^+_j\, \mathbf{p}^{(j)})_{|\Gamma^+_j} &= \delta \mathbf{v}_j(r), \quad r \in \mathbb{R}_+, \quad j = 1,2,...,m_+ - 1, \\ (\mathbf{p}^{(j+1)} - \mathbf{p}^{(j)})_{|\Gamma^+_j} &= \delta \mathbf{p}_j(r), \quad r \in \mathbb{R}_+; \end{aligned}$$

$$(2.6)\qquad \begin{aligned} (\mathbf{w}^{(k+1)} - \mathbf{w}^{(k)} - r\boldsymbol{\tau}^-_k\, \mathbf{q}^{(k)})_{|\Gamma^-_k} &= \delta \mathbf{w}_k(r), \quad r \in \mathbb{R}_+, \quad k = 1,2,...,m_- - 1, \\ (\mathbf{q}^{(k+1)} - \mathbf{q}^{(k)})_{|\Gamma^-_k} &= \delta \mathbf{q}_k(r), \quad r \in \mathbb{R}_+, \end{aligned}$$

where $\boldsymbol{\tau}^+_j$, $\boldsymbol{\tau}^-_k$ are diagonal matrices similar to $\boldsymbol{\tau}_i$.

Finally, the last of the interfacial conditions between different geometry regions (along the boundaries $\Gamma^+_{m_+}$, Γ^-_0) are characterized by given discontinuities of the displacements and tractions:

$$(2.7)\qquad \begin{aligned} (\mathbf{u}^{(1)} - \mathbf{v}^{(m_+)})_{|\Gamma^+_{m_+}} &= \delta \mathbf{u}_+(x_1), \\ (\boldsymbol{\sigma}^{(1)} - \mathbf{p}^{(m_+)})_{|\Gamma^+_{m_+}} &= \delta \boldsymbol{\sigma}_+(x_1), \qquad x_1 > 0; \end{aligned}$$

$$(2.8)\qquad \begin{aligned} (\mathbf{u}^{(1)} - \mathbf{w}^{(1)})_{|\Gamma^-_0} &= \delta \mathbf{u}_-(-x_1), \\ (\boldsymbol{\sigma}^{(1)} + \mathbf{q}^{(1)})_{|\Gamma^-_0} &= \delta \boldsymbol{\sigma}_-(-x_1), \qquad x_1 < 0. \end{aligned}$$

The direction of normals to the boundaries is taken into account in (2.8).

Now we define the exterior boundary conditions for the domain Ω. So, on the wedge boundaries Γ^+_0, $\Gamma^-_{m_-}$ one from the following relations holds:

$$(2.9)\qquad \begin{aligned} &\text{(a)} \quad \mathbf{v}^{(1)}{}_{|\Gamma^+_0} = \delta \mathbf{v}_0(r), && && r \in \mathbb{R}_+, \\ &\text{(b)} \quad \mathbf{p}^{(1)}{}_{|\Gamma^+_0} = \delta \mathbf{p}_0(r), && && r \in \mathbb{R}_+, \\ &\text{(c)} \quad v^{(1)}_\theta{}_{|\Gamma^+_0} = \delta v_3(r), && p^{(1)}_{r\theta}{}_{|\Gamma^+_0} = \delta p_3(r), && r \in \mathbb{R}_+, \\ &\text{(d)} \quad v^{(1)}_r{}_{|\Gamma^+_0} = \delta v_4(r), && p^{(1)}_{\theta\theta}{}_{|\Gamma^+_0} = \delta p_4(r), && r \in \mathbb{R}_+; \end{aligned}$$

$$
(2.10)\quad
\begin{aligned}
&(a) && \mathbf{w}^{(m_-)}|_{\Gamma^-_{m_-}} = -\delta\mathbf{w}_0(r), && && r\in\mathbb{R}_+,\\
&(b) && \mathbf{q}^{(m_-)}|_{\Gamma^-_{m_-}} = -\delta\mathbf{q}_0(r), && && r\in\mathbb{R}_+,\\
&(c) && w_\theta^{(m_-)}|_{\Gamma^-_{m_-}} = -\delta w_3(r), && q_{r\theta}^{(m_-)}|_{\Gamma^-_{m_-}} = -\delta q_3(r), && r\in\mathbb{R}_+,\\
&(d) && w_r^{(m_-)}|_{\Gamma^-_{m_-}} = -\delta w_4(r), && q_{\theta\theta}^{(m_-)}|_{\Gamma^-_{m_-}} = -\delta q_4(r), && r\in\mathbb{R}_+.
\end{aligned}
$$

Let us note that in the limiting case of crack ($\theta_0^+ = \theta_{m_-}^-$) there is no contact between the crack surfaces. Some of such problems are considered in [20, 35] for homogeneous unbounded media.

On the exterior boundary Γ_n we shall consider conditions (a), (b), (c), (d) analogous to (2.9), (2.10) and the relation (e):

$$
(2.11)\quad
\begin{aligned}
&(a) && \mathbf{u}^{(n)}|_{\Gamma_n} = -\delta\mathbf{u}_0(x_1), && && x_1\in\mathbb{R},\\
&(b) && \boldsymbol{\sigma}^{(n)}|_{\Gamma_n} = -\delta\boldsymbol{\sigma}_0(x_1), && && x_1\in\mathbb{R},\\
&(c) && u_{x_2}^{(n)}|_{\Gamma_n} = -\delta u_3(x_1), && \sigma_{x_1x_2}^{(n)}|_{\Gamma_n} = -\delta\sigma_3(x_1), && x_1\in\mathbb{R},\\
&(d) && u_{x_1}^{(n)}|_{\Gamma_n} = -\delta u_4(x_1), && \sigma_{x_2x_2}^{(n)}|_{\Gamma_n} = -\delta\sigma_4(x_1), && x_1\in\mathbb{R},\\
&(e) && \lim_{x_2\to\infty}\mathbf{u}^{(n+1)} = 0. && && x_1\in\mathbb{R}.
\end{aligned}
$$

In the last case (e) we assume that the region Ω_{n+1} is a half-plane. Then the condition $(2.11)_e$ means that the solution of the problem decreases to zero in the direction $x_2\to\infty$ as well as in the other one $x_1\to\infty$. Consequently, we have here ninety different combinations of the exterior conditions. The corresponding problems (2.1) – (2.8) with the boundary conditions from (2.9) – (2.11) will be denoted by $(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$, where ($\mathcal{J}^+ = 1-4$, $\mathcal{J}^- = 1-4$, $\mathcal{J} = 1-5$). Here the value of $\mathcal{J}^+$ is equal to 1 (2, 3, 4) if the conditions $(2.9)_a$ ($(2.9)_b$, $(2.9)_c$, $(2.9)_d$) hold. In an analogous way one can define the values of $\mathcal{J}^-$, $\mathcal{J}$ from the conditions (2.10) and (2.11), respectively.

We assume that all of the known functions which are presented in the equations and the boundary conditions are sufficiently smooth:

$$
(2.12)\quad
\begin{aligned}
&X_\alpha\in\mathbf{C}(\overline{G}), \quad \Theta\in\mathbf{C}^1(\overline{G}), \quad \delta\mathbf{u}_i\in\mathbf{C}_2^2(\mathbb{R}),\\
&\qquad\qquad \delta\boldsymbol{\sigma}_i\in\mathbf{C}_2^1(\mathbb{R}), \quad i=0,1,...,n-1,\\
&\delta u_3, \delta u_4\in\mathbf{C}^2(\mathbb{R}), \quad \delta\mathbf{v}_j, \delta\mathbf{w}_k, \delta\mathbf{u}_\pm\in\mathbf{C}_2^2(\mathbb{R}_+), \quad j=0,1,...,m_+-1,\\
&\delta\sigma_3, \sigma_4\in\mathbf{C}^1(\mathbb{R}), \quad \delta\mathbf{p}_j, \delta\mathbf{q}_k, \delta\boldsymbol{\sigma}_\pm\in\mathbf{C}_2^1(\mathbb{R}_+), \quad k=0,1,...,m_--1,\\
&\delta v_3, \delta v_4, \delta w_3, \delta w_4\in\mathbf{C}^2(\mathbb{R}_+), \quad \delta p_3, \delta p_4, \delta q_3, \delta q_4\in\mathbf{C}^1(\mathbb{R}_+),\\
&\delta\mathbf{u}_i, r\delta\boldsymbol{\sigma}_i, \delta u_3, \delta u_4, \delta\mathbf{v}_j, \delta\mathbf{w}_k, \delta\mathbf{u}_\pm, r\delta\sigma_3, r\sigma_4, r\delta\mathbf{p}_j, r\delta\mathbf{q}_k, r\delta\boldsymbol{\sigma}_\pm, \delta v_3,\\
&\quad \delta v_4, \delta w_3, \delta w_4, r\delta p_3, r\delta p_4, r\delta q_3, r\delta q_4, r\Theta, r^2X_\alpha = o(r^{-1-\varepsilon}), \quad r\to\infty,\\
&\delta\mathbf{u}_\pm, \delta\mathbf{v}_0, \delta\mathbf{w}_0, \delta v_3, \delta v_4, \delta w_3, \delta w_4 = o(r^{1+\varepsilon}), \quad r\to 0,\\
&\delta\boldsymbol{\sigma}_\pm, \delta\mathbf{p}_0, \delta\mathbf{q}_0, \delta p_3, \delta p_4, \delta q_3, \delta q_4, \Theta, rX_\alpha = o(r^\varepsilon), \quad r\to 0.
\end{aligned}
$$

Here G denotes any region Ω_j, $\Omega_j^{\pm}$ from Ω, but ε is some positive constant. The suppositions concerning the defined functions are assumed in the forms (2.12) in order to exhibit the singularities of solutions, connected with the internal properties of the problems only.

REMARK 1. As it follows from [23, 24], the temperature Θ is a function of the class $\mathbf{C}^2(G)$, at least, in any region G, and the asymptotics is true:

$$\Theta(r,\theta) = a + b\ln r + f_1^{\infty}(\theta)r^{-\omega_1^{\infty}} + f_2^{\infty}(\theta)r^{-\omega_2^{\infty}} + o(r^{-2-\varepsilon}), \qquad r \to \infty,$$

$$\Theta(r,\theta) = \begin{cases} f_0^0(\theta)r^{\omega_0^0} + \mathcal{O}(r), \\ f_1^0(\theta)r^{\omega_1^0} + f_2^0(\theta)r^{\omega_2^0} + o(r), \qquad r \to 0, \\ f_3^0(\theta)r\ln r + \mathcal{O}(r^2), \end{cases}$$

where ω_1^{∞}, $\omega_2^{\infty} - 1$, $\omega_i^0 \in (0,1)$ are certain constants; functions f_i^{∞}, f_i^0 depend on the geometry and exterior boundary conditions. Three different forms of the asymptotics of the temperature Θ near zero point depend on the type of interfacial boundary conditions. The first one corresponds to "ideal" interfacial conditions; the second – to "nonideal" contact through a thin heat conducting wedge; and, finally, the third term corresponds to interaction between the materials through a thin heat conducting layer. Besides, the constant $b = 0$ when there is a balance of the heat flow. When all known functions are equal to zero, except Θ, we can easily find particular solutions of the problems (2.1) – (2.11) in the neighbourhood of zero and infinity points using suitable asymptotics. Then using the property of linearity of the problems, we obtain the solutions of the initial problems as a sum of the mentioned solution and the solution of the problems (2.1) – (2.11) under assumptions (2.12).

We shall seek for the classical solutions of the problems $(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ in a class of vector-functions $\mathbf{LW}(\Omega)$ such that $\mathbf{u} \in \mathbf{LW}(\Omega)$ if the following relations are true:

$$\text{(2.13)} \qquad \begin{aligned} &1. \quad \mathbf{u}_{|G} \in \mathbf{C}^2(G), \\ &2. \quad \mathbf{u}(r,\theta) = \mathcal{O}(r^{-\gamma_1}), \qquad \underline{\sigma}(r,\theta) = \mathcal{O}(r^{-\gamma_2-1}), \qquad r \to \infty, \\ &3. \quad \mathbf{u}(r,\theta) = \mathbf{u}_* + \mathcal{O}(r^{\gamma_0}), \quad \underline{\sigma}(r,\theta) = \mathcal{O}(r^{\gamma_0-1}), \qquad r \to 0, \end{aligned}$$

where, as before, by G we denote any region from Ω, and γ_0, γ_1, γ_2 are certain constants such that $0 < \gamma_0 < 1$, γ_1, $\gamma_2 > 0$. Besides, in the cases when displacements (or one of its components) are prescribed at least on one of the wedge boundaries, the corresponding components of the vector $\mathbf{u}_*$ are equal to zero due to the assumptions (2.12) (the corresponding relations are presented in (3.29)). Precise values of the parameters $\gamma_0 = \gamma_0(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$, $\gamma_1 = \gamma_1(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$, $\gamma_2 = \gamma_2(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ and $\mathbf{u}_*$ will be obtained by solving the problems. Let us note that the value of γ_0 defines the order of stress singularity in the neighbourhood of zero and plays an important role in physical applications [8, 26].

REMARK 2. It can be shown that the solution of the problem $(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ from $\mathbf{LW}(\Omega)$ with the values of the parameters as given above belongs to an energetic space of the corresponding linear boundary problem [26]. Therefore, the problems in the class $\mathbf{LW}(\Omega)$ have unique solutions.

REMARK 3. Instead of the interfacial conditions (2.7), (2.8) between the domains of different geometry (layer and wedges), the other ones can be considered which are generalizations of the conditions (2.4), (2.5), (2.6). But, as it has been shown in [24] just for Mode III problem, this significantly complicates the problems, and such new conditions should be investigated separately.

3. Sweep method in the layered domain

Applying to Eqs. (2.1), (2.2) the Fourier transform of the form:

$$\overline{f}(\lambda, x) = \mathcal{F}[f(x_1, x_2);\ x_1 \to \lambda] \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(i\lambda x_1) u(x_1, x_2)\, dx_1, \tag{3.1}$$

we obtain (see [37]) the following relations inside each of the layered domains Ω_j, $(j = 1, ..., n)$:

$$-i\lambda\overline{\sigma}_{1\alpha}^{(j)} + \frac{\partial}{\partial x_2}\overline{\sigma}_{2\alpha}^{(j)} + \overline{X}_\alpha^{(j)} = 0, \qquad \alpha = 1, 2; \tag{3.2}$$

$$\begin{aligned} -i\lambda\overline{u}_1^{(j)} &= \frac{1}{2\mu_j}\left[\overline{\sigma}_{11}^{(j)} - \nu_j\overline{\sigma}^{(j)}\right] + \beta_j\overline{\Theta}^{(j)}, \\ \frac{\partial}{\partial x_2}\overline{u}_2^{(j)} &= \frac{1}{2\mu_j}\left[\overline{\sigma}_{22}^{(j)} - \nu_j\overline{\sigma}^{(j)}\right] + \beta_j\overline{\Theta}^{(j)}, \\ \frac{\partial}{\partial x_2}\overline{u}_1^{(j)} - i\lambda\overline{u}_2^{(j)} &= \frac{1}{\mu_j}\overline{\sigma}_{12}, \qquad \sigma^{(j)} = \sigma_{11}^{(j)} + \sigma_{22}^{(j)}, \end{aligned} \tag{3.3}$$

where $\beta_j = \gamma_j/(2\mu_j + \lambda_j)$, but the constant ν_j is defined in (2.2). For the function $\overline{\sigma}_{22}^{(j)}$ the equation can be found:

$$\left(\frac{\partial^2}{\partial x_2^2} - \lambda^2\right)^2 \overline{\sigma}_{22}^{(j)} = \left(\frac{\partial^2}{\partial x_2^2} - \lambda^2\right) g_1^{(j)} + \lambda^2 g_2^{(j)}, \tag{3.4}$$

where

$$g_1^{(j)} = -\frac{\partial}{\partial x_2}\overline{X}_2^{(j)} - i\lambda\overline{X}_1^{(j)} + \lambda^2\frac{2\mu_j\beta_j}{1-\nu_j}\overline{\Theta}^{(j)}, \qquad g_2^{(j)} = \frac{1}{1-\nu_j}\left[\frac{\partial}{\partial x_2}\overline{X}_2^{(j)} - i\lambda\overline{X}_1^{(j)}\right].$$

The corresponding solution is of the form:

$$\overline{\sigma}_{22}^{(j)} = C_1^j(\lambda)e^{-|\lambda|x_2} + x_2C_2^j(\lambda)e^{-|\lambda|x_2} + C_3^j(\lambda)e^{|\lambda|x_2} + x_2C_4^j(\lambda)e^{|\lambda|x_2} + \sigma_*^{(j)}, \tag{3.5}$$

where

$$\sigma_*^{(j)} = -\frac{1}{2}\int\limits_{y_{j-1}}^{x_2}(\xi - x_2)g_2^{(j)}(\lambda,\xi)\text{ch}[(\xi - x_2)\lambda]\,d\xi + \frac{1}{2\lambda}\int\limits_{y_{j-1}}^{x_2} g_2^{(j)}(\lambda,\xi)\text{sh}[(\xi - x_2)\lambda]\,d\xi - \frac{1}{\lambda}\int\limits_{y_{j-1}}^{x_2} g_1^{(j)}(\lambda,\xi)\text{sh}[(\xi - x_2)\lambda]\,d\xi.$$

Using relations (3.2) and (3.3), Fourier transforms of all the remaining components of stress tensor and vector of displacements can be calculated in terms of functions C_k^j $(k = 1, ..., 4)$.

Following [16], we denote new unknown vector-functions:

$$\begin{aligned} &\overline{\boldsymbol{\sigma}}_t^j(\lambda) = \overline{\boldsymbol{\sigma}}^{(j)}(\lambda, x_2)_{|\Gamma_j}, \qquad \overline{\mathbf{u}}_t^j(\lambda) = \overline{\mathbf{u}}^{(j)}(\lambda, x_2)_{|\Gamma_j},\\ &\overline{\boldsymbol{\sigma}}_b^j(\lambda) = \overline{\boldsymbol{\sigma}}^{(j)}(\lambda, x_2)_{|\Gamma_{j-1}}, \qquad \overline{\mathbf{u}}_b^j(\lambda) = \overline{\mathbf{u}}^{(j)}(\lambda, x_2)_{|\Gamma_{j-1}}, \qquad j = 1, 2, ..., n. \end{aligned} \tag{3.6}$$

Further on we omit all overbars and upper brackets.

From *a priori* estimates (2.13) for vector-functions of class $\mathbf{LW}(\Omega)$ and from the properties of the Fourier transform it can be shown that the vector-functions defined above should satisfy the relations:

$$\begin{aligned} &[\mathbf{u}_{t(b)}^j(\lambda)]_+ = \begin{cases} \mathcal{O}(\lambda^{\gamma_1 - 1}), & 0 < \gamma_1 < 1,\\ \mathcal{O}(\ln(\lambda)), & \gamma_1 = 1,\\ \text{Const} + \mathcal{O}(\lambda^{\gamma_1 - 1}), & \gamma_1 > 1, \quad \lambda \to 0; \end{cases}\\ &[\mathbf{u}_{t(b)}^j(\lambda)]_- = \mathcal{O}(\lambda^{\gamma_1 - 1}), \quad \lambda \to 0;\\ &[\boldsymbol{\sigma}_{t(b)}^j(\lambda)]_+ = \text{Const} + \mathcal{O}(\lambda^{\gamma_2}), \qquad [\boldsymbol{\sigma}_{t(b)}^j(\lambda)]_- = \mathcal{O}(\lambda^{\gamma_2}), \quad \lambda \to 0;\\ &\lambda\frac{\partial}{\partial\lambda}\boldsymbol{\sigma}_{t(b)}^j(\lambda) = \mathcal{O}(\lambda^{\gamma_2}), \qquad \lambda \to 0, \quad j = 1, 2, ..., n;\\ &\lambda\mathbf{u}_{t(b)}^j,\ \lambda\mathbf{u}_t^1,\ \boldsymbol{\sigma}_{t(b)}^j,\ \sigma_t^1,\ \lambda\frac{\partial}{\partial\lambda}\boldsymbol{\sigma}_{t(b)}^j,\ \lambda\frac{\partial}{\partial\lambda}\boldsymbol{\sigma}_t^1 = \mathcal{O}(\lambda^{-2}),\\ &\qquad\qquad \lambda \to \infty, \quad j = 2, 3, ..., n;\\ &\lambda\mathbf{u}_b^1(\lambda),\ \boldsymbol{\sigma}_b^1(\lambda),\ \lambda\frac{\partial}{\partial\lambda}\boldsymbol{\sigma}_b^1(\lambda) = \mathcal{O}(\lambda^{-\gamma_0}), \qquad \lambda \to \infty. \end{aligned} \tag{3.7}$$

Here by $[\mathbf{f}(\lambda)]_\pm$ we understand the even (odd) part of a vector-function $\mathbf{f}$. All constants in (3.7) are different, in general.

Substituting the relations (3.6) in (3.2), (3.3) and eliminating the functions of $C_k^j(\lambda)$, we obtain the relations between vector-functions $\mathbf{u}^j_{t(b)}$ and $\boldsymbol{\sigma}^j_{t(b)}$ in the form:

$$\begin{aligned} \mathbf{u}^j_t &= \mathbf{R}^j_{tt}\boldsymbol{\sigma}^j_t + \mathbf{R}^j_{tb}\boldsymbol{\sigma}^j_b + \mathbf{u}^j_{t0}, \\ \mathbf{u}^j_b &= \mathbf{R}^j_{bt}\boldsymbol{\sigma}^j_t + \mathbf{R}^j_{bb}\boldsymbol{\sigma}^j_b + \mathbf{u}^j_{b0}, \qquad j = 1, 2, ..., n, \end{aligned} \tag{3.8}$$

where coefficients are calculated in the following way:

$$\mathbf{R}^j_{tt}(\lambda) = \frac{1}{2i\lambda\mu_j}\mathbf{E}_2 - \frac{1-\nu_j}{\lambda^2\mu_j}\mathbf{R}^j_1\mathbf{E}_1,$$

$$\mathbf{R}^j_{bb}(\lambda) = \frac{1}{2i\lambda\mu_j}\mathbf{E}_2 + \frac{1-\nu_j}{\lambda^2\mu_j}\mathbf{E}_1\mathbf{R}^j_1,$$

$$\mathbf{R}^j_{tb}(\lambda) = \frac{1-\nu_j}{\lambda^2\mu_j}\mathbf{R}^j_2\mathbf{E}_1,$$

$$\mathbf{R}^j_{bt}(\lambda) = -\frac{1-\nu_j}{\lambda^2\mu_j}\mathbf{E}_1\mathbf{R}^j_2, \qquad \mathbf{E}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\mathbf{R}^j_1 = r_j(\lambda)\begin{pmatrix} h_j + \dfrac{1}{2\lambda}\mathrm{sh}2\lambda h_j & \dfrac{i}{\lambda}\mathrm{sh}^2\lambda h_j \\ \dfrac{i}{\lambda}\mathrm{sh}^2\lambda h_j & h_j - \dfrac{1}{2\lambda}\mathrm{sh}2\lambda h_j \end{pmatrix}, \qquad \mathbf{E}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{3.9}$$

$$\mathbf{R}^j_2 = r_j(\lambda)\begin{pmatrix} h_j\mathrm{ch}\lambda h_j + \dfrac{1}{\lambda}\mathrm{sh}\lambda h_j & -ih_j\mathrm{sh}\lambda h_j \\ ih_j\mathrm{sh}\lambda h_j & h_j\mathrm{ch}\lambda h_j - \dfrac{1}{\lambda}\mathrm{sh}\lambda h_j \end{pmatrix},$$

$$r_j(\lambda) = \frac{\lambda^2}{\lambda^2 h_j^2 - \mathrm{sh}^2\lambda h_j}, \qquad \mathbf{u}^j_{t(b)0} = \frac{1-\nu_j}{\lambda^2\mu_j}r_j(\lambda)$$

$$\times\begin{pmatrix} f^1_{t(b)} + \left[\dfrac{1}{2(1-\nu_j)r_j(\lambda)} - \dfrac{\mathrm{sh}^2\lambda h_j}{\lambda^2}\right]\overline{X}_{2t(b)} - \dfrac{h_j}{\lambda}\mathrm{sh}\lambda h_j\overline{X}_{2b(t)} \\ i\left(f^2_{t(b)} \pm \left[h_j - \dfrac{\mathrm{sh}2\lambda h_j}{2\lambda}\right]\overline{X}_{2t(b)} \mp \left[h_j\mathrm{ch}\lambda h_j - \dfrac{1}{\lambda}\mathrm{sh}\lambda h_j\right]\overline{X}_{2b(t)}\right) \end{pmatrix},$$

where $y_b = y_{i-1}$, $y_t = y_i$, $\overline{X}_{2t(b)} = \overline{X}^{(j)}_2(\lambda, y_{t(b)})$, but the remaining functions are defined by

$$f^1_{t(b)}(\lambda) = \int\limits_{y_{j-1}}^{y_j} [g_2 - 2g_1](\lambda,\xi)\left\{\frac{h_j}{2\lambda}\mathrm{sh}[\lambda(\xi - y_{t(b)})] + \frac{\mathrm{sh}\lambda h_j}{2\lambda^2}\mathrm{sh}[\lambda(\xi - y_{b(t)})]\right\}d\xi$$

$$-\int\limits_{y_{j-1}}^{y_j} g_2(\lambda,\xi)\left\{\frac{\mathrm{sh}\lambda h_j}{2\lambda}(\xi - y_{t(b)})\mathrm{ch}[\lambda(\xi - y_{b(t)})] + \frac{h_j}{2}(\xi - y_{b(t)})\mathrm{ch}[\lambda(\xi - y_{t(b)})]\right\}d\xi,$$

$$f^2_{t(b)} = \frac{h_j}{2}\int\limits_{y_{j-1}}^{y_j}(\xi - y_{b(t)})g_2\mathrm{sh}[\lambda(\xi - y_{t(b)})]d\xi - \frac{\mathrm{sh}\lambda h_j}{\lambda^2}\int\limits_{y_{j-1}}^{y_j} g_1\mathrm{ch}[\lambda(\xi - y_{b(t)})]d\xi$$
$$-\frac{\mathrm{sh}\lambda h_j}{2\lambda}\int\limits_{y_{j-1}}^{y_j}(\xi - y_{t(b)})g_2\mathrm{sh}[\lambda(\xi - y_{b(t)})]d\xi + \frac{h_j}{\lambda}\int\limits_{y_{j-1}}^{y_j} g_1\mathrm{ch}[\lambda(\xi - y_{t(b)})]d\xi.$$

Note that the matrix-functions $\mathbf{R}^j_{t(b)t(b)}(\lambda)$ and the vector-functions $\mathbf{u}^j_{b(t)0}(\lambda)$ can be estimated like $\mathcal{O}(\lambda^{-4})$ when $\lambda \to 0$. But in view of (3.7), the unknown vectors $\mathbf{u}^j_{t(b)}(\lambda)$, $\boldsymbol{\sigma}^j_{t(b)}(\lambda)$ are bounded near this point. Hence, by investigating the main terms of asymptotics ($\lambda \to 0$) of the right-hand side of relations (3.8) it can be obtained that vector-functions $\boldsymbol{\sigma}^j_{t(b)}(\lambda)$ should satisfy the following additional relations:

$$\boldsymbol{\sigma}^j_t(0) - \boldsymbol{\sigma}^j_b(0) + \int\limits_{y_{j-1}}^{y_j}\overline{\mathbf{X}}^j(0,\xi)d\xi = 0,$$
$$\mathbf{X}^j(x_1,x_2) = [X^{(j)}_2(x_1,x_2), X^{(j)}_1(x_1,x_2)]^\top,$$
$$(3.10)\qquad [1,0]i\frac{d}{d\lambda}\left\{\boldsymbol{\sigma}^j_t(\lambda) - \boldsymbol{\sigma}^j_b(\lambda) + \int\limits_{y_{j-1}}^{y_j}\overline{\mathbf{X}}^j(\lambda,\xi)d\xi\right\}\Bigg|_{\lambda=0}$$
$$+ [0,1]\left\{y_j\boldsymbol{\sigma}^j_t(0) - y_{j-1}\boldsymbol{\sigma}^j_b(0) + \int\limits_{y_{j-1}}^{y_j}\xi\overline{\mathbf{X}}^j(0,\xi)d\xi\right\} = 0.$$

Let us note, that the mentioned equations are the usual equilibrium conditions of the j-th layer.

Now we apply the Fourier transform to the interfacial contact conditions along boundaries Γ_j, $j = 1,2,...,n-1$. The corresponding equations can be written in terms of vector-functions $\mathbf{u}^j_{t(b)}$, $\boldsymbol{\sigma}^j_{t(b)}$ defined above

$$(3.11)\qquad \begin{aligned}\mathbf{u}^{j+1}_b - \mathbf{u}^j_t - \boldsymbol{\tau}_j\boldsymbol{\sigma}^j_t &= \Delta\mathbf{u}_j,\\ \boldsymbol{\sigma}^{j+1}_b - \boldsymbol{\sigma}^j_t &= \Delta\boldsymbol{\sigma}_j, \qquad j = 1,2,...,n-1.\end{aligned}$$

Here $\Delta\mathbf{u}_j(\lambda) = \mathcal{F}[\delta\mathbf{u}_j](\lambda)$, $\Delta\boldsymbol{\sigma}_j(\lambda) = \mathcal{F}[\delta\boldsymbol{\sigma}_j](\lambda)$ are the Fourier transforms of known functions.

As it has been shown in [16], relations (3.8) and (3.11) make it possible to eliminate the unknown functions, either $\mathbf{u}^j_{t(b)}$ or $\boldsymbol{\sigma}^j_{t(b)}$, and to obtain formulas for the remaining ones. We present the relations for vector-functions $\boldsymbol{\sigma}^j_{t(b)}$. Substitute (3.8) in (3.11), then two systems of difference equations $(3.12)_\mathrm{a}$, $(3.12)_\mathrm{b}$ for vector-functions $\boldsymbol{\sigma}^j_{t(b)}$ are obtained

$$\text{(3.12)}\qquad \begin{aligned} &\text{(a)}\quad \mathbf{A}^j_\sigma \boldsymbol{\sigma}^{j-1}_t - \mathbf{C}^j_\sigma \boldsymbol{\sigma}^j_t + \mathbf{B}^j_\sigma \boldsymbol{\sigma}^{j+1}_t + \mathbf{F}^j_{\sigma t} = \mathbf{0}, \qquad j = 2,3,...,n-1,\\ &\text{(b)}\quad \mathbf{A}^j_\sigma \boldsymbol{\sigma}^j_b - \mathbf{C}^j_\sigma \boldsymbol{\sigma}^{j+1}_b + \mathbf{B}^j_\sigma \boldsymbol{\sigma}^{j+2}_b + \mathbf{F}^j_{\sigma b} = \mathbf{0}, \qquad j = 1,2,...,n-2, \end{aligned}$$

where

$$\text{(3.13)}\qquad \begin{aligned} &\mathbf{A}^j_\sigma = -\mathbf{R}^j_{tb}, \qquad \mathbf{C}^j_\sigma = \boldsymbol{\tau}_j + \mathbf{R}^j_{tt} - \mathbf{R}^{j+1}_{bb}, \qquad \mathbf{B}^j_\sigma = \mathbf{R}^{j+1}_{bt},\\ &\mathbf{F}^j_{\sigma b} = \mathbf{u}^{j+1}_{b0} - \mathbf{u}^j_{t0} - \Delta\mathbf{u}_j + (\mathbf{R}^j_{tt} + \boldsymbol{\tau}_j)\Delta\boldsymbol{\sigma}_j - \mathbf{R}^{j+1}_{bt}\Delta\boldsymbol{\sigma}_{j+1},\\ &\mathbf{F}^j_{\sigma t} = \mathbf{u}^{j+1}_{b0} - \mathbf{u}^j_{t0} - \Delta\mathbf{u}_j + \mathbf{R}^{j+1}_{bb}\Delta\boldsymbol{\sigma}_j - \mathbf{R}^j_{tb}\Delta\boldsymbol{\sigma}_{j-1}. \end{aligned}$$

Equations (3.12) are identical in the case when all jumps of vector-functions $\boldsymbol{\sigma}_j$ across the interfaces ($\Delta\boldsymbol{\sigma}_j = 0$, $j = 1,2,...,n-1$) are equal to zero.

In order to solve any of these systems of difference equations, it is necessary to have exterior boundary ("initial" in this sense) conditions for the first and the last surface of the package of layers. At boundary Γ_n one of the conditions from (2.11) is defined. Applying the Fourier transform to the corresponding boundary condition and taking into account (3.8), (3.11), we rewrite it in a form similar to (3.12):

$$\text{(3.14)}\qquad \mathbf{A}^n_\sigma \boldsymbol{\sigma}^{n-1}_t - \mathbf{C}^n_\sigma \boldsymbol{\sigma}^n_t + \mathbf{F}^n_{\sigma t} = 0,$$

where the functions $\mathbf{A}^n_\sigma$, $\mathbf{C}^n_\sigma$, $\mathbf{F}^n_{\sigma t}$ are defined for each of the conditions $(2.11)_\mathrm{a}$–$(2.11)_\mathrm{e}$:

$$\text{(3.15)}\qquad \begin{aligned} &\mathcal{J} = 1:\\ &\qquad \mathbf{A}^n_\sigma = -\mathbf{R}^n_{tb}, \qquad \mathbf{C}^n_\sigma = \mathbf{R}^n_{tt},\\ &\qquad \mathbf{F}^n_{\sigma t} = -\Delta\mathbf{u}_n - \mathbf{u}^n_{t0} - \mathbf{R}^n_{tb}\Delta\boldsymbol{\sigma}_{n-1},\\ &\mathcal{J} = 2:\\ &\qquad \mathbf{A}^n_\sigma = \mathbf{0}, \qquad \mathbf{C}^n_\sigma = \mathbf{I}, \qquad \mathbf{F}^n_{\sigma t} = -\Delta\boldsymbol{\sigma}_n,\\ &\mathcal{J} = 3,4:\\ &\qquad \mathbf{A}^n_\sigma = -\begin{pmatrix} \delta_{\mathcal{J}3} & 0\\ 0 & \delta_{\mathcal{J}4}\end{pmatrix}\mathbf{R}^n_{tb},\\ &\qquad \mathbf{C}^n_\sigma = \begin{pmatrix} \delta_{\mathcal{J}3} & 0\\ 0 & \delta_{\mathcal{J}4}\end{pmatrix}\mathbf{R}^n_{tt} + \begin{pmatrix} \delta_{\mathcal{J}4} & 0\\ 0 & \delta_{\mathcal{J}3}\end{pmatrix},\\ &\qquad \mathbf{F}^n_{\sigma t} = -\begin{pmatrix} \delta_{\mathcal{J}3} & 0\\ 0 & \delta_{\mathcal{J}4}\end{pmatrix}\left(\mathbf{u}^n_{t0} + \mathbf{R}^n_{tb}\Delta\boldsymbol{\sigma}_{n-1}\right) - \Delta\mathbf{h}_{\mathcal{J}},\\ &\qquad \Delta\mathbf{h}_{\mathcal{J}} = \begin{pmatrix} \overline{\delta u}_{\mathcal{J}}(\lambda)\\ \overline{\delta\sigma}_{\mathcal{J}}(\lambda)\end{pmatrix},\\ &\mathcal{J} = 5:\\ &\qquad \mathbf{A}^n_\sigma = -\mathbf{R}^n_{tb}, \qquad \mathbf{C}^n_\sigma = \boldsymbol{\tau}_n + \mathbf{R}^n_{tt} - \mathbf{R}^{n+1}_\infty,\\ &\qquad \mathbf{F}^n_{\sigma t} = -\Delta\mathbf{u}_n - \mathbf{u}^n_{t0} - \mathbf{R}^n_{tb}\Delta\boldsymbol{\sigma}_{n-1} + \mathbf{R}^{n+1}_\infty\Delta\boldsymbol{\sigma}_n + \mathbf{u}^{n+1}_\infty, \end{aligned}$$

where $\delta_{\mathcal{J}j}$ is the Kronecker symbol, but

$$\mathbf{R}_{\infty}^{n+1}(\lambda) = \lim_{h_{n+1}\to\infty} \mathbf{R}_{bb}^{n+1}(\lambda, h_{n+1}) = -\frac{1-\nu_{n+1}}{\mu_{n+1}|\lambda|}\mathbf{I} + i\frac{1-2\nu_{n+1}}{2\mu_{n+1}\lambda}\mathbf{E}_2,$$

$$\mathbf{u}_{\infty}^{n+1}(\lambda) = \lim_{h_{n+1}\to\infty} \mathbf{u}_{b0}^{n+1}(\lambda) = \frac{1-\nu_{n+1}}{2\mu_{n+1}\lambda^2}$$

$$\times\begin{pmatrix} \displaystyle\int_{y_n}^{\infty}[g_2-2g_1](\lambda,\xi)e_n\,d\xi + |\lambda|\int_{y_n}^{\infty} g_2(\lambda,\xi)(\xi-y_n)e_n\,d\xi + \frac{3-2\nu_{n+1}}{1-\nu_{n+1}}\overline{X}_{2b} \\ i\operatorname{sign}\lambda\left[2\displaystyle\int_{y_n}^{\infty} g_1(\lambda,\xi)e_n\,d\xi - |\lambda|\int_{y_n}^{\infty} g_2(\lambda,\xi)(\xi-y_n)e_n\,d\xi - 2\overline{X}_{2b}\right]\end{pmatrix},$$

where $e_n = e^{|\lambda|(y_n-\xi)}$.

Let us note that relation (3.14) corresponding to (9e) is obtained by passing to the limit $h_{n+1}\to\infty$ in (3.8) and taking into account boundary condition (3.12) for $j=n$. This fact allows us to consider this condition in a common scheme with the other ones.

In order to complete the difference equations (3.12) it is necessary to know the second boundary condition along boundary Γ_0. Such a condition is absent (the solution along Γ_0 is not known and is connected with the solutions inside the wedge-shaped regions $\Omega_{m_+}^+$, Ω_0^-). To overcome the mentioned difficulty, let us assume that this condition has the form:

(3.16) $$\mathbf{z}(x_1) = \boldsymbol{\sigma}^{(1)}(x_1,x_2)|_{\Gamma_0},$$

with some unknown vector-function $\mathbf{z}(x_1)$. Then the missing relation can be written:

(3.17) $$-\mathbf{C}_{\sigma}^0\boldsymbol{\sigma}_b^1 + \mathbf{B}_{\sigma}^0\boldsymbol{\sigma}_b^2 + \mathbf{F}_{\sigma b}^0 = \mathbf{0},$$

where

$$\mathbf{C}_{\sigma}^0 = \mathbf{I},\qquad \mathbf{B}_{\sigma}^0 = \mathbf{0},\qquad \mathbf{F}_{\sigma b}^0 = \overline{\mathbf{z}}_+(\lambda) + \overline{\mathbf{z}}_-(\lambda).$$

Here we present the vector-function $\mathbf{z}(x_1)$ as a sum of even and odd vector-functions $\mathbf{z}_+(x_1)$, $\mathbf{z}_-(x_1)$.

As one could expect, the boundary conditions (3.14) and (3.17) are prescribed for different vector-functions $\boldsymbol{\sigma}_b^j$, $\boldsymbol{\sigma}_t^j$. In order to solve any of the systems of difference equations $(3.12)_{\rm a}$, $(3.12)_{\rm b}$, these equations should be rewritten in terms of the common type vector-functions. So (3.17) can be written in the form:

(3.18) $$-\mathbf{C}_{\sigma}^0\boldsymbol{\sigma}_t^0 + \mathbf{B}_{\sigma}^0\boldsymbol{\sigma}_t^1 + \mathbf{F}_{\sigma t}^0 = \mathbf{0},$$

where $\mathbf{C}_{\sigma}^0$, $\mathbf{B}_{\sigma}^0$ are defined above, $\mathbf{F}_{\sigma t}^0 = \mathbf{F}_{\sigma b}^0$, and the vector-function $\boldsymbol{\sigma}_t^0$ is defined by the relation similar to $(3.11)_2$:

(3.19) $$\boldsymbol{\sigma}_t^0 = \boldsymbol{\sigma}_b^1 \qquad (\Delta\boldsymbol{\sigma}_0 = \mathbf{0}).$$

Now we can solve the difference equation $(3.12)_a$ with boundary conditions (3.14), (3.18) by the sweep method. Following [16], we define the auxiliary matrix-functions $\boldsymbol{\alpha}_\sigma^j$ and vector-functions $\boldsymbol{\beta}_\sigma^j$:

$$(3.20) \qquad \boldsymbol{\alpha}_\sigma^n = (\mathbf{C}_\sigma^n)^{-1}\mathbf{A}_\sigma^n, \qquad \boldsymbol{\beta}_\sigma^n = (\mathbf{C}_\sigma^n)^{-1}\mathbf{F}_{\sigma t}^t,$$

and in the next steps

$$(3.21) \qquad \begin{aligned} \boldsymbol{\alpha}_\sigma^j &= (\mathbf{C}_\sigma^j - \mathbf{B}_\sigma^j\boldsymbol{\alpha}_\sigma^{j+1})^{-1}\mathbf{A}_\sigma^j, \\ \boldsymbol{\beta}_\sigma^j &= (\mathbf{C}_\sigma^j - \mathbf{B}_\sigma^j\boldsymbol{\alpha}_\sigma^{j+1})^{-1}(\mathbf{F}_{\sigma t}^j + \mathbf{B}_\sigma^j\boldsymbol{\beta}_\sigma^{j+1}), \qquad j = n-1, ..., 2, 1. \end{aligned}$$

Then the solutions of this problem are in the form:

$$(3.22) \qquad \boldsymbol{\sigma}_t^0 = \boldsymbol{\beta}_\sigma^0 = \overline{\mathbf{z}}_+(\lambda) + \overline{\mathbf{z}}_-(\lambda),$$

$$(3.23) \qquad \boldsymbol{\sigma}_t^j = \boldsymbol{\alpha}_\sigma^j\boldsymbol{\sigma}_t^{j-1} + \boldsymbol{\beta}_\sigma^j, \qquad j = 1, 2, ..., n.$$

If vector-functions $\overline{\mathbf{z}}_+(\lambda)$, $\overline{\mathbf{z}}_-(\lambda)$ are known, then the values of $\boldsymbol{\sigma}_t^j$ will be found from (3.22), (3.23). Moreover, the values of $\boldsymbol{\sigma}_b^j$ and $\mathbf{u}_{b(t)}^j$ can be obtained from (3.8), $(3.11)_2$. Corresponding relations are of the form

$$(3.24) \qquad \begin{aligned} \mathbf{u}_t^j &= \mathbf{D}_{\sigma t}^j\boldsymbol{\sigma}_t^j + \mathbf{d}_{\sigma t}^j, \qquad j = 1, 2, ..., n, \\ \mathbf{u}_b^j &= \mathbf{D}_{\sigma b}^j\boldsymbol{\sigma}_t^j + \mathbf{d}_{\sigma b}^j, \qquad j = 1, 2, ..., n-1, \end{aligned}$$

where

$$\begin{aligned} \mathbf{D}_{\sigma t}^j &= \mathbf{R}_{tt}^j + \mathbf{R}_{tb}^j(\boldsymbol{\alpha}_\sigma^j)^{-1}, \quad & \mathbf{d}_{\sigma t}^j &= \mathbf{u}_{t0}^j + \mathbf{R}_{tb}^j(\Delta\boldsymbol{\sigma}_{j-1} - (\boldsymbol{\alpha}_\sigma^j)^{-1}\boldsymbol{\beta}_\sigma^j), \\ \mathbf{D}_{\sigma b}^j &= \mathbf{R}_{bt}^j + \mathbf{R}_{bb}^j(\boldsymbol{\alpha}_\sigma^j)^{-1}, \quad & \mathbf{d}_{\sigma b}^j &= \mathbf{u}_{b0}^j + \mathbf{R}_{bb}^j(\Delta\boldsymbol{\sigma}_{j-1} - (\boldsymbol{\alpha}_\sigma^j)^{-1}\boldsymbol{\beta}_\sigma^j). \end{aligned}$$

Further, we shall need the relation between the Fourier transforms of vector-functions $\mathbf{u}_b^1$, and tractions $\boldsymbol{\sigma}_b^1$ $(= \boldsymbol{\sigma}_t^0)$, along the exterior (with respect to the layered part of the domain) boundary Γ_0:

$$(3.25) \qquad \mathbf{u}_b^1 = \mathbf{M}_\sigma\boldsymbol{\sigma}_b^1 + \mathbf{m}_\sigma = \mathbf{M}_\sigma(\lambda)(\overline{\mathbf{z}}_+(\lambda) + \overline{\mathbf{z}}_-(\lambda)) + \mathbf{m}_\sigma,$$

where matrix-function $\mathbf{M}_\sigma$ and vector-function $\mathbf{m}_\sigma$ are of the form

$$(3.26) \qquad \mathbf{M}_\sigma = \mathbf{R}_{bt}^1\boldsymbol{\alpha}_\sigma^1 + \mathbf{R}_{bb}^1, \qquad \mathbf{m}_\sigma = \mathbf{u}_{b0}^1 + \mathbf{R}_{bt}^1\boldsymbol{\beta}_\sigma^1.$$

Relations (3.25), (3.26) will be necessary to satisfy the contact conditions along the boundary Γ_0.

LEMMA 1. Matrix-function $\mathbf{M}_\sigma(\lambda)$ has negative components on the main diagonal ($m_{\sigma kk}(\lambda) < 0$, $k = 1, 2$), and is nondegenerate ($\det \mathbf{M}_\sigma(\lambda) > 0$) for arbitrary $\lambda \in \mathbb{R}_+$. Both the matrix-function $\mathbf{M}_\sigma$ and vector-function $\mathbf{m}_\sigma$ belong to the corresponding class $\mathbf{C}^\infty(\mathbb{R}_+)$, and the following estimates at infinity are valid for any exterior boundary conditions under consideration ($\mathcal{J} = 1-5$):

$$\mathbf{M}_\sigma(\lambda) = -\frac{1-\nu_1}{\mu_1|\lambda|}\mathbf{I} + i\frac{1-2\nu_1}{2\mu_1\lambda}\mathbf{E}_2 + \mathcal{O}(P_3(|\lambda|)e^{-2|\lambda|h_1}),$$
$$\mathbf{m}_\sigma(\lambda) = o(|\lambda|^{-3}), \qquad |\lambda| \to \infty;$$

but in the neighbourhood of zero point ($\lambda \to 0$), they depend on the exterior boundary conditions in the following manner:

$\mathcal{J} = 1:$
$$\mathbf{M}_\sigma(\lambda) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\lambda) \\ \mathcal{O}(\lambda) & \mathcal{O}(1) \end{pmatrix}, \qquad \mathbf{m}_\sigma(\lambda) = \mathcal{O}(1),$$

$\mathcal{J} = 2:$
$$\mathbf{M}_\sigma(\lambda) = c_2\lambda^{-4}\begin{pmatrix} 2+\mathcal{O}(\lambda^2) & i\lambda a_2 + \mathcal{O}(\lambda^3) \\ -i\lambda a_2 + \mathcal{O}(\lambda^3) & \lambda^2 b_2 + \mathcal{O}(\lambda^4) \end{pmatrix},$$
$$\mathbf{m}_\sigma(\lambda) = -c_2\lambda^{-4}\begin{pmatrix} 2\xi_1 + 2i\lambda\Upsilon_L - i\lambda h_n\xi_2 + \mathcal{O}(\lambda^2) \\ -i\lambda a_2\xi_1 + \lambda^2 a_2\Upsilon_L + \lambda^2(b_2 - y_n a_2)\xi_2 + \mathcal{O}(\lambda^3) \end{pmatrix},$$

$\mathcal{J} = 3:$
$$\mathbf{M}_\sigma(\lambda) = c_3\lambda^{-2}\begin{pmatrix} \mathcal{O}(\lambda^2) & i\lambda a_3 + \mathcal{O}(\lambda^3) \\ -i\lambda b_3 + \mathcal{O}(\lambda^3) & 1+\mathcal{O}(\lambda^2) \end{pmatrix},$$
$$\mathbf{m}_\sigma(\lambda) = -c_3\lambda^{-2}\begin{pmatrix} i\lambda a_3\xi_2 + \mathcal{O}(\lambda^2) \\ \xi_2 + \mathcal{O}(\lambda) \end{pmatrix},$$

$\mathcal{J} = 4:$
$$\mathbf{M}_\sigma(\lambda) = c_4\lambda^{-4}\begin{pmatrix} 1+\mathcal{O}(\lambda^2) & i\lambda a_4 + \mathcal{O}(\lambda^3) \\ -i\lambda a_4 + \mathcal{O}(\lambda^3) & \lambda^2 a_4^2 + \mathcal{O}(\lambda^4) \end{pmatrix},$$
$$\mathbf{m}_\sigma(\lambda) = -c_4\lambda^{-4}\begin{pmatrix} \xi_1 + i\lambda\Upsilon_L + \mathcal{O}(\lambda^2) \\ -i\lambda a_4\xi_1 + \lambda^2 a_4\Upsilon_L + \mathcal{O}(\lambda^3) \end{pmatrix},$$

$\mathcal{J} = 5:$
$$\mathbf{M}_\sigma(\lambda) = c_5\lambda^{-2}\begin{pmatrix} |\lambda|(1+\mathcal{O}(|\lambda|)) & i\lambda a_5(1+\mathcal{O}(|\lambda|)) \\ -i\lambda a_5(1+\mathcal{O}(|\lambda|)) & |\lambda|(1+\mathcal{O}(|\lambda|)) \end{pmatrix},$$
$$\mathbf{m}_\sigma(\lambda) = -c_5\lambda^{-2}\begin{pmatrix} |\lambda|\xi_1 + i\lambda a_5\xi_2 + \mathcal{O}(\lambda^2) \\ -i\lambda a_5\xi_1 + |\lambda|\xi_2 + \mathcal{O}(\lambda^2) \end{pmatrix}.$$

Here constants $a_\mathcal{J} = a_\mathcal{J}^{(1)}$, $b_\mathcal{J} = b_\mathcal{J}^{(1)}$, $c_\mathcal{J} = c_\mathcal{J}^{(1)}$ ($\mathcal{J} = 2,3,4$) are defined from the recurrent relations:

$$c_2^{(j)} = c_2^{(j+1)}\left(1 + c_2^{(j+1)}\left[2(a_2^{(j+1)})^2 + 2h_j a_2^{(j)} + 2a_2^{(j+1)}a_2^{(j)}\right]\frac{\mu_j h_j}{3(1-\nu_j)}\right)^{-1},$$
$$a_2^{(j)} = a_2^{(j+1)} + 2h_j, \qquad b_2^{(j)} = b_2^{(j+1)} + h_j\left(a_2^{(j)} + a_2^{(j+1)}\right),$$
$$a_2^{(n)} = h_n, \qquad b_2^{(n)} = \frac{2}{3}h_n^2, \qquad c_2^{(n)} = -\frac{3(1-\nu_n)}{\mu_n h_n^3},$$

$$a_3^{(j)} = a_3^{(j+1)} - h_j \frac{3+\nu_j}{1-\nu_j}, \qquad b_3^{(j)} = b_3^{(j+1)} + h_j,$$

$$c_3^{(j)} = c_3^{(j+1)} h_j^{-1} \left[1 - 2c_3^{(j+1)}\right]^{-1},$$

$$a_3^{(n)} = b_3^{(n)} = -h_n \frac{3+\nu_n}{1-\nu_n}, \qquad c_3^{(n)} = -\frac{1-\nu_n}{2\mu_n h_n},$$

$$c_4^{(j)} = c_4^{(j+1)}(1-\nu_j) h_j^{-1} \left(1 - \nu_j - \mu_j c_4^{(j+1)} \left[2a_4^{(j+1)} a_4^{(j)} + \frac{2}{3} h_j^2\right]\right)^{-1},$$

$$a_4^{(j)} = a_4^{(j+1)} + h_j, \qquad a_4^{(n)} = h_n, \qquad c_4^{(n)} = -\frac{3(1-\nu_n)}{2\mu_n h_n^3},$$

$$a_5 = \frac{1-2\nu_{n+1}}{2(1-\nu_{n+1})}, \qquad c_5 = -\frac{(1-\nu_{n+1})}{\mu_{n+1}}.$$

Besides, vector Ξ_L and constant Υ_L from the right-hand sides of the relations are:

$$\Xi_L = [\xi_1, \xi_2]^\top = \sum_{j=1}^{n} \int_{y_{j-1}}^{y_j} \bar{\mathbf{X}}^j(0,x)\,dx - \sum_{j=1}^{n} \Delta\boldsymbol{\sigma}_j(0),$$

$$\Upsilon_L = i[1,0] \frac{\partial}{\partial\lambda} \left(\sum_{j=1}^{n} \Delta\boldsymbol{\sigma} j(\lambda) - \sum_{j=1}^{n} \int_{y_{j-1}}^{y_j} \bar{\mathbf{X}}^j(\lambda,x)\,dx \right) \Big|_{\lambda=0}$$
$$- [0,1] \left(\sum_{j=1}^{n} (y_n - y_j) \Delta\boldsymbol{\sigma}_j(0) + \sum_{j=1}^{n} \int_{y_{j-1}}^{y_j} (x - y_n) \bar{\mathbf{X}}^j(0,x)\,dx \right),$$

for all types of the exterior boundary conditions along the boundary Γ_n ($\mathcal{J} = 2,3,4$). But in the case $\mathcal{J} = 5$ (when the last layer is a half-plane), in these relations it is necessary to replace number n by $n+1$, and to assume that $\Delta\boldsymbol{\sigma}_{n+1} = 0$, $y_{n+1} = \infty$.

To prove this Lemma 1 we need recurrent relations for the matrix-function $\mathbf{M}_\sigma = \mathbf{M}_\sigma^1$ and the vector-function $\mathbf{m}_\sigma = \mathbf{m}_\sigma^1$ in the form:

$$\begin{aligned}
\mathbf{M}_\sigma^j &= -\mathbf{R}_{bt}^j \left[\boldsymbol{\tau}_j + \mathbf{R}_{tt}^j - \mathbf{M}_\sigma^{j+1}\right]^{-1} \mathbf{R}_{tb}^j + \mathbf{R}_{bb}^j,\\
\mathbf{M}_\sigma^n &= \mathbf{R}_{bt}^n \boldsymbol{\alpha}_\sigma^n + \mathbf{R}_{bb}^n, \qquad j = n-1, ..., 2, 1;\\
\mathbf{m}_\sigma^j &= \mathbf{u}_{b0}^j - (\mathbf{M}_\sigma^j - \mathbf{R}_{bb}^j)(\mathbf{R}_{tb}^j)^{-1}(\mathbf{m}_\sigma^{j+1} + \mathbf{M}_\sigma^{j+1} \Delta\boldsymbol{\sigma}_j - \mathbf{u}_{t0}^j - \Delta\mathbf{u}_j),\\
\mathbf{m}_\sigma^n &= \mathbf{R}_{bt}^n \boldsymbol{\beta}_\sigma^n + \mathbf{u}_{b0}^n, \qquad j = n-1, ..., 2, 1.
\end{aligned}$$

Here $\boldsymbol{\alpha}_\sigma^n, \boldsymbol{\beta}_\sigma^n$ are defined in (3.20). Then the results follow by induction on j. We do not present all the details, taking into account the volume of the paper and its technical character.

In the process of proving Lemma 1 we also obtain all properties of the even and odd components $\mathbf{M}_\sigma^\pm$, $\mathbf{m}_\sigma^\pm$ of matrix-function $\mathbf{M}_\sigma$ and vector-function $\mathbf{m}_\sigma$. Thus, matrix $\mathbf{M}_\sigma^+$ has only nonzero elements on the main diagonal and they are real even functions of λ, but $\mathbf{M}_\sigma^-$ has nonzero elements on the second diagonal and they are imaginary odd functions of λ.

COROLLARY 1. From (3.7) and Lemma 1 we can rewrite *a priori* estimates for the unknown vector-functions $\mathbf{z}_+$, $\mathbf{z}_-$:

$$
\begin{aligned}
&\bar{\mathbf{z}}_\pm(\lambda),\ \lambda\frac{\partial}{\partial\lambda}\bar{\mathbf{z}}_\pm(\lambda) = \mathcal{O}(\lambda^{-\gamma_0}), \qquad \lambda\to\infty;\\
&\bar{\mathbf{z}}_+(\lambda) = \mathbf{z}_*^+ + \mathcal{O}(\lambda^{\gamma_2^+}), \qquad \lambda\frac{\partial}{\partial\lambda}\bar{\mathbf{z}}_\pm(\lambda) = \mathcal{O}(\lambda^{\gamma_2^\pm}), \qquad \lambda\to 0;\\
(3.27)_1 \qquad &\bar{\mathbf{z}}_-(\lambda) = \mathbf{z}_*^-\lambda^{\gamma_2^-} + \mathcal{O}(\lambda^{\gamma_2^-+\varepsilon}), \qquad \lambda\to 0;\\
&\lambda(\mathbf{M}_\sigma^+(\lambda)\bar{\mathbf{z}}_+(\lambda) + \mathbf{M}_\sigma^-(\lambda)\bar{\mathbf{z}}_-(\lambda) + \mathbf{m}_\sigma^+(\lambda)) = \mathcal{O}(\lambda^{\min\{1,\gamma_1^+\}}), \qquad \lambda\to 0;\\
&\lambda(\mathbf{M}_\sigma^+(\lambda)\bar{\mathbf{z}}_-(\lambda) + \mathbf{M}_\sigma^-(\lambda)\bar{\mathbf{z}}_+(\lambda) + \mathbf{m}_\sigma^-(\lambda)) = \mathcal{O}(\lambda^{\gamma_1^-}), \qquad \lambda\to 0;\\
&\lambda(\mathbf{M}_\sigma^+(\lambda)\bar{\mathbf{z}}_\pm(\lambda) + \mathbf{M}_\sigma^-(\lambda)\bar{\mathbf{z}}_\mp(\lambda) + \mathbf{m}_\sigma^\pm(\lambda)) = \mathcal{O}(\lambda^{\gamma_0}), \qquad \lambda\to\infty;
\end{aligned}
$$

here

$$
\begin{aligned}
&\mathcal{J}=1: \quad \gamma_1^+ \ge 1, \qquad \gamma_1^- = 2,\\
&\mathcal{J}=2: \quad \begin{cases} \gamma_2^+ = 2, \quad \gamma_2^- = 1,\\ \mathbf{z}_*^+ = \Xi_L, \quad [1,0]\mathbf{z}_*^- = i\Upsilon_L - i[0,1]y_n\Xi_L, \end{cases}\\
(3.27)_2 \qquad &\mathcal{J}=3: \quad \begin{cases} \gamma_2^- = \gamma_1^+ = 1,\\ [0,1](\mathbf{z}_*^+ - \Xi_L) = 0, \end{cases}\\
&\mathcal{J}=4: \quad \begin{cases} \gamma_2^+ = 2, \quad \gamma_2^- = 1,\\ [1,0](\mathbf{z}_*^+ - \Xi_L) = 0, \quad [1,0]\mathbf{z}_*^- = i\Upsilon_L - i[0,1]y_n\Xi_L, \end{cases}\\
&\mathcal{J}=5: \quad \begin{cases} \gamma_2^+ = \gamma_1^+, \quad \gamma_2^- = 1, \quad \gamma_1^- \ge 1,\\ \mathbf{z}_*^+ = \Xi_L. \end{cases}
\end{aligned}
$$

Let us note that the even and odd components of the solution decrease at infinity in a different way (the corresponding orders are $\gamma_1^\pm$, $\gamma_2^\pm$). So by γ_1, γ_2 in (2.13) we shall understand the largest of them. Besides, the values of constant vector $\mathbf{z}_*^+$ in the case $\mathcal{J}=1$, and the first (second) component of $\mathbf{z}_*^+$ in the case $\mathcal{J}=3$ ($\mathcal{J}=4$) are unknown as yet and will be obtained below. This is important to note that the corresponding relations (3.27) present the usual equilibrium conditions for the layered part of the domain and consequently, vector $\mathbf{z}_*^+$ (or one of its components) is defined from *a priori* estimations. They follow from (3.10) and the interfacial conditions (3.11):

$$
\boldsymbol{\sigma}_t^n(0) - \boldsymbol{\sigma}_b^1(0) + \sum_{j=1}^{n}\int_{y_{j-1}}^{y_j}\bar{\mathbf{X}}(0,x)dx - \sum_{j=1}^{n-1}\Delta\boldsymbol{\sigma}_j(0) = 0,
$$

$$i[1,0]\frac{\partial}{\partial\lambda}\left(\boldsymbol{\sigma}_t^n(\lambda)-\boldsymbol{\sigma}_b^1(\lambda)+\sum_{j=1}^{n}\int_{y_{j-1}}^{y_j}\bar{\mathbf{X}}^j(\lambda,x)dx-\sum_{j=1}^{n-1}\Delta\boldsymbol{\sigma}j(\lambda)\right)\Big|_{\lambda=0}$$
$$+[0,1]\left(\sum_{j=1}^{n}\int_{y_{j-1}}^{y_j}x\bar{\mathbf{X}}^j(0,x)dx-\sum_{j=1}^{n-1}y_j\Delta\boldsymbol{\sigma}_j(0)+y_n\boldsymbol{\sigma}_t^n(0)\right)=0.$$

The second equation can also be rewritten in an equivalent form:

$$i[1,0]\frac{\partial}{\partial\lambda}\left(\boldsymbol{\sigma}_t^n(\lambda)-\boldsymbol{\sigma}_b^1(\lambda)+\sum_{j=1}^{n}\int_{y_{j-1}}^{y_j}\bar{\mathbf{X}}^j(\lambda,x)dx-\sum_{j=1}^{n-1}\Delta\boldsymbol{\sigma}j(\lambda)\right)\Big|_{\lambda=0}$$
$$+[0,1]\left(\sum_{j=1}^{n}\int_{y_{j-1}}^{y_j}(x-y_n)\bar{\mathbf{X}}^j(0,x)dx+\sum_{j=1}^{n-1}(y_n-y_j)\Delta\boldsymbol{\sigma}_j(0)+y_n\boldsymbol{\sigma}_b^1(0)\right)=0.$$

Constant u_* from (2.13) can be calculated from the equation:

$$\mathbf{u}_*=2\int_0^\infty(\mathbf{M}_\sigma^+(\lambda)\bar{\mathbf{z}}_+(\lambda)+\mathbf{M}_\sigma^-(\lambda)\bar{\mathbf{z}}_-(\lambda)+\mathbf{m}_\sigma^+(\lambda))d\lambda. \tag{3.28}$$

Moreover, if any component of the displacement is prescribed along one of the exterior wedge surfaces, then we have (see (2.13)) the additional relation for the corresponding component of vector $\mathbf{u}_*$:

$$\begin{aligned}
\mathbf{u}_*(1,\mathcal{J}^-,\mathcal{J})=0,\quad \mathbf{u}_*(\mathcal{J}^+,1,\mathcal{J})&=0,\quad \mathcal{J}^\pm=1-4,\quad \mathcal{J}=1-5,\\
[1,0]\mathbf{S}(\theta_0^+)\mathbf{u}_*(3,\mathcal{J}^-,\mathcal{J})&=0,\quad \mathcal{J}^-=2-4,\quad \mathcal{J}=1-5,\\
[0,1]\mathbf{S}(\theta_0^+)\mathbf{u}_*(4,\mathcal{J}^-,\mathcal{J})&=0,\quad \mathcal{J}^-=2-4,\quad \mathcal{J}=1-5,\\
[1,0]\mathbf{S}(\theta_{m_-}^-)\mathbf{u}_*(\mathcal{J}^+,3,\mathcal{J})&=0,\quad \mathcal{J}^+=2-4,\quad \mathcal{J}=1-5,\\
[0,1]\mathbf{S}(\theta_{m_-}^-)\mathbf{u}_*(\mathcal{J}^+,4,\mathcal{J})&=0,\quad \mathcal{J}^+=2-4,\quad \mathcal{J}=1-5.
\end{aligned}\tag{3.29}$$

Here matrix-function $\mathbf{S}(\phi)$ is defined in (4.9). Besides, for the next problems we can conclude that for any $\mathcal{J}=1-5$

$$\begin{aligned}
\mathbf{u}_*(3,4,\mathcal{J})=0,\quad \mathbf{u}_*(4,3,\mathcal{J})=0,\quad \theta_0^+\neq\theta_{m_-}^-+\pi/2,\\
\mathbf{u}_*(3,3,\mathcal{J})=0,\quad \mathbf{u}_*(4,4,\mathcal{J})=0,\quad \theta_0^+\neq\theta_{m_-}^-.
\end{aligned}$$

Finally, if equalities arise instead of the inequalities in the last four problems, only one of the respective conditions (3.29) which should be satisfied is linearly independent.

4. Sweep method in the wedge-shaped domain

Rewriting Eqs. (2.1), (2.2) in the polar coordinates and applying the Mellin transform in the form:

$$\tilde{\mathbf{u}}(s,\theta) = \int_0^\infty \mathbf{u}(r,\theta) r^{s-1} dr,$$

$$\tilde{\boldsymbol{\sigma}}, \tilde{\Theta}(s,\theta) = \int_0^\infty \boldsymbol{\sigma}, \Theta(r,\theta) r^{s} dr,$$

$$\tilde{\mathbf{X}}(s,\theta) = \int_0^\infty \mathbf{X}(r,\theta) r^{s+1} dr,$$

we obtain [37] the following relations in the respective regions:

$$\begin{aligned} -s\tilde{\sigma}_{rr} + \frac{\partial}{\partial\theta}\tilde{\sigma}_{r\theta} - \tilde{\sigma}_{\theta\theta} + \tilde{X}_r &= 0, \\ -(s-1)\tilde{\sigma}_{r\theta} + \frac{\partial}{\partial\theta}\tilde{\sigma}_{\theta\theta} + \tilde{X}_\theta &= 0; \end{aligned} \tag{4.1}$$

$$\begin{aligned} -s\tilde{u}_r &= \frac{1}{2\mu}\left[\tilde{\sigma}_{rr} - \nu\tilde{\sigma}\right] + \beta\tilde{\Theta}, \\ \tilde{u}_r + \frac{\partial}{\partial\theta}\overline{u}_\theta &= \frac{1}{2\mu}\left[\tilde{\sigma}_{\theta\theta} - \nu\tilde{\sigma}\right] + \beta\tilde{\Theta}, \\ \frac{\partial}{\partial\theta}\overline{u}_r - (s+1)\tilde{u}_\theta &= \frac{1}{\mu}\tilde{\sigma}_{r\theta}, \qquad \sigma = \sigma_{rr} + \sigma_{\theta\theta}, \end{aligned} \tag{4.2}$$

where constants μ, $\beta = \gamma/(2\mu+\lambda)$, and ν (defined in (2.2)) are different in regions Ω_j^+, Ω_j^-. For function $\tilde{\sigma}_{\theta\theta}$ the equation can be found:

$$\begin{aligned} &\left(\frac{\partial^2}{\partial\theta^2} + (s+1)^2\right)\left(\frac{\partial^2}{\partial\theta^2} + (s-1)^2\right)\tilde{\sigma}_{\theta\theta} \\ &\qquad = \left(\frac{\partial^2}{\partial\theta^2} + (s+1)^2\right) h_1 + s(s-1)h_2\,, \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} h_1(s,\theta) &= -\frac{\partial}{\partial\theta}\tilde{X}_\theta - (s-1)\tilde{X}_r - s(s-1)\frac{2\mu\beta}{1-\nu}\tilde{\Theta}, \\ h_2(s,\theta) &= \frac{1}{1-\nu}\left[(s+1)\tilde{X}_r - \frac{\partial}{\partial\theta}\tilde{X}_\theta\right]. \end{aligned}$$

The corresponding solution is of the form:

$$\tilde{\sigma}_{\theta\theta}^{(j\pm)}(\theta,s) = C_1^{j\pm}(s)\cos[\theta(s+1)] + C_2^{j\pm}(s)\cos[\theta(s-1)] + C_3^{j\pm}(s)\sin[\theta(s+1)] + \frac{s-1}{4(s+1)}\int\limits_{\theta_{j-1}^{\pm}}^{\theta} h_2^{(j\pm)}(s,\phi)\sin[(\phi-\theta)(s+1)]d\phi + C_4^{j\pm}(s)\sin[\theta(s-1)] - \frac{1}{4}\int\limits_{\theta_{j-1}^{\pm}}^{\theta}\left[h_2^{(j\pm)} + \frac{4}{(s-1)}h_1^{(j\pm)}\right]\sin[(\phi-\theta)(s-1)]d\phi. \tag{4.4}$$

Using a line of reasoning similar to that applied in (3.8), we can obtain the relations between the Mellin transforms of the vectors of displacements and tractions along the corresponding boundaries of the wedges.

$$\begin{aligned} \mathbf{v}_t^j &= \mathbf{P}_{tt}^j\mathbf{p}_t^j + \mathbf{P}_{tb}^j\mathbf{p}_b^j + \mathbf{v}_{t0}^j\,, \\ \mathbf{v}_b^j &= \mathbf{P}_{bt}^j\mathbf{p}_t^j + \mathbf{P}_{bb}^j\mathbf{p}_b^j + \mathbf{v}_{b0}^j\,, \qquad j = 1,2,...,m_+\,; \end{aligned} \tag{4.5}$$

$$\begin{aligned} \mathbf{w}_t^{(k)} &= \mathbf{Q}_{tt}^k\mathbf{q}_t^k + \mathbf{Q}_{tb}^k\mathbf{q}_b^k + \mathbf{w}_{t0}^k\,, \\ \mathbf{w}_b^k &= \mathbf{Q}_{bt}^k\mathbf{q}_t^k + \mathbf{Q}_{bb}^k\mathbf{q}_b^k + \mathbf{w}_{b0}^k\,, \qquad k = 1,2,...,m_-\,. \end{aligned} \tag{4.6}$$

From the properties of the Mellin transform and from *a priori* estimates (2.13) of vector-functions of class $\mathbf{LW}(\Omega)$ it follows that vector-functions $\mathbf{p}_{b(t)}^j(s)$, $\mathbf{q}_{b(t)}^k(s)$ are analytical in the strip $-\gamma_0 < \Re s < \gamma_2$, but $\mathbf{w}_{b(t)}^k(s)$, $\mathbf{v}_{b(t)}^j(s)$ are analytical in domain $0 < \Re s < \gamma_1$, in general. In point $s = 0$ they can have a common simple pole. Besides, the relations hold:

$$\mathbf{v}_{t(b)}^j(s) - \frac{1}{s}\mathbf{u}_*, \qquad \mathbf{w}_{t(b)}^k(s) - \frac{1}{s}\mathbf{u}_* = \mathcal{O}(1), \qquad s \to 0, \tag{4.7}$$

and the vector-functions from the left-hand side of (4.7) are analytical in the whole strip $-\gamma_0 < \Re s < \gamma_1$. All of the discussed vector-functions decrease to zero along imaginary axis inside the corresponding domains.

Here the coefficients in (4.5) are found from the formulae:

$$\begin{aligned} \mathbf{P}_{tt}^j(s) &= \frac{1}{2s\mu_j^+}[\mathbf{E}_2 - \mathbf{P}_1^j\mathbf{E}_1], \qquad \mathbf{P}_{tb}^j(s) = \frac{1}{2s\mu_j^+}\mathbf{P}_2^j\mathbf{E}_1\,, \\ \mathbf{P}_{bb}^j(s) &= \frac{1}{2s\mu_j^+}[\mathbf{E}_2 + \mathbf{E}_1\mathbf{P}_1^j], \qquad \mathbf{P}_{bt}^j(s) = -\frac{1}{2s\mu_j^+}\mathbf{E}_1\mathbf{P}_2^j\,, \\ \mathbf{P}_1^j &= p_j(s)\begin{pmatrix} s\sin 2\phi_j^+ + \sin 2s\phi_j^+ & 2\sin^2 s\phi_j^+ - 2s\sin^2\phi_j^+ \\ 2\sin^2 s\phi_j^+ + 2s\sin^2\phi_j^+ & s\sin 2\phi_j^+ - \sin 2s\phi_j^+ \end{pmatrix}, \end{aligned}$$

$$(4.8) \qquad p_j(s) = \frac{1-\nu_j^+}{\sin^2 s\phi_j^+ - s^2\sin^2\phi_j^+},$$

$$2s\mu_j^+ \mathbf{v}_{t0}^j = \mathbf{P}_1^j \mathbf{E}_1 H_t - \mathbf{P}_2^j \mathbf{E}_1 H_b - \frac{1}{s+1}\begin{pmatrix} \tilde{X}_{\theta t} - (1-\nu_j^+)\displaystyle\int\limits_{\theta_{j-1}^+}^{\theta_j^+} h_2 \cos[(\phi-\theta_j^+)(s+1)]d\phi \\ -(1-\nu_j^+)\displaystyle\int\limits_{\theta_{j-1}^+}^{\theta_j^+} h_2 \sin[(\phi-\theta_j^+)(s+1)]d\phi \end{pmatrix},$$

$$2s\mu_j^+ \mathbf{v}_{b0}^j = \mathbf{E}_1 \mathbf{P}_2^j H_t - \mathbf{E}_1 \mathbf{P}_1^j H_b - \frac{1}{s+1}\begin{pmatrix} \tilde{X}_{\theta b} \\ 0 \end{pmatrix}, \qquad H_b = \frac{1}{s-1}\begin{pmatrix} 0 \\ \tilde{X}_{\theta b} \end{pmatrix},$$

$$\mathbf{P}_2^j = p_j(s)\begin{pmatrix} 2s\sin\phi_j^+\cos s\phi_j^+ + 2\sin s\phi_j^+\cos\phi_j^+ & -2(s-1)\sin\phi_j^+\sin s\phi_j^+ \\ 2(s+1)\sin\phi_j^+\sin s\phi_j^+ & 2s\sin\phi_j^+\cos s\phi_j^+ - 2\sin s\phi_j^+\cos\phi_j^+ \end{pmatrix},$$

where $\phi_j^+ = \theta_j^+ - \theta_{j-1}^+$, $\tilde{X}_{\theta b} = \tilde{X}_\theta(s, \theta_{j-1}^+)$, $\tilde{X}_{\theta t} = \tilde{X}_\theta(s, \theta_j^+)$, the functions $h_j(s,\theta)$ and the matrices $\mathbf{E}_1$, $\mathbf{E}_2$ are defined in (4.3) and (3.8), respectively, but the remaining vector-function H_t is:

$$H_t = \frac{1}{4}\begin{pmatrix} h_t^{(1)} \\ h_t^{(2)} \end{pmatrix},$$

$$h_t^{(1)} = \frac{s-1}{s+1}\int\limits_{\theta_{j-1}^+}^{\theta_j^+} h_2 \sin[(\phi-\theta_j^+)(s+1)]d\phi - \int\limits_{\theta_{j-1}^+}^{\theta_j^+}\left[h_2 + \frac{4h_1}{s-1}\right]\sin[(\phi-\theta_j^+)(s-1)]d\phi,$$

$$h_t^{(2)} = \frac{4}{s-1}\tilde{X}_{\theta t} - \int\limits_{\theta_{j-1}^+}^{\theta_j^+} h_2\cos[(\phi-\theta_j^+)(s+1)]d\phi + \int\limits_{\theta_{j-1}^+}^{\theta_j^+}\left[h_2 + \frac{4h_1}{s-1}\right]\cos[(\phi-\theta_j^+)(s-1)]d\phi.$$

Note that matrix-functions $\mathbf{P}_{t(b)t(b)}^j(s)$ and vector-functions $\mathbf{v}_{b(t)0}^j(s)$ can be estimated to be of $\mathcal{O}(s^{-2})$ when $s \to 0$. But the unknown vectors $\mathbf{v}^j_{t(b)}(s)$ can have in this point a simple pole only. Hence, by investigating the main terms of asymptotics ($s \to 0$) of the right-hand side of the relations (4.5) it can be obtained that the bounded vector-functions $\mathbf{p}^j_{t(b)}(s)$ should satisfy the following additional relations:

$$\begin{aligned} \mathbf{p}^j_t(0) &= \mathbf{S}(\phi^+_j)\mathbf{p}^j_b(0) - \int\limits_{\theta^+_{j-1}}^{\theta^+_j} \mathbf{S}(\theta^+_j - \phi)\tilde{\mathbf{X}}^j(0,\phi)d\phi, \\ \mathbf{S}(\phi) &= \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}, \end{aligned} \tag{4.9}$$

where $\mathbf{X}^j(r,\theta) = [X^j_\theta(r,\theta), X^j_r(r,\theta)]^\top$. This relation is the usual equilibrium condition of the j_+-th wedge. Moreover, passing to the limit $s \to 1$ in (4.5) we obtain

$$[1,0]\left\{\mathbf{p}^j_t(1) - \mathbf{p}^j_b(1) - \int\limits_{\theta^+_{j-1}}^{\theta^+_j} \tilde{\mathbf{X}}^j(1,\phi)d\phi\right\} = 0, \tag{4.10}$$

what is the torque balance condition for the j_+-th wedge.

What concerns the formulae for coefficients $\mathbf{Q}^j_{t(b)t(b)}(s)$, $\mathbf{w}^j_{t(b)0}(s)$ from (4.6), they are similar to those in (4.8) if all upper indices "+" are replaced by "−" (for example μ^+_j, θ^+_j, ϕ^+_j by μ^-_j, θ^-_j, ϕ^-_j).

Further on we rewrite the internal boundary conditions as above:

$$\begin{aligned} \mathbf{v}^{j+1}_b - \mathbf{v}^j_t - \boldsymbol{\tau}^+_j\mathbf{p}^j_t &= \Delta\mathbf{v}_j, \\ \mathbf{p}^{j+1}_b - \mathbf{p}^j_t &= \Delta\mathbf{p}_j, \qquad j = 1,...,m_+ - 1, \\ \mathbf{w}^{k+1}_b - \mathbf{w}^k_t - \boldsymbol{\tau}^-_k\mathbf{q}^k_t &= \Delta\mathbf{w}_k, \\ \mathbf{q}^{k+1}_b - \mathbf{q}^k_t &= \Delta\mathbf{q}_k, \qquad k = 1,...,m_- - 1, \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} \Delta\mathbf{v}_j(s) &= \mathcal{M}[\delta\mathbf{v}_j](s), \qquad \Delta\mathbf{p}_j(s) = \mathcal{M}[\delta\mathbf{p}_j](s+1), \\ \Delta\mathbf{w}_k(s) &= \mathcal{M}[\delta\mathbf{w}_k](s), \qquad \Delta\mathbf{q}_k(s) = \mathcal{M}[\delta\mathbf{q}_k](s+1), \end{aligned} \tag{4.12}$$

are the Mellin transforms of the respective vector-functions (see (2.5), (2.6)).

So the net result for wedge-shaped domain Ω^- is of the similar form as (3.22), (3.23):

$$\begin{aligned} \mathbf{q}^0_t &= \mathbf{q}^1_b = \tilde{\mathbf{z}}_-(s+1) - \tilde{\mathbf{z}}_+(s+1) + \Delta\mathbf{q}_*(s), \\ \mathbf{q}^k_t &= \boldsymbol{\alpha}^k_q\mathbf{q}^{k-1}_t + \boldsymbol{\beta}^k_q, \qquad k = 1,...,m_-, \\ \mathbf{w}^k_t &= D^k_{qt}\mathbf{q}^k_t + \mathbf{d}^k_{qt}, \qquad \mathbf{w}^k_b = D^k_{qb}\mathbf{q}^k_t + \mathbf{d}^k_{qb}. \end{aligned} \tag{4.13}$$

Let us now write the relations necessary to calculate the coefficients of (4.13):

$$
\begin{aligned}
\boldsymbol{\alpha}_q^{m_-} &= (\mathbf{C}_q^{m_-})^{-1}\mathbf{A}_q^{m_-}, \qquad \boldsymbol{\beta}_q^{m_-} = (\mathbf{C}_q^{m_-})^{-1}\mathbf{F}_{qt}^{m_-},\\
\boldsymbol{\alpha}_q^{k} &= (\mathbf{C}_q^{k} - \mathbf{B}_q^{k}\boldsymbol{\alpha}_q^{k+1})^{-1}\mathbf{A}_q^{k},\\
\boldsymbol{\beta}_q^{k} &= (\mathbf{C}_q^{k} - \mathbf{B}_q^{k}\boldsymbol{\alpha}_q^{k+1})^{-1}(\mathbf{F}_{qt}^{k} + \mathbf{B}_q^{k}\boldsymbol{\beta}_q^{k+1}),\\
\mathbf{D}_{qt}^{k} &= \mathbf{Q}_{tt}^{k} + \mathbf{Q}_{tb}^{k}(\boldsymbol{\alpha}_q^{k})^{-1}, \qquad \mathbf{d}_{qt}^{k} = \mathbf{w}_{t0}^{k} + \mathbf{Q}_{tb}^{k}(\Delta\mathbf{q}_{k-1} - (\boldsymbol{\alpha}_q^{k})^{-1}\boldsymbol{\beta}_q^{k}),\\
\mathbf{D}_{qb}^{k} &= \mathbf{Q}_{bt}^{k} + \mathbf{Q}_{bb}^{k}(\boldsymbol{\alpha}_q^{k})^{-1}, \qquad \mathbf{d}_{qb}^{k} = \mathbf{w}_{b0}^{k} + \mathbf{Q}_{bb}^{k}(\Delta\mathbf{q}_{k-1} - \boldsymbol{\alpha}_q^{k})^{-1}\boldsymbol{\beta}_q^{k}),\\
\mathbf{A}_q^{k} &= -\mathbf{Q}_{tb}^{k}, \qquad \mathbf{C}_q^{k} = \boldsymbol{\tau}_k^{-} + \mathbf{Q}_{tt}^{k} - \mathbf{Q}_{bb}^{k+1}, \qquad \mathbf{B}_q^{k} = \mathbf{Q}_{bt}^{k+1},\\
\mathbf{F}_{qb}^{k} &= \mathbf{w}_{b0}^{k+1} - \mathbf{w}_{t0}^{k} - \Delta\mathbf{w}_k + (\mathbf{Q}_{tt}^{k} + \boldsymbol{\tau}_k^{-})\Delta\mathbf{q}_k - \mathbf{Q}_{bt}^{k+1}\Delta\mathbf{q}_{k+1},\\
\mathbf{F}_{qt}^{k} &= \mathbf{w}_{b0}^{k+1} - \mathbf{w}_{t0}^{k} - \Delta\mathbf{w}_k + \mathbf{Q}_{bb}^{k+1}\Delta\mathbf{q}_k - \mathbf{Q}_{tb}^{k}\Delta\mathbf{q}_{k-1},\\
\Delta\mathbf{q}_*(s) &= \mathcal{M}[\delta\boldsymbol{\sigma}_-](s+1), \qquad k = m_- - 1, ..., 1.
\end{aligned}
\tag{4.14}
$$

The constants at the first step of the sweep method are defined by one of boundary conditions (2.10)

$\mathcal{J} = 1:$

$$
\mathbf{C}_q^{m_-} = \mathbf{Q}_{tt}^{m_-}, \qquad \mathbf{A}_q^{m_-} = -\mathbf{Q}_{tb}^{m_-},
$$
$$
\mathbf{F}_{qt}^{m_-} = -\Delta\mathbf{w}_{m_-} - \mathbf{w}_{t0}^{m_-} - \mathbf{Q}_{tb}^{m_-}\Delta\mathbf{q}_{m_- -1},
$$

$\mathcal{J} = 2:$

$$
\mathbf{C}_q^{m_-} = \mathbf{I}, \qquad \mathbf{A}_q^{m_-} = \mathbf{0}, \qquad \mathbf{F}_{qt}^{m_-} = -\Delta\mathbf{q}_{m_-},
$$

$\mathcal{J} = 3, 4:$

$$
\mathbf{C}_q^{m_-} = \begin{pmatrix} \delta_{\mathcal{J}3} & 0 \\ 0 & \delta_{\mathcal{J}4} \end{pmatrix}\mathbf{Q}_{tt}^{m_-} + \begin{pmatrix} \delta_{\mathcal{J}4} & 0 \\ 0 & \delta_{\mathcal{J}3} \end{pmatrix},
$$
$$
\mathbf{A}_q^{m_-} = -\begin{pmatrix} \delta_{\mathcal{J}3} & 0 \\ 0 & \delta_{\mathcal{J}4} \end{pmatrix}\mathbf{Q}_{tb}^{m_-}, \qquad \Delta\mathbf{h}_{\mathcal{J}} = \begin{pmatrix} \widetilde{\delta w}_{\mathcal{J}}(s) \\ \widetilde{\delta q}_{\mathcal{J}}(s+1) \end{pmatrix},
$$
$$
\mathbf{F}_{qt}^{m_-} = -\begin{pmatrix} \delta_{\mathcal{J}3} & 0 \\ 0 & \delta_{\mathcal{J}4} \end{pmatrix}\left(\mathbf{w}_{t0}^{m_-} + \mathbf{Q}_{tb}^{m_-}\Delta\mathbf{q}_{m_- -1}\right) - \Delta\mathbf{h}_{\mathcal{J}}.
$$

Here $\delta_{\mathcal{J}j}$ is the Kronecker symbol. The remaining initial conditions along the boundary Γ_0^- follow from (2.5) and from assumption (3.16):

$$
\mathbf{C}_q^0 = \mathbf{I}, \qquad \mathbf{B}_q^0 = \mathbf{0}, \qquad \Delta\mathbf{q}_0 = \mathbf{0}, \qquad \mathbf{F}_{qt}^0 = \widetilde{\mathbf{z}}_-(s+1) - \widetilde{\mathbf{z}}_+(s+1) + \Delta\mathbf{q}_*.
\tag{4.15}
$$

In an analogous way we obtain the relations

$$
\begin{aligned}
\mathbf{p}_b^{m_+ +1} &= \mathbf{p}_t^{m_+} = \widetilde{\mathbf{z}}_-(s+1) + \widetilde{\mathbf{z}}_+(s+1) - \Delta\mathbf{p}_*(s),\\
\mathbf{p}_b^{j} &= \boldsymbol{\alpha}_p^{j}\mathbf{p}_b^{j+1} + \boldsymbol{\beta}_p^{j}, \qquad j = m_+, ..., 1,\\
\mathbf{v}_t^{j} &= \mathbf{D}_{pt}^{j}\mathbf{p}_b^{j} + \mathbf{d}_{pt}^{j}, \qquad \mathbf{v}_b^{j} = \mathbf{D}_{pb}^{j}\mathbf{p}_p^{j} + \mathbf{d}_{pb}^{j},
\end{aligned}
\tag{4.16}
$$

for wedge region Ω^+, where the necessary coefficients are of the form:

$$
\begin{aligned}
\boldsymbol{\alpha}_p^1 &= (\mathbf{C}_p^0)^{-1}\mathbf{B}_p^0, \qquad \boldsymbol{\beta}_p^1 = (\mathbf{C}_p^0)^{-1}\mathbf{F}_{pb}^0,\\
\boldsymbol{\alpha}_p^{j+1} &= (\mathbf{C}_p^j - \mathbf{A}_p^j\boldsymbol{\alpha}_p^j)^{-1}\mathbf{B}_p^j,\\
\boldsymbol{\beta}_p^{j+1} &= (\mathbf{C}_p^j - \mathbf{A}_p^j\boldsymbol{\alpha}_p^j)^{-1}(\mathbf{F}_{pb}^j + \mathbf{A}_p^j\boldsymbol{\beta}_p^j),\\
\mathbf{D}_{pt}^j &= \mathbf{P}_{tb}^j + \mathbf{P}_{tt}^j\boldsymbol{\alpha}_p^j)^{-1}, \qquad \mathbf{d}_{pt}^j = \mathbf{v}_{t0}^j - \mathbf{P}_{tt}^j(\Delta\mathbf{p}_j + \boldsymbol{\alpha}_p^j)^{-1}\boldsymbol{\beta}_p^j),\\
\mathbf{D}_{pb}^j &= \mathbf{P}_{bb}^j + \mathbf{P}_{bt}^j\boldsymbol{\alpha}_p^j)^{-1}, \qquad \mathbf{d}_{pb}^j = \mathbf{v}_{b0}^j - \mathbf{P}_{bt}^j(\Delta\mathbf{p}_j + (\boldsymbol{\alpha}_p^j)^{-1}\boldsymbol{\beta}_p^j),\\
\mathbf{A}_p^j &= -\mathbf{P}_{tb}^j, \qquad \mathbf{C}_p^j = \boldsymbol{\tau}_j^+ + \mathbf{P}_{tt}^j - \mathbf{P}_{bb}^{j+1}, \qquad \mathbf{B}_p^j = \mathbf{P}_{bt}^{j+1},\\
\mathbf{F}_{pb}^j &= \mathbf{v}_{b0}^{j+1} - \mathbf{v}_{t0}^j - \Delta\mathbf{v}_j + (\mathbf{P}_{tt}^j + \boldsymbol{\tau}_j^+)\Delta\mathbf{p}_j - \mathbf{P}_{bt}^{j+1}\Delta\mathbf{p}_{j+1},\\
\mathbf{F}_{pt}^j &= \mathbf{v}_{b0}^{j+1} - \mathbf{v}_{t0}^j - \Delta\mathbf{v}_j + \mathbf{P}_{bb}^{j+1}\Delta\mathbf{p}_j - \mathbf{P}_{tb}^j\Delta\mathbf{p}_{j-1},\\
\Delta\mathbf{p}_*(s) &= \mathcal{M}[\delta\boldsymbol{\sigma}_+](s+1), \qquad j = 1,2,...,l-1.
\end{aligned} \tag{4.17}
$$

Boundary conditions for the corresponding system of difference equations follow from (2.9), $(2.7)_{\mathrm{b}}$ and assumption (3.16):

$\mathcal{J} = 1$:

$$
\begin{aligned}
\mathbf{B}_p^0 &= -\mathbf{P}_{bt}^1, \qquad \mathbf{C}_p^0 = \mathbf{P}_{bb}^1,\\
\mathbf{F}_{bt}^0 &= \Delta\mathbf{v}_0 - \mathbf{v}_{b0}^1 + \mathbf{P}_{bt}^1\Delta\mathbf{p}_1,
\end{aligned}
$$

$\mathcal{J} = 2$:

$$
\mathbf{B}_p^0 = \mathbf{0}, \qquad \mathbf{C}_p^0 = \mathbf{I}, \qquad \mathbf{F}_{pb}^0 = \Delta\mathbf{p}_0,
$$

$\mathcal{J} = 3,4$:

$$
\begin{aligned}
\mathbf{B}_p^0 &= -\begin{pmatrix}\delta_{\mathcal{J}3} & 0\\ 0 & \delta_{\mathcal{J}4}\end{pmatrix}\mathbf{P}_{bt}^1,\\
\mathbf{C}_p^0 &= \begin{pmatrix}\delta_{\mathcal{J}3} & 0\\ 0 & \delta_{\mathcal{J}4}\end{pmatrix}\mathbf{P}_{bb}^1 + \begin{pmatrix}\delta_{\mathcal{J}4} & 0\\ 0 & \delta_{\mathcal{J}3}\end{pmatrix},\\
\mathbf{F}_{pb}^0 &= -\begin{pmatrix}\delta_{\mathcal{J}3} & 0\\ 0 & \delta_{\mathcal{J}4}\end{pmatrix}\left(\mathbf{v}_{b0}^0 - \mathbf{P}_{bt}^1\Delta\mathbf{p}_1\right) + \Delta\mathbf{h}_{\mathcal{J}},\\
\Delta\mathbf{h}_{\mathcal{J}} &= \begin{pmatrix}\widetilde{\delta v}_{\mathcal{J}}(s)\\ \widetilde{\delta p}_{\mathcal{J}}(s+1)\end{pmatrix},\\
\mathbf{C}_p^{m_+} &= \mathbf{I}, \qquad \mathbf{A}_p^{m_+} = \mathbf{0}, \qquad \Delta\mathbf{p}_{m_+} = \mathbf{0},\\
\mathbf{F}_{pt}^{m_+} &= \tilde{\mathbf{z}}_+(s+1) + \tilde{\mathbf{z}}_-(s+1) - \Delta\mathbf{p}_*.
\end{aligned}
$$

Besides we can find relations similar to (3.25) connecting the Mellin transforms of displacements and tractions along boundaries Γ_0^-, $\Gamma_{m_+}^+$:

$$
\mathbf{v}_t^{m_+} = \mathbf{M}_p\mathbf{p}_t^{m_+} + \mathbf{m}_p = \mathbf{M}_p(\tilde{\mathbf{z}}_+(s+1) + \tilde{\mathbf{z}}_-(s+1) - \Delta\mathbf{p}_*(s)) + \mathbf{m}_p, \tag{4.18}
$$

$$
\mathbf{w}_b^1 = \mathbf{M}_q\mathbf{q}_b^1 + \mathbf{m}_q = \mathbf{M}_q(\tilde{\mathbf{z}}_-(s+1) - \tilde{\mathbf{z}}_+(s+1) + \Delta\mathbf{q}_*(s)) + \mathbf{m}_q, \tag{4.19}
$$

where

$$\mathbf{M}_p(s) = \mathbf{P}_{tb}^{m_+}\boldsymbol{\alpha}_p^{m_+} + \mathbf{P}_{tt}^{m_+}, \qquad \mathbf{m}_p(s) = \mathbf{v}_{t0}^{m_+} + \mathbf{P}_{tb}^{m_+}\boldsymbol{\beta}_p^{m_+},$$
(4.20)
$$\mathbf{M}_q(s) = \mathbf{Q}_{bt}^{1}\boldsymbol{\alpha}_q^{1} + \mathbf{Q}_{bb}^{1}, \qquad \mathbf{m}_q(s) = \mathbf{w}_{b0}^{1} + \mathbf{Q}_{bt}^{1}\boldsymbol{\beta}_q^{1}.$$

LEMMA 2. Matrix-functions $\mathbf{M}_p(s)$, $\mathbf{M}_q(s)$ and vector-functions $\mathbf{m}_p(s)$, $\mathbf{m}_q(s)$ are analytical in the strip $|\Re s| < 1$ except maybe one point $s = 0$. For matrix-functions $\mathbf{M}_p(s)$, $\mathbf{M}_q(s)$ and for their components $m_{pkk}(s)$, $m_{qkk}(s)$ $(k = 1, 2)$, the following relations are true:

$$\mathbf{M}_{p(q)}(-s) = \mathbf{M}_{p(q)}^{\top}(s), \qquad \overline{\mathbf{M}_{p(q)}(it)} = \mathbf{M}_{p(q)}^{\top}(it), \qquad t \in \mathbb{R},$$
$$\det \mathbf{M}_{p(q)}(it) > 0, \qquad m_{pkk}(it) > 0, \qquad m_{qkk}(it) < 0, \qquad t \in \mathbb{R}_+.$$

Besides, for any exterior boundary conditions ($\mathcal{J}^{\pm} = 1 - 4$) asymptotic expansions near the infinity point hold true ($|\Im s| \to \infty$):

$$\mathbf{M}_p(s) = -\frac{1}{2s\mu_{m_+}^{+}}\Big[(1-2\nu_{m_+}^{+})\mathbf{E}_2 - 2(1-\nu_{m_+}^{+})\mathrm{tg}(\phi_{m_+}^{+}s)\mathbf{I}\Big] + O\Big(P_3(s)e^{-2|\Im s|\phi_{m_+}^{+}}\Big),$$
$$\mathbf{M}_q(s) = -\frac{1}{2s\mu_1^{-}}\Big[(1-2\nu_1^{-})\mathbf{E}_2 + 2(1-\nu_1^{-})\mathrm{tg}(\phi_1^{-}s)\mathbf{I}\Big] + O\Big(P_4(s)e^{-2|\Im s|\phi_1^{-}}\Big),$$

but in the neighbourhood of the zero point ($s \to 0$), they depend on the exterior boundary conditions along the wedge surfaces:

$\mathcal{J}^{\pm} = 1:$
$$\mathbf{M}_p(s),\ \mathbf{M}_q(s) = \mathcal{O}(1), \qquad \mathbf{m}_p(s),\ \mathbf{m}_q(s) = \mathcal{O}(1),$$

$\mathcal{J}^{\pm} = 2:$
$$\mathbf{M}_p(s) \sim s^{-2}\mathbf{T}_2^{+}, \qquad \mathbf{M}_q(s) \sim s^{-2}\mathbf{T}_2^{-},$$
$$\mathbf{m}_p(s) \sim -s^{-2}\mathbf{T}_2^{+}\left(\boldsymbol{\Xi}_+ + \mathbf{S}(-\theta_0^{+})\Delta\mathbf{p}_0(0)\right),$$
$$\mathbf{m}_q(s) \sim -s^{-2}\mathbf{T}_2^{-}\left(\boldsymbol{\Xi}_- - \mathbf{S}(-\theta_{m_-}^{-})\Delta\mathbf{q}_{m_-}(0)\right),$$

$\mathcal{J}^{\pm} = 3:$
$$\mathbf{M}_p(s) \sim c_3^p s^{-2}\mathbf{T}_3(\theta_0^{+}), \qquad \mathbf{M}_q(s) \sim c_3^q s^{-2}\mathbf{T}_3(\theta_{m_-}^{-}),$$
$$\mathbf{m}_p(s) \sim -2c_3^p s^{-2}\left(\xi_1^{+}\sin\theta_0^{+} + \xi_2^{+}\cos\theta_0^{+} + \Delta q_0^{(2)}(0)\right)\begin{pmatrix}\sin\theta_0^{+}\\ \cos\theta_0^{+}\end{pmatrix},$$
$$\mathbf{m}_q(s) \sim -2c_3^q s^{-2}\left(\xi_1^{-}\sin\theta_{m_-}^{-} + \xi_2^{-}\cos\theta_{m_-}^{-} - \Delta q_{m_-}^{(2)}(0)\right)\begin{pmatrix}\sin\theta_{m_-}^{-}\\ \cos\theta_{m_-}^{-}\end{pmatrix},$$

$\mathcal{J}^{\pm} = 4:$
$$\mathbf{M}_p(s) \sim c_4^p s^{-2}\mathbf{T}_4(\theta_0^{+}), \qquad \mathbf{M}_q(s) \sim c_4^q s^{-2}\mathbf{T}_4(\theta_{m_-}^{-}),$$
$$\mathbf{m}_p(s) \sim -2c_4^p s^{-2}\left(\xi_1^{+}\cos\theta_0^{+} - \xi_2^{+}\sin\theta_0^{+} + \Delta q_0^{(1)}(0)\right)\begin{pmatrix}\cos\theta_0^{+}\\ -\sin\theta_0^{+}\end{pmatrix},$$
$$\mathbf{m}_q(s) \sim -2c_4^q s^{-2}\left(\xi_1^{-}\cos\theta_{m_-}^{-} - \xi_2^{-}\sin\theta_{m_-}^{-} - \Delta q_{m_-}^{(1)}(0)\right)\begin{pmatrix}\cos\theta_{m_-}^{-}\\ -\sin\theta_{m_-}^{-}\end{pmatrix},$$

where the next terms in these relations are estimated as being of $\mathcal{O}(s^{-1})$ as $s \to 0$. Besides, near point $s = 1$ the following equations hold true:

$\mathcal{J}^{\pm} = 1, 3:$
$$\mathbf{M}_p(s),\ \mathbf{M}_q(s) = \mathcal{O}(1), \qquad \mathbf{m}_p(s),\ \mathbf{m}_q(s) = \mathcal{O}(1), \qquad s \to 1,$$

$\mathcal{J}^{\pm} = 2:$
$$\mathbf{M}_{p(q)}(s) \sim \frac{1}{s-1}\begin{pmatrix} d_{12}^{p(q)} & 0 \\ d_{22}^{p(q)} & 0 \end{pmatrix}, \qquad \mathbf{m}_{p(q)}(s) \sim -\frac{\Upsilon_{p(q)}}{s-1}\begin{pmatrix} d_{12}^{p(q)} \\ d_{22}^{p(q)} \end{pmatrix},$$

$\mathcal{J}^{\pm} = 4:$
$$\mathbf{M}_{p(q)}(s) \sim \frac{1}{s-1}\begin{pmatrix} d_{4}^{p(q)} & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{m}_{p(q)}(s) \sim -\frac{\Upsilon_{p(q)}}{s-1}\begin{pmatrix} d_{4}^{p(q)} \\ 0 \end{pmatrix}.$$

Here $P_3(s)$, $P_4(s)$ are some polynomials of the second degree, matrix-functions $\mathbf{T}_3(\phi)$, $\mathbf{T}_4(\phi)$ are of the form:

$$\mathbf{T}_3(\phi) = \begin{pmatrix} 2\sin^2\phi & \sin 2\phi \\ \sin 2\phi & 2\cos^2\phi \end{pmatrix}, \qquad \mathbf{T}_4(\phi) = \begin{pmatrix} 2\cos^2\phi & -\sin 2\phi \\ -\sin 2\phi & 2\sin^2\phi \end{pmatrix},$$

but $\Delta p_0^{(j)}(0)$, $\Delta q_{m_-}^{(j)}(0)$ $(j = 1, 2)$ are the components of vectors $\Delta\mathbf{p}_0(0)$, $\Delta\mathbf{q}_{m_-}(0)$ defined in (4.12). Vectors $\mathbf{\Xi}_{\pm}$, and constants Υ_p, Υ_q are defined like this:

$$\mathbf{\Xi}_{+} = [\xi_1^{+}, \xi_2^{+}]^{\top} = \sum_{j=1}^{m_+-1} \mathbf{S}(\theta_{m_+}^{+} - \theta_j^{+})\Delta\mathbf{p}_j(0) - \sum_{j=1}^{m_+} \int_{\theta_{j-1}^{+}}^{\theta_j^{+}} \mathbf{S}(\theta_{m_+}^{+} - \phi)\tilde{\mathbf{X}}^j(0, \phi)d\phi,$$

$$\mathbf{\Xi}_{-} = [\xi_1^{-}, \xi_2^{-}]^{\top} = -\sum_{j=1}^{m_--1} \mathbf{S}(\theta_0^{-} - \theta_j^{-})\Delta\mathbf{q}_j(0) + \sum_{j=1}^{m_-} \int_{\theta_{j-1}^{-}}^{\theta_j^{-}} \mathbf{S}(\theta_0^{-} - \phi)\tilde{\mathbf{X}}^j(0, \phi)d\phi,$$

$$\Upsilon_p = [1, 0]\left\{\sum_{j=0}^{m_+-1} \Delta\mathbf{p}_j(1) - \sum_{j=1}^{m_+} \int_{\theta_{j-1}^{+}}^{\theta_j^{+}} \tilde{\mathbf{X}}^j(1, \phi)d\phi\right\},$$

$$\Upsilon_q = [1, 0]\left\{\sum_{j=1}^{m_-} \int_{\theta_{j-1}^{-}}^{\theta_j^{-}} \tilde{\mathbf{X}}^j(1, \phi)d\phi - \sum_{j=1}^{m_+} \Delta\mathbf{q}_j(1)\right\}.$$

Constants $c_3^p = -a_{+}^{m_+}$, $c_4^p = a_{-}^{m_+}$, $c_3^q = b_{+}^1$, $c_4^q = -b_{-}^1$ are calculated by recurrent relations:

$$a_{\pm}^{j+1} = a_{\pm}^{j}\left\{1 + \frac{2\mu_{j+1}^{+} a_{\pm}^{j}}{1 - \nu_{j+1}^{+}}\left[\phi_{j+1}^{+} \pm \sin\phi_{j+1}^{+} \cos\left(\phi_{j+1}^{+} + 2\sum_{k=1}^{j}\phi_k^{+}\right)\right]\right\}^{-1},$$

$$b_{\pm}^{j-1} = b_{\pm}^{j}\left\{1+\frac{2\mu_{j-1}^{-}a_{\pm}^{j}}{1-\nu_{j-1}^{-}}\left[\phi_{j-1}^{-}\pm\sin\phi_{j-1}^{-}\cos\left(\phi_{j-1}^{-}+2\sum_{k=j}^{m_-}\phi_k^{-}\right)\right]\right\}^{-1},$$

$$a_{\pm}^{1} = \frac{1-\nu_1^{+}}{\mu_1^{+}[2\phi_1^{+}\pm\sin 2\phi_1^{+}]}, \qquad b_{\pm}^{m_-} = \frac{1-\nu_{m_-}^{-}}{\mu_{m_-}^{-}[2\phi_{m_-}^{-}\pm\sin 2\phi_{m_-}^{-}]},$$

but matrices $\mathbf{T}_2^{\pm}$ are defined by the following recurrent matrix equations:

$$\mathbf{T}_2^{+} = \frac{2(1-\nu_{m_+}^{+})}{\mu_{m_+}^{+}}\mathbf{B}_{m_+}^{+}\mathbf{D}_{m_+}^{+}, \qquad \mathbf{T}_2^{-} = \frac{2(1-\nu_1^{-})}{\mu_1^{-}}\mathbf{B}_1^{-}\mathbf{D}_1^{-},$$

$$\mathbf{B}_j^{+} = \mathbf{I}-\left\{\mathbf{I}+\frac{\mu_j^{+}(1-\nu_{j-1}^{+})}{\mu_{j-1}^{+}(1-\nu_j^{+})}\mathbf{S}(\phi_j^{+})\mathbf{B}_{j-1}^{+}\mathbf{D}_{j-1}^{+}\mathbf{E}_1\left(\mathbf{D}_j^{+}\right)^{-1}\mathbf{E}_1\mathbf{S}^{-1}(\phi_j^{+})\right\}^{-1},$$

$$\mathbf{B}_j^{-} = \mathbf{I}-\left\{\mathbf{I}+\frac{\mu_j^{-}(1-\nu_{j+1}^{-})}{\mu_{j+1}^{-}(1-\nu_j^{-})}\mathbf{S}^{-1}(\phi_j^{-})\mathbf{B}_{j+1}^{-}\mathbf{D}_{j+1}^{-}\mathbf{E}_1\left(\mathbf{D}_j^{-}\right)^{-1}\mathbf{E}_1\mathbf{S}(\phi_j^{-})\right\}^{-1},$$

$$\mathbf{B}_1^{+},\ \mathbf{B}_{m_-}^{-} = \mathbf{I}, \qquad \mathbf{D}_j^{\pm} = \frac{\pm 1}{4[(\phi_j^{\pm})^2-\sin^2\phi_j^{\pm}]}\begin{pmatrix} 2\phi_j^{\pm}+\sin 2\phi_j^{\pm} & \mp 2\sin^2\phi_j^{\pm} \\ \mp 2\sin^2\phi_j^{\pm} & 2\phi_j^{\pm}-\sin 2\phi_j^{\pm}\end{pmatrix}.$$

Taking into account the volume of the paper, we do not present here the corresponding recurrent relations for the constants $d_{ij}^{p(q)}$ in the asymptotics near point $s = 1$, because they are not directly used in the analysis of systems of integral equations.

COROLLARY 2. In the process of proving Lemma 2 it can be also shown that matrix-functions $\mathbf{M}_p(s)$, $\mathbf{M}_q(s)$ can be represented in the following form:

$$\mathbf{M}_{p(q)}(s) = \mathbf{M}_{p(q)}^{+}(s) + \mathbf{M}_{p(q)}^{-}(s),$$

where matrix-functions $\mathbf{M}_{p(q)}^{\pm}(s)$ satisfy the relations:

$$\mathbf{M}_{p(q)}^{\pm}(s) = \pm\mathbf{M}_{p(q)}^{\pm}(-s), \qquad \mathbf{M}_{p(q)}^{+}(s) = [\mathbf{M}_{p(q)}^{+}(s)]^{\top}, \qquad \mathbf{M}_{p(q)}^{-}(s) = -[\mathbf{M}_{p(q)}^{-}(s)]^{\top}.$$

COROLLARY 3. Let us assume that if the following statements hold true:

1. Domain Ω is symmetric with respect to $0X_2$ axis ($m_+ = m_-$, $\theta_j^{+}+\theta_{m_+-j}^{-}+\pi = 0$),

2. The constants in Eqs. (2.2) and in interior boundary conditions (2.5), (2.6) for the corresponding wedges are identical ($\mu_j^{+} = \mu_{m_+-j+1}^{-}$, $\nu_j^{+} = \nu_{m_+-j+1}^{-}$, $\boldsymbol{\tau}_j^{+} = \boldsymbol{\tau}_{m_+-j}^{-}$),

3. The types of the boundary conditions on both of the exterior wedge surfaces are identical ($\mathcal{J}^{+} = \mathcal{J}^{-}$);

then it can be shown that:

$$\mathbf{M}_p(s)+\mathbf{M}_q(s) = \begin{pmatrix} 0 & f_1(s) \\ f_1(-s) & 0\end{pmatrix}, \qquad \mathbf{M}_p(s)-\mathbf{M}_q(s) = \begin{pmatrix} f_2(s) & 0 \\ 0 & f_3(s)\end{pmatrix},$$

where $f_j(s)$ $(j = 1-3)$ are such functions that $f_j(-it) = \overline{f_j(it)}$, and $sf_1(s)$, $f_2(s)$, $f_3(s)$ are even real functions.

REMARK 4. The following relations should be true if the tractions are prescribed along the external wedge boundaries Γ_0^+, $\Gamma_{m_-}^-$:

$$(4.21)\quad \begin{aligned} &\mathcal{J}^+ = 2: && \mathbf{p}_t^{m+}(0) = \Xi_+ + \mathbf{S}(-\theta_0^+)\Delta\mathbf{p}_0(0),\\ &\mathcal{J}^- = 2: && \mathbf{p}_b^1(0) = \Xi_- - \mathbf{S}(-\theta_{m_-}^-)\Delta\mathbf{p}_{m_-}(0),\\ &\mathcal{J}^+ = 3: && [0,1]\{\Delta\mathbf{p}_0(0) + \mathbf{S}(\theta_0^+)[\Xi_+ - \mathbf{p}_t^{m+}(0)]\} = 0,\\ &\mathcal{J}^- = 3: && [0,1]\{\Delta\mathbf{q}_{m_-}(0) - \mathbf{S}(\theta_{m_-}^-)[\Xi_- - \mathbf{q}_b^1(0)]\} = 0,\\ &\mathcal{J}^+ = 4: && [1,0]\{\Delta\mathbf{p}_0(0) + \mathbf{S}(\theta_0^+)[\Xi_+ - \mathbf{p}_t^{m+}(0)]\} = 0,\\ &\mathcal{J}^- = 4: && [1,0]\{\Delta\mathbf{q}_{m_-}(0) - \mathbf{S}(\theta_{m_-}^-)[\Xi_- - \mathbf{q}_b^1(0)]\} = 0. \end{aligned}$$

They follow from relations (4.18), (4.19), because their left-hand sides can have a simple pole only at point $s = 0$, but the right-hand sides have a second degree pole, in general.

REMARK 5. Let us note that $\widetilde{\mathbf{z}}^+(1) = \pi\overline{\mathbf{z}}^+(0)$. Then, using the first relations from (4.13), (4.16) for $\mathbf{q}_b^1$, $\mathbf{p}_t^{m+}$ we obtain:

$$(4.22)\quad \begin{aligned} &(2,2,\mathcal{J}): && 2\pi\mathbf{z}_*^+ - \Xi_W = 0,\\ &(2,3,\mathcal{J}): && [0,1]\mathbf{S}(\theta_{m_-}^-)\{2\pi\mathbf{z}_*^+ - \Xi_W\} = 0,\\ &(2,4,\mathcal{J}): && [1,0]\mathbf{S}(\theta_{m_-}^-)\{2\pi\mathbf{z}_*^+ - \Xi_W\} = 0, \qquad \mathcal{J} = 1-5,\\ &(3,2,\mathcal{J}): && [0,1]\mathbf{S}(\theta_0^+)\{2\pi\mathbf{z}_*^+ - \Xi_W\} = 0,\\ &(4,2,\mathcal{J}): && [1,0]\mathbf{S}(\theta_0^+)\{2\pi\mathbf{z}_*^+ - \Xi_W\} = 0. \end{aligned}$$

Here we have introduced the following notation:

$$(4.23)\quad \Xi_W = \Xi_+ - \Xi_- + \Delta\mathbf{p}_*(0) + \Delta\mathbf{q}_*(0) + \mathbf{S}(-\theta_0^+)\mathbf{p}_0(0) + \mathbf{S}(-\theta_{m_-}^-)\mathbf{q}_{m_-}(0),$$

where Ξ_W is the principal vector of all forces and tractions acting on the wedge-shaped part of body $\Omega^+ \cup \Omega^-$. This Remark allows us to find the values of the constant vector $\mathbf{z}_*^+ = \overline{\mathbf{z}}^+(0)$ (or one of its components) for the corresponding problems.

The proofs of the Remarks follow from the fact that functions $\mathbf{v}_t^{m+}(s)$, $\mathbf{w}_b^1(s)$ can have only a simple pole at point $s = 0$. Besides, for certain values of the exterior wedge angles θ_0^+, $\theta_{m_-}^-$ we can also conclude that for any $\mathcal{J} = 1-5$

$$(4.24)\quad \begin{aligned} &(3,3,\mathcal{J}): && [1,0]\mathbf{S}(\theta_0^+)\{2\pi\mathbf{z}_*^+ - \Xi_W\} = 0,\\ &(4,4,\mathcal{J}): && [0,1]\mathbf{S}(\theta_0^+)\{2\pi\mathbf{z}_*^+ - \Xi_W\} = 0, \end{aligned} \qquad \theta_0^+ = \theta_{m_-}^-;$$

$$\begin{aligned} &(3,4,\mathcal{J}): && [0,1]\mathbf{S}(\theta_0^+)\{2\pi\mathbf{z}_*^+ - \Xi_W\} = 0,\\ &(4,3,\mathcal{J}): && [1,0]\mathbf{S}(\theta_0^+)\{2\pi\mathbf{z}_*^+ - \Xi_W\} = 0, \end{aligned} \qquad \theta_0^+ - \theta_{m_-}^- = \pi/2.$$

Let us note that for an arbitrary problem $(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ there are exactly two additional equations for constant vectors $\mathbf{u}_*$ or $\mathbf{z}_*^+$ (or their components) in (3.27), (3.29), (4.22) and (4.24).

REMARK 6. The following additional relations should be true if the normal component of tractions are prescribed along the external wedge boundaries Γ_0^+, $\Gamma_{m_-}^-$ and $\gamma_1, \gamma_2 > 1$ (see (4.7)):

$$\mathcal{J}^+ = 2,4: \quad [1,0]\mathbf{p}_t^{m+}(1) = \Upsilon_p, \qquad \mathcal{J}^- = 2,4: \quad [1,0]\mathbf{q}_b^1(1) = \Upsilon_q. \tag{4.25}$$

They are the usual torque balance equations, and they follow from relations (4.18), (4.19), because their right-hand sides can have a simple pole at point $s = 1$, but the left-hand sides are analytic in this point.

Besides, in problems $(2,2,\mathcal{J})$, $(2,4,\mathcal{J})$, $(4,2,\mathcal{J})$, $(4,4,\mathcal{J})$ we can additionally obtain from (4.18) and (4.25) that:

$$-2\pi i[1,0]\mathbf{z}_*^- = [1,0]\{\Delta\mathbf{p}_*(1) - \Delta\mathbf{q}_*(1)\} + \Upsilon_p + \Upsilon_q, \tag{4.26}$$

where $\mathbf{z}_*^-$ is defined in (3.27). To this end, the identity

$$\widetilde{\mathbf{z}}_-(2) = \int_0^\infty r\mathbf{z}_-(r)\,dr = -i\pi\left(\frac{2i}{2\pi}\int_0^\infty \mathbf{z}_-(r)\sin r\lambda\,dr\right)'\Bigg|_{\lambda=0} = -i\pi\mathbf{z}_*^-,$$

is used.

5. Conclusions

So, we have investigated the solutions of the problems of both (layered and wedge-shaped) parts of the domain. All interfacial conditions along the regions of similar geometry (2.4) ("layer – layer") and (2.5), (2.6) ("wedge – wedge") have been satisfied. Now, it is necessary to take into account interfacial conditions (2.7), (2.8) along the "layer – wedges" interfaces.

Let us note here that each of the relations (3.25), (4.18) and (4.19) as well as Lemma 1 and Lemma 2 are important. They are necessary to solve the boundary value problems for layered and wedge regions, separately. Namely, if we have arbitrary boundary conditions on the boundary Γ_0 of the types 1 – 5 (2.11), then we can find the respective integral transform of the corresponding solution in a closed form. Moreover, if the boundary conditions are of a general form (contact with the other body and so on), then the respective relation (3.25), (4.18) and (4.19) makes it possible to investigate the corresponding problem along boundary Γ_0 only. Then the information on the behaviour of the auxiliary matrix-functions and vector-functions (Lemma 1 and Lemma 2) will play an important role (for example, to reduce the problem to integral equations).

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2-D boundary value problems of thermoelasticity in a multi-wedge – multi-layered region Part 2. Systems of integral equations

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IN THE PAPER, arbitrary 2-D BVP of termoelasticity in a wedge-shaped – layered region are reduced to special systems of singular integral equations with fixed point singularities. For this purpose, the Fourier and Mellin integral transforms of the solutions in the layered and wedge-shaped parts of the domain are "fitted" together along the common interface. This interface is characterized by the conditions of given discontinuities of displacements and tractions. The theory, developed by the author elsewhere, is applied to investigate the systems obtained. The results, concerning existence and properties of the solutions are presented depending on the exterior boundary conditions. The numerical method applied to solve the systems of equations is justified.

1. Introduction

INTEGRAL TRANSFORMS are often applied to solve boundary value problems in infinite domains. So the Fourier transform in layered regions and the Mellin transform in wedge-shaped ones make it possible to find solutions of some problems in closed forms (see [19, 20, 21] and others). In the other cases, integral transforms allow us to reduce problems to integral equations (singular integral equations) which are very effective in solving numerous boundary value problems of the theory of elasticity (see [3, 9]). Thus, as it follows from [3, 9], if the problems under consideration have piecewise smooth boundaries, then singular integral equations with fixed point singularities appear, as a rule.

Previously the idea of using the Fourier and Mellin transforms simultaneously in order to solve arbitrary two-dimensional boundary value problems for the Poisson's equation in combined domains was presented in [10, 12, 13]. All of the problems were reduced to a class of singular equations (systems of the singular equations) on the semi-axis with fixed point singularities in the neighbourhood of zero and at infinity. To this end, it was essential that the composition of sine or cosine Fourier and the Mellin transforms should be represented in the form of a product of the Mellin transform with modified argument and a certain function of the argument. Some integral equations with fixed and moving singularities are considered in [3, 9], but they can not be applied to singular equations obtained in [10, 12, 13].

In [11] the mentioned class of singular integral equations with fixed point singularities on a half-axis are investigated. Conditions of solvability of the integral equations in some Banach spaces with a relevant weight were obtained, and the convergence of projective methods to find their numerical solutions was proved. Corresponding theorems for the systems of integral equations were presented in Appendix [10] without proofs. The results obtained were based on the theory of the integral operators of the Wiener-Hopf type which had been constructed in [4, 5, 7, 18]. As it was indicated in [10, 14], the process of the numerical approximation of the solutions of such equations is very stable.

In [10, 12, 13] it was shown that symbols of the obtained systems of integral equations depend essentially on the type of the interfacial and exterior boundary conditions. In turn, this is a consequence of asymptotic behaviour of functions (matrix-functions) appearing in the kernels of the integral operators.

This paper deals with the boundary value problems of thermoelasticity, and is a continuation of the previous paper [15]. Corresponding formulation of the problems is presented precisely in the second section of [15]. All definitions and notations from [15] are still valid in this part of the paper. We show that the "sweep method" [8] and the method of integral transforms similar to that used in the paper [10] make it possible to reduce all linear boundary value problems in combined domains to systems of singular integral equations with fixed point singularities, for different types of the interior and exterior conditions along the boundaries. Then the estimations made in the previous part of the paper [15] allow us to calculate the symbols of the integral operators, and to regularize those of the integral equations the indices of which are not equal to zero.

In the second section of this paper, fitting of the Fourier and Mellin transformations along common boundaries between the first layer and wedges is drawn. As a result, systems of functional equations are obtained. In the next section, necessary conditions for solvability of some boundary value problems under consideration are discussed from the mechanical and mathematical points of view.

In the fourth section, the mentioned systems of functional equations are reduced to systems of singular integral equations. The process of reduction depends essentially on the types of the exterior and interior boundary conditions. In the fifth section, analysis of the corresponding systems of integral equations is presented, their symbols are calculated depending on the types of the boundary conditions, and the parameters of functional spaces in which these equations are investigated. These parameters determine the behaviour of the solutions of the boundary value problems near singular points of the domain (at infinity and in the neighbourhood of the wedge tip). In the Appendix, some necessary formulas are presented which have complicated forms.

2. Fitting of the Fourier and Mellin transforms along the common boundary Γ_0

First of all we mentally cut the solid under consideration into three (layered - Ω_L and wedge-shaped - $\Omega^{\pm}$) parts, and apply the Fourier and Mellin transforms to Eqs. (2.1) – (2.2) of the paper [15], and to the exterior and interior boundary conditions (2.3) – (2.12) in [15], in the respective domains. Using the sweep method, the relations between the corresponding transformations of unknown vectors of displacements and tractions along the common boundary $\Gamma_0 = \Gamma_0^- \cup \Gamma_{m_+}^+$ between the domains of different geometry have been obtained:

$$\overline{\mathbf{u}}_b^1(\lambda) = \mathbf{M}_\sigma(\lambda)\overline{\boldsymbol{\sigma}}_b^1(\lambda) + \mathbf{m}_\sigma(\lambda), \tag{2.1}$$

$$\widetilde{\mathbf{v}}_t^{m_+}(s) = \mathbf{M}_p(s)\widetilde{\mathbf{p}}_t^{m_+}(s) + \mathbf{m}_p(s), \tag{2.2}$$

$$\widetilde{\mathbf{w}}_b^1(s) = \mathbf{M}_q(s)\widetilde{\mathbf{q}}_b^1(s) + \mathbf{m}_q(s) \tag{2.3}$$

(see Eqs. (3.25), (4.18) and (4.19) in [15]). Here matrix-functions and vector-functions $\mathbf{M}_\sigma(\lambda)$, $\mathbf{m}_\sigma(\lambda)$ and $\mathbf{M}_p(s)$, $\mathbf{m}_p(s)$, $\mathbf{M}_q(s)$, $\mathbf{m}_q(s)$ calculated in [15] contain all information about the layered part and the wedge-shaped parts of the domain, respectively. They can be effectively calculated and their asymptotics near zero point depend in an essential way on the exterior boundary conditions (see Lemma 1, Lemma 2 in [15]). Besides, *a priori* estimations (2.13) in [15] lead to conditions (3.7) in [15] for unknown vector-functions $\overline{\mathbf{u}}_b^1(\lambda)$, $\overline{\boldsymbol{\sigma}}_b^1(\lambda)$, in particular. On the other hand, the vector-functions $\widetilde{\mathbf{q}}_b^1(s)$, $\widetilde{\mathbf{p}}_t^{m_+}(s)$ should be analytic in the strip $-\gamma_0 < \Re s < \gamma_2$ in view of the mentioned *a priori* assumptions, but $\widetilde{\mathbf{v}}_t^{m_+}(s)$, $\widetilde{\mathbf{w}}_b^1(s)$ are analytic in $0 < \Re s < \gamma_1$, in general.

Returning to Eqs. (3.16) in [15], let us consider new unknown odd and even vector-functions $\mathbf{z}_-$, $\mathbf{z}_+$ defined by the relation:

$$\boldsymbol{\sigma}_b^1(x_1) = \mathbf{z}_+(x_1) + \mathbf{z}_-(x_1). \tag{2.4}$$

Then, using the second equations of the interfacial conditions (2.7) and (2.8) from [15], relations (2.1) – (2.3) are rewritten in the form:

$$\overline{\mathbf{u}}_b^1(\lambda) = \left[\mathbf{M}_\sigma^+(\lambda) + \mathbf{M}_\sigma^-(\lambda)\right]\left[\overline{\mathbf{z}}_+(\lambda) + \overline{\mathbf{z}}_-(\lambda)\right] + \mathbf{m}_\sigma^+(\lambda) + \mathbf{m}_\sigma^-(\lambda), \tag{2.5}$$

$$\widetilde{\mathbf{v}}_t^{m_+}(s) = \mathbf{M}_p(s)\left[\widetilde{\mathbf{z}}_+(s+1) + \widetilde{\mathbf{z}}_-(s+1) - \widetilde{\delta\boldsymbol{\sigma}}_+(s+1)\right] + \mathbf{m}_p(s), \tag{2.6}$$

$$\widetilde{\mathbf{w}}_b^1(s) = \mathbf{M}_q(s)\left[\widetilde{\mathbf{z}}_-(s+1) - \widetilde{\mathbf{z}}_+(s+1) + \widetilde{\delta\boldsymbol{\sigma}}_-(s+1)\right] + \mathbf{m}_q(s), \tag{2.7}$$

where $\mathbf{M}_\sigma^+(\lambda)$, $\mathbf{m}_\sigma^+(\lambda)$, $\mathbf{M}_\sigma^-(\lambda)$, $\mathbf{m}_\sigma^-(\lambda)$ are even and odd components of $\mathbf{M}_\sigma(\lambda)$, $\mathbf{m}_\sigma(\lambda)$.

Now, we have the situation when all equations and interior and exterior boundary conditions of the problems are satisfied, except the first equations of the interfacial conditions (see (2.7) – (2.8) in [15]) which can be rewritten in the form:

$$\mathbf{u}^{(1)}\big|_{\Gamma_{m_+}^+}(x_1) = (\mathbf{v}^{(m_+)} + \delta\mathbf{u}_+)\big|_{\Gamma_{m_+}^+}(r), \qquad x_1 = r, \tag{2.8}$$

$$\mathbf{u}^{(1)}\big|_{\Gamma_0^-}(x_1) = (\mathbf{w}^{(1)} + \delta\mathbf{u}_-)\big|_{\Gamma_0^-}(r), \qquad x_1 = -r. \tag{2.9}$$

Let us represent vector-function $\mathbf{u}^{(1)}$ along boundary Γ_0 in (2.8), (2.9) by the inverse Fourier transform

$$\mathbf{u}^{(1)}(x_1)_{|\Gamma_0} = \mathcal{F}^{-1}[\overline{\mathbf{u}}_b^1](x_1),$$

or taking into account (2.5) and parity of vector-functions $\mathbf{M}_\sigma^\pm(\lambda)$, $\mathbf{m}_\sigma^\pm(\lambda)$:

$$\begin{aligned}(2.10)\qquad \mathbf{u}^{(1)}_{|\Gamma_0} = 2\mathcal{F}_c^{-1}\left[\mathbf{M}_\sigma^+\overline{\mathbf{z}}_+ + \mathbf{M}_\sigma^-\overline{\mathbf{z}}_- + \mathbf{m}_\sigma^+\right](x_1)\\ - 2i\mathcal{F}_s^{-1}\left[\mathbf{M}_\sigma^+\overline{\mathbf{z}}_- + \mathbf{M}_\sigma^-\overline{\mathbf{z}}_+ + \mathbf{m}_\sigma^-\right](x_1),\end{aligned}$$

where

$$\mathcal{F}_c^{-1}[f(\lambda);\ \ \lambda \to x_1] \equiv \int_0^\infty f(\lambda)\cos(\lambda x_1)d\lambda,$$

$$\mathcal{F}_s^{-1}[f(\lambda);\ \ \lambda \to x_1] \equiv \int_0^\infty f(\lambda)\sin(\lambda x_1)d\lambda,$$

are the sine and cosine Fourier transforms [19].

Replacing in (2.8), (2.9) argument x_1 by r and $-r$, respectively, and applying the Mellin transform to both sides of the equations, we obtain

$$(2.11)\qquad \begin{aligned}&2\mathcal{M}\mathcal{F}_c^{-1}\left[\mathbf{M}_\sigma^+\overline{\mathbf{z}}_+ + \mathbf{M}_\sigma^-\overline{\mathbf{z}}_- + \mathbf{m}_\sigma^+\right](s)\\ &\qquad - 2i\mathcal{M}\mathcal{F}_s^{-1}\left[\mathbf{M}_\sigma^+\overline{\mathbf{z}}_- + \mathbf{M}_\sigma^-\overline{\mathbf{z}}_+ + \mathbf{m}_\sigma^-\right](s) = \widetilde{\mathbf{v}}_t^{m+}(s) + \widetilde{\delta\mathbf{u}}_+(s),\\ &2\mathcal{M}\mathcal{F}_c^{-1}\left[\mathbf{M}_\sigma^+\overline{\mathbf{z}}_+ + \mathbf{M}_\sigma^-\overline{\mathbf{z}}_- + \mathbf{m}_\sigma^+\right](s)\\ &\qquad + 2i\mathcal{M}\mathcal{F}_s^{-1}\left[\mathbf{M}_\sigma^+\overline{\mathbf{z}}_- + \mathbf{M}_\sigma^-\overline{\mathbf{z}}_+ + \mathbf{m}_\sigma^-\right](s) = \widetilde{\mathbf{w}}_b^1(s) + \widetilde{\delta\mathbf{u}}_-(s).\end{aligned}$$

Substituting $\widetilde{\mathbf{v}}_t^{m+}$, $\widetilde{\mathbf{w}}_b^1$ from (2.2), (2.3) in these equations, they can be rewritten in the form:

$$(2.12)\qquad \begin{aligned}&2\mathcal{M}\mathcal{F}_c^{-1}\left[\mathbf{M}_\sigma^+\overline{\mathbf{z}}_+ + \mathbf{M}_\sigma^-\overline{\mathbf{z}}_- + \mathbf{m}_\sigma^+\right](s) = \mathbf{M}_+(s)\widetilde{\mathbf{z}}_-(s+1)\\ &\qquad\qquad + \mathbf{M}_-(s)\widetilde{\mathbf{z}}_+(s+1) + \mathbf{d}_+(s),\\ &2i\mathcal{M}\mathcal{F}_s^{-1}\left[\mathbf{M}_\sigma^+\overline{\mathbf{z}}_- + \mathbf{M}_\sigma^-\overline{\mathbf{z}}_+ + \mathbf{m}_\sigma^-\right](s) = -\mathbf{M}_-(s)\widetilde{\mathbf{z}}_-(s+1)\\ &\qquad\qquad - \mathbf{M}_+(s)\widetilde{\mathbf{z}}_+(s+1) + \mathbf{d}_-(s).\end{aligned}$$

Here we denote matrix-functions $\mathbf{M}_\pm(s)$ and vector-functions $\mathbf{d}_\pm(s)$ as follows:

$$\begin{aligned}2\mathbf{M}_\pm(s) &= \mathbf{M}_p(s) \pm \mathbf{M}_q(s),\\ 2\mathbf{d}_\pm(s) &= \mathbf{M}_q\widetilde{\delta\boldsymbol{\sigma}}_-(s+1) \mp \mathbf{M}_p\widetilde{\delta\boldsymbol{\sigma}}_+(s+1) + \mathbf{m}_q(s) \pm \mathbf{m}_p(s) + \widetilde{\delta\mathbf{u}}_-(s) \pm \widetilde{\delta\mathbf{u}}_+(s).\end{aligned}$$

As it was mentioned above, Eqs. (2.12) hold in the strip $0 < \Re s < \gamma_1$, in general. They constitute the system of two functional equations of vector-functions $\mathbf{z}_\pm(\lambda)$. We can conveniently consider the system in terms of vector-functions $\overline{\mathbf{z}}_\pm(\lambda)$. For this purpose, we proceed as in [12] and represent unknown vector-funtions $\widetilde{\mathbf{z}}_\pm(s+1)$ from the right-hand sides of the equations in the form:

$$\widetilde{\mathbf{z}}_+(s+1) = 2\mathcal{M}\mathcal{F}_c^{-1}[\overline{\mathbf{z}}_+](s+1), \qquad \widetilde{\mathbf{z}}_-(s+1) = -2i\mathcal{M}\mathcal{F}_s^{-1}[\overline{\mathbf{z}}_-](s+1). \tag{2.13}$$

In [12] it is shown that operators $\mathcal{M}\mathcal{F}_c^{-1}$, $\mathcal{M}\mathcal{F}_s^{-1}$ can be represented in the forms of products of one integral operator – the Mellin transform with a modified argument, and certain functions of the argument. Namely, for any $0 < \alpha, \beta < 1$ the identities hold

$$\begin{aligned} \mathcal{M}\mathcal{F}_c^{-1}[f_+](s) &= \Gamma(s)\cos\frac{\pi s}{2}\mathcal{M}[f_+](1-s), \qquad 0 < \Re s < \beta, \\ \mathcal{M}\mathcal{F}_s^{-1}[f_-](s) &= \Gamma(s)\sin\frac{\pi s}{2}\mathcal{M}[f_-](1-s), \qquad -\alpha < \Re s < \beta, \end{aligned} \tag{2.14}$$

where functions $f_\pm$ should be summable on $\mathbb{R}_+$, and satisfy the estimates:

$$\begin{aligned} f_\pm(\lambda) &= o(\lambda^{-1+\beta}), \qquad \lambda \to 0, \\ f_\pm(\lambda) &= o(\lambda^{-1-\alpha}), \qquad \lambda \to \infty. \end{aligned} \tag{2.15}$$

Besides, the first equation of (2.14) can be extended to a wider strip than that mentioned above. Really, it can be seen that the right-hand side of $(2.14)_1$ is an analytic function in the strip $-\alpha < \Re s < \beta$, except maybe one point $s = 0$. At this point it can have a simple pole, connected with the behaviour of the Gamma-function. From (2.15) and properties of the Fourier transform we can obtain

$$\mathcal{F}_c^{-1}[f_+] = \text{Const} + \mathcal{O}(x^\alpha), \qquad x \to 0.$$

This fact makes it possible to extend analytically the left-hand side of equation $(2.14)_1$ to the whole strip $-\alpha < \Re s < \beta$.

As it follows from Corollary 1 of [15], terms $[\mathbf{M}_\sigma^+\overline{\mathbf{z}}_\pm + \mathbf{M}_\sigma^-\overline{\mathbf{z}}_\mp + \mathbf{m}_\sigma^\pm]$ satisfy exactly conditions (2.15) with $\alpha = \gamma_0$ and $\beta = \min\{1, \gamma_1\}$. Consequently, the left-hand sides of Eqs. (2.12) can be reduced to the form:

$$\begin{aligned} &2\mathcal{M}\mathcal{F}_c^{-1}\left[\mathbf{M}_\sigma^+\overline{\mathbf{z}}_+ + \mathbf{M}_\sigma^-\overline{\mathbf{z}}_- + \mathbf{m}_\sigma^+\right](s) \\ &\qquad = 2\Gamma(s)\cos\frac{\pi s}{2}\mathcal{M}\left[\mathbf{M}_\sigma^+\overline{\mathbf{z}}_+ + \mathbf{M}_\sigma^-\overline{\mathbf{z}}_- + \mathbf{m}_\sigma^+\right](1-s), \\ &2i\mathcal{M}\mathcal{F}_s^{-1}\left[\mathbf{M}_\sigma^+\overline{\mathbf{z}}_- + \mathbf{M}_\sigma^-\overline{\mathbf{z}}_+ + \mathbf{m}_\sigma^-\right](s) \\ &\qquad = 2i\Gamma(s)\sin\frac{\pi s}{2}\mathcal{M}\left[\mathbf{M}_\sigma^+\overline{\mathbf{z}}_- + \mathbf{M}_\sigma^-\overline{\mathbf{z}}_+ + \mathbf{m}_\sigma^-\right](1-s), \end{aligned} \tag{2.16}$$

where identities hold in the strip $-\gamma_0 < \Re s < \min\{1, \gamma_1\}$ and it is possible that a simple pole exists at point $s = 0$:

$$(2.17)\qquad 2\mathcal{M}\mathcal{F}_c^{-1}\left[\mathbf{M}_\sigma^+\overline{\mathbf{z}}_+ + \mathbf{M}_\sigma^-\overline{\mathbf{z}}_- + \mathbf{m}_\sigma^+\right](s) = \frac{1}{s}\mathbf{u}_* + \mathcal{O}(1), \qquad s \to 0.$$

Identities (2.14) are not directly adapted to reduce operators $\mathcal{M}\mathcal{F}_c^{-1}$, $\mathcal{M}\mathcal{F}_s^{-1}$ which are in the right-hand sides of (2.12). The reason is that the arguments of the operators are situated in another region, and the conditions as (2.15) are not satisfied for vector-functions $\mathbf{z}_\pm(\lambda)$. However, as it is shown in [12], relations similar to (2.12) hold in this case also. Namely:

$$(2.18)\qquad \begin{aligned} \mathcal{M}\mathcal{F}_c^{-1}[f_+](s+1) &= -\Gamma(s+1)\sin\frac{\pi s}{2}\mathcal{M}[f_+^*](-s) + \frac{\pi}{2}f_+(0)\Gamma(s+1), \\ \mathcal{M}\mathcal{F}_s^{-1}[f_-](s+1) &= \Gamma(s+1)\cos\frac{\pi s}{2}\mathcal{M}[f_-](-s), \end{aligned}$$

in the strip $-\alpha < \Re s < \beta$, when the following estimates are satisfied:

$$\begin{aligned} f_\pm(\lambda),\ \ \lambda\frac{\partial}{\partial\lambda}f_\pm(\lambda) &= o(\lambda^{-\alpha}), \qquad \lambda \to \infty, \\ f_-(\lambda),\ \ \lambda\frac{\partial}{\partial\lambda}f_\pm(\lambda) &= o(\lambda^{\beta}), \qquad \lambda \to 0, \\ f_+(\lambda) &= f_+(0) + o(\lambda^{\beta}), \qquad \lambda \to 0. \end{aligned}$$

Here we choose function $f_+^*(\lambda)$ in the form

$$(2.19)\qquad f_+^*(\lambda) = f_+(\lambda) - f_+(0)(1+\lambda^2)^{-1},$$

so that the following relations are true:

$$f_+^*(\lambda) = \mathcal{O}(\lambda^{\min\{\beta,2\}}), \quad \lambda \to 0, \qquad f_+^*(\lambda) = \mathcal{O}(\lambda^{-\min\{\alpha,2\}}), \quad \lambda \to \infty.$$

It remains now to verify whether the necessary conditions for equality (2.18) are identical with the estimations presented in Corollary 1 [15] for vector-functions $\mathbf{z}_\pm(\lambda)$ with $\alpha = \gamma_0$, $\beta = \gamma_2$. Hence the system of functional equations (2.12) can be reduced to the following form:

$$(2.20)\qquad \begin{aligned} \widehat{\mathbf{Y}}(s) &= \mathbf{\Phi}(s)\widehat{\mathbf{Z}}(s) + \mathbf{F}(s), \qquad -\gamma_0 < \Re s < \gamma_\infty, \\ \mathbf{Y}(\lambda) &= \mathbf{L}(\lambda)\mathbf{Z}(\lambda) + \mathbf{l}(\lambda), \qquad 0 < \lambda < \infty, \end{aligned}$$

where we have introduced the notations:

$$\widehat{f}(s) = \widetilde{f}(-s) \equiv \mathcal{M}[f](-s), \qquad \gamma_\infty = \min\{1, \gamma_1, \gamma_2\},$$

and

$$\mathbf{Z}(\lambda) = \begin{pmatrix} \overline{\mathbf{z}}^*_+(\lambda) \\ i\overline{\mathbf{z}}_-(\lambda) \end{pmatrix}, \qquad \mathbf{F}(s) = \frac{\mu_1}{\Gamma(s)\sin\pi s} \begin{pmatrix} (\mathbf{d}_+(s) + \mathbf{M}_-\mathbf{d}_*(s))\sin\dfrac{\pi s}{2} \\ (\mathbf{d}_-(s) - \mathbf{M}_+\mathbf{d}_*(s))\cos\dfrac{\pi s}{2} \end{pmatrix},$$

$$\mathbf{L}(\lambda) = \mu_1\lambda \left(\begin{array}{c|c} \mathbf{M}^+_\sigma & -i\mathbf{M}^-_\sigma \\ \hline i\mathbf{M}^-_\sigma & \mathbf{M}^+_\sigma \end{array}\right), \qquad \mathbf{l}(\lambda) = \mu_1\lambda \begin{pmatrix} \mathbf{M}^+_\sigma \mathbf{z}^+_*(1+\lambda^2)^{-1} + \mathbf{m}^+_\sigma \\ i\mathbf{M}^-_\sigma \mathbf{z}^+_*(1+\lambda^2)^{-1} + i\mathbf{m}^-_\sigma \end{pmatrix},$$

$$\mathbf{d}_*(s) = \mathbf{z}^+_*\pi\Gamma(s+1), \qquad \overline{\mathbf{z}}^*_+(\lambda) = \overline{\mathbf{z}}_+(\lambda) - \mathbf{z}^+_*(1+\lambda^2)^{-1},$$

$$\mathbf{\Phi}(s) = \mu_1 s \left(\begin{array}{c|c} -\mathbf{M}_-(s)\mathrm{tg}\dfrac{\pi s}{2} & -\mathbf{M}_+(s) \\ \hline \mathbf{M}_+(s) & \mathbf{M}_-(s)\mathrm{ctg}\dfrac{\pi s}{2} \end{array}\right).$$

We normalize the relations by μ_1 so that the vector-functions $\mathbf{Y}(\lambda)$, $\mathbf{Z}(\lambda)$ consisting of four components, and corresponding to Fourier transforms of the vectors of displacements and tractions along the interfacial boundary Γ_0, have similar dimensions. Besides, 4×4-matrix-function $\mathbf{\Phi}(s)$ has no physical dimensions, and consists of four blocks of 2×2-matrix-functions (as well as matrix-function $\mathbf{L}(\lambda)$). Value of the unknown constant vector $\mathbf{z}^+_* = \overline{\mathbf{z}}_+(0)$ depends on the combination of the boundary conditions, and will be defined later. Note that vector $\mathbf{z}_+(x_1)$ can be easily calculated, and $\mathbf{z}_+(x_1) = \mathbf{z}^*_+(x_1) + \mathbf{z}^+_*\pi\exp(-|x_1|)$.

The form of the first equation in (2.20) makes it possible to consider vector-functions $\mathbf{Y}(\lambda)$, $\mathbf{Z}(\lambda)$ along the half-axis $\mathbb{R}_+$ only. Then the value of these vector-functions for negative magnitudes of λ can be found due to parity (the first two components are even functions, but the last ones are odd functions of λ).

As one can conclude from the *a priori* estimations given in Corollary 1 in [15], $\widehat{\mathbf{Y}}(s)$, $\widehat{\mathbf{Z}}(s)$ should be analytical in the strips $-\gamma_0 < \Re s < \gamma_1$ and $-\gamma_0 < \Re s < \gamma_2$, respectively. From Lemma 1 in [15] and definition (2.20) it follows that vector-function $\mathbf{F}(s)$ and matrix-function $\mathbf{\Phi}(s)$ are analytical in the strip $|\Re s| < 1$, at least, except maybe point $s = 0$, where a second degree pole can appear.

So, once the vector-functions $\mathbf{Z}(\lambda)$ (or $\mathbf{Y}(\lambda)$) will be obtained from systems (2.20), then all vectors of displacements and tractions $\mathbf{u}^{(i)}(x_1, x_2)$, $\mathbf{v}^{(j)}(r,\theta)$, $\mathbf{w}^{(k)}(r,\theta)$, $\boldsymbol{\sigma}^{(i)}_b(\lambda)$, $\mathbf{p}^{(j)}_t(s)$ and $\mathbf{q}^{(k)}_b(s)$ can be calculated by formulas (3.22)–(3.24), (4.13), (4.16) from [15] and the inverse Fourier and Mellin transforms.

3. Satisfaction of equilibrium conditions

The right-hand side of the first equation in (2.20) has, in general, a pole of the second degree at the zero point, if the boundary conditions along the exterior wedge surfaces are not of the first type, i.e. $\mathcal{J}^{\pm} \neq 1$ (see Lemma 2 in [15]), but the left-hand side should be always analytical at this point. Consequently, the corresponding additional conditions (see (4.22), (4.24) from [15]) for the unknown vector $\mathbf{z}_*^+$ should be satisfied. On the other hand, the right-hand side of the second equation of (2.20) has a singularity near the zero point or does not equal zero, depending on the exterior conditions along boundary Γ_n of the last layer ($\mathcal{J} = 2-5$, see Lemma 1 [15]), but the left-hand side tends to zero at $\lambda \to 0$. Then the respective additional conditions $(3.27)_2$ of [15] for vector $\mathbf{z}_*^+$ should be true. In the case, when the value of $\mathbf{z}_*^+$ should satisfy both of the mentioned conditions simultaneously, we have some relations connecting all exterior forces and tractions. They are the usual equilibrium conditions.

We shall not write here all the equilibrium equations depending on the possible combinations of the exterior boundary conditions, and consider only some of them as examples. Let us consider problem $(2,2,1)$ where the displacements are prescribed along the exterior boundary of the layers, but along the exterior wedge surfaces the tractions are given. Then from (4.22) of [15] we have $\mathbf{z}_*^+ = (2\pi)^{-1}\boldsymbol{\Xi}_W$, and no equilibrium conditions are obtained. But if $\mathcal{J} = 5$, $\mathcal{J}^{\pm} = 2$ (the problem where the last layer is a half-plane, and along the exterior wedge surfaces the tractions are prescribed), then $\mathbf{z}_*^+ = (2\pi)^{-1}\boldsymbol{\Xi}_W$, and $\mathbf{z}_*^+ = \boldsymbol{\Xi}_L$ from $(3.27)_2$ in [15]. Consequently, the following equilibrium conditions follow:

$$2\pi\boldsymbol{\Xi}_L - \boldsymbol{\Xi}_W = 0. \tag{3.1}$$

Here $-2\pi\boldsymbol{\Xi}_L$ and $\boldsymbol{\Xi}_W$ defined in Lemma 1 and Lemma 2 of [15] are the principal vectors of all exterior forces and tractions acting on the layered and wedge-shaped parts of the body, respectively. The same equilibrium equations occur in the case of problem $(2,2,2)$. If we consider problem $(3,3,5)$ (or $(3,3,2)$), then $\mathbf{z}_*^+ = \boldsymbol{\Xi}_L$, and these conditions do not appear in (4.22) of [15], in general. However, when wedge-shaped parts $\Omega^{\pm}$ of the body contain the angles: $\theta_0^+ = \theta_{m_-}^- = -\pi/2$ (it corresponds, in particular, to the situation when the crack is perpendicular to the bimaterial interface), the relation $[0,1]\{\mathbf{z}_*^+ - (2\pi)^{-1}\boldsymbol{\Xi}_W\} = 0$ follows from Eq. (4.24) of [15]. Hence the following equilibrium equation is obtained $[0,1]\{2\pi\boldsymbol{\Xi}_L - \boldsymbol{\Xi}_W\} = 0$. For the problem $(4,4,2)$ or $(4,4,5)$, in the case $\theta_0^+ = \theta_{m_-}^- = -\pi/2$, we obtain $[1,0]\{2\pi\boldsymbol{\Xi}_L - \boldsymbol{\Xi}_W\} = 0$. As the last example, we consider the problem $(2,2,3)$. Then from (3.27) and (4.22) in [15] it follows that $[0,1]\{\mathbf{z}_*^+ - \boldsymbol{\Xi}_L\}$ and $\mathbf{z}_*^+ = (2\pi)^{-1}\boldsymbol{\Xi}_W$. Hence the equilibrium equation $[0,1]\{2\pi\boldsymbol{\Xi}_L - \boldsymbol{\Xi}_W\} = 0$ should be true.

Nevertheless, there are combinations of the boundary conditions when constant vector $\mathbf{z}_*^+$ (or one of its components) can not be found (for example problems $(1,1,1)$, $(1,3,1)$ or problem $(3,4,1)$ with the restriction $\theta_0^+ - \theta_{m_-}^- \neq \pi/2$). In such cases we shall use additional conditions (3.29) from [15] for the displacements near the wedge tip to calculate the unknown value of vector $\mathbf{z}_*^+$.

Besides, if the stresses tend to zero at infinity in such a manner that the torque has a sense, (i.e. $\gamma_2 > 1$ in the *a priori* assumption (2.13) of [15] and $(3.27)_2$ of

[15] for the cases $\mathcal{J} = 2, 4$), then the additional torque balance condition

$$2\pi\Big\{\Upsilon_L - [0,1]y_n\Xi_L\Big\} = [1,0]\Big\{\Delta\mathbf{p}_*(1) - \Delta\mathbf{q}_*(1)\Big\} + \Upsilon_p + \Upsilon_q, \tag{3.2}$$

follows for problems $(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$; $\mathcal{J}, \mathcal{J}^\pm = 2, 4$ from $(3.27)_2$ and (4.26) of [15].

4. Reducing the problems to systems of integral equations

Let us note that from Lemma 1 of [15] and definition (2.19) for component $\bar{\mathbf{z}}^*_+(\lambda)$, it follows:

$$\mathbf{L}(\lambda) + \mathbf{L}_\infty = \mathcal{O}(\lambda^{-2}), \qquad \lambda \to \infty, \tag{4.1}$$

$$\mathbf{Y}(\lambda) + \mathbf{L}_\infty\mathbf{Z}(\lambda) = \mathcal{O}(\lambda^{-2-\gamma_0}), \qquad \lambda \to \infty. \tag{4.2}$$

Here a 4×4-matrix $\mathbf{L}_\infty$ is the limit value of matrix-function $-\mathbf{L}(\lambda)$ at infinity $(\lambda \to \infty)$, and is constructed by the identity 2×2-matrix $\mathbf{I}$ and 2×2-matrix $\mathbf{E}_2$ defined in Eq. (3.9) of [15]:

$$\mathbf{L}_\infty = \mathbf{A}(1 - \nu_1, 1/2 - \nu_1), \tag{4.3}$$

$$\mathbf{A}(\xi,\eta) = \left(\begin{array}{c|c} \xi\mathbf{I} & -\eta\mathbf{E}_2 \\ \hline \eta\mathbf{E}_2 & \xi\mathbf{I} \end{array}\right), \qquad \mathbf{B}(\xi,\eta) = \left(\begin{array}{c|c} \xi\mathbf{E}_2 & -\eta\mathbf{I} \\ \hline \eta\mathbf{I} & \xi\mathbf{E}_2 \end{array}\right). \tag{4.4}$$

Let us note here that 4×4 block-matrices $\mathbf{A}(a,b)$ and $\mathbf{B}(c,d)$ construct a commutative algebra for arbitrary values of parameters a, b, c, d:

$$\begin{aligned}
\mathbf{A}(a,b)\mathbf{A}(c,d) &= \mathbf{A}(ac + bd, ad + bc),\\
\mathbf{B}(a,b)\mathbf{B}(c,d) &= \mathbf{A}(-ac - bd, ad + bc),\\
\mathbf{A}(a,b)\mathbf{B}(c,d) &= \mathbf{B}(ac - bd, ad - bc),\\
\mathbf{A}^{-1}(a,b) &= \mathbf{A}\left(\frac{a}{a^2 - b^2}, \frac{-b}{a^2 - b^2}\right),\\
\mathbf{B}^{-1}(a,b) &= \mathbf{B}\left(\frac{-a}{a^2 - b^2}, \frac{b}{a^2 - b^2}\right), \qquad a \neq b.
\end{aligned}$$

This fact as well as the matrix $\mathbf{B}$ itself will be used below.

Taking this fact into account, we rewrite Eqs. (2.20) in the form:

$$\begin{aligned}
\left[\widehat{\mathbf{Y}} + \mathbf{L}_\infty\widehat{\mathbf{Z}}\right](s) &= \mathbf{\Phi}_*(s)\widehat{\mathbf{Z}}(s) + \mathbf{F}(s), \qquad \mathbf{\Phi}_*(s) = \mathbf{L}_\infty + \mathbf{\Phi}(s),\\
\mathbf{Y}(\lambda) &= \mathbf{L}(\lambda)\mathbf{Z}(\lambda) + \mathbf{l}(\lambda).
\end{aligned} \tag{4.5}$$

From (4.2) one can conclude that the left-hand side of the first equation of (4.5) is an analytic vector–function in the strip $-2-\gamma_0 < \Re s < \gamma_\infty$, which is wider than the analyticity strips of $\widehat{\mathbf{Y}}(s)$, $\widehat{\mathbf{Z}}(s)$.

For further analysis we need the following estimations of matrix-functions $\mathbf{\Phi}_*$ at infinity which are true for arbitrary problems $(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ under consideration:

$$\mathbf{\Phi}_*(s) = \mathbf{N}_1^0 + \mathbf{N}_2^0 \operatorname{tg}\frac{\pi s}{2} + \mathbf{N}_3^0(s), \qquad \mathbf{N}_3^0 = \mathcal{O}(e^{-\varepsilon|\Im s|}), \qquad |\Im s| \to \infty. \tag{4.6}$$

Here $\varepsilon = \min\{\phi_{m_+}^+, \phi_1^-\}$, but values of matrices $\mathbf{N}_1^0$, $\mathbf{N}_2^0$ can be calculated basing on the results of Lemma 2 from the previous paper [15]:

$$\begin{aligned} \mathbf{N}_1^0 &= \mathbf{L}_\infty + \frac{\mu_1}{2}\mathbf{A}(\chi_1^+ + \chi_1^-, -\chi_2^+ - \chi_2^-), \\ \mathbf{N}_2^0 &= \frac{\mu_1}{2}\mathbf{B}(\chi_2^+ - \chi_2^-, \chi_1^+ - \chi_1^-), \end{aligned} \tag{4.7}$$

where the constants were defined as follows:

$$\chi_1^+ = \frac{1-\nu_{m_+}^+}{\mu_{m_+}^+}, \qquad \chi_1^- = \frac{1-\nu_1^-}{\mu_1^-}, \qquad \chi_2^+ = \frac{1-2\nu_{m_+}^+}{2\mu_{m_+}^+}, \qquad \chi_2^- = \frac{1-2\nu_1^-}{2\mu_1^-}.$$

Using the fact that matrices $\mathbf{N}_1^0$, $\mathbf{N}_2^0$ belong to the commutative algebra, we can obtain:

$$\begin{aligned} \mathbf{\Phi}_*^{-1}(s) &= \mathbf{N}_1^\infty + \mathbf{N}_2^\infty \operatorname{tg}\frac{\pi s}{2} + \mathbf{N}_3^\infty(s), \\ \mathbf{N}_3^\infty(s) &= \mathcal{O}(e^{-\varepsilon|\Im s|}), \qquad |\Im s| \to \infty, \end{aligned} \tag{4.8}$$

where

$$\mathbf{N}_1^\infty = \left[(\mathbf{N}_1^0)^2 + (\mathbf{N}_2^0)^2\right]^{-1}\mathbf{N}_1^0, \qquad \mathbf{N}_2^\infty = -\left[(\mathbf{N}_1^0)^2 + (\mathbf{N}_2^0)^2\right]^{-1}\mathbf{N}_2^0. \tag{4.9}$$

Besides, the inverse matrix in these relations exists in view of (4.7).

From the mentioned Lemma 2, we can also find the following asynptotics near the zero point, depending on the external boundary conditions. Thus for problems $(1, 1, \mathcal{J})$, when $\mathcal{J} = 1-5$, we obtain

$$\begin{aligned} \mathbf{\Phi}_*(s) &= \mathbf{L}_\infty + \left(\begin{array}{c|c} \mathbf{0} & -s\mu_1\mathbf{M}_+^0 \\ \hline s\mu_1\mathbf{M}_+^0 & \dfrac{2\mu_1}{\pi}\mathbf{M}_-^0 \end{array}\right) + \mathcal{O}(s^2), \\ \mathbf{F}(s) &= \mathcal{O}(1), \qquad s \to 0, \end{aligned} \tag{4.10}$$

but for all remaining problems $(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$, $\mathcal{J}^+, \mathcal{J}^- = 1-4$, $\mathcal{J}^+\mathcal{J}^- > 1$, $\mathcal{J} = 1-5$, the corresponding asymptotic expansions have the form:

$$\mathbf{\Phi}_*(s) = \mathbf{L}_\infty + \frac{1}{s^2}\left(\begin{array}{c|c} -\dfrac{\pi\mu_1 s^2}{2}\mathbf{M}_-^0 & -s\mu_1\mathbf{M}_+^0 \\ \hline s\mu_1\mathbf{M}_+^0 & \dfrac{2\mu_1}{\pi}\mathbf{M}_-^0 \end{array}\right) + \mathcal{O}(s), \tag{4.11}$$

$$\mathbf{F}(s) = \mathcal{O}(s^{-2}), \qquad s \to 0.$$

Here matrices $\mathbf{M}_\pm^0$ are calculated on the basis of the limiting behaviour of matrix-functions $\mathbf{M}_\pm(s)$ near the zero point.

We shall not present here the exact formulas for asymptotics of matrix-functions $\mathbf{\Phi}_*^{-1}(s)$ and vector $\mathbf{F}(s)$ near the zero point (basing on Lemma 1 and Lemma 2 of [15], some results are presented in the Appendix). Let us only note that for any problems under consideration the estimations hold true

$$\mathbf{\Phi}_*^{-1}(s)\mathbf{F}(s) = \mathcal{O}(1), \qquad s \to 0,$$

in view of the additional conditions determining the value of constant vector $\mathbf{z}_*^+$ (see the third section). When $\mathbf{z}_*^+$ cannot be found from the *a priori* estimations, $\mathbf{\Phi}_*^{-1}(s)\mathbf{F}(s)$ is analytical near the zero point for any values of $\mathbf{z}_*^+$.

REMARK 1. By direct verification, it can be concluded that function $\det \mathbf{\Phi}_*(s)$ is not equal to zero near point $s = 0$ and $s \to \pm i\infty$ for problems under considerations. Taking into account the fact that vector-functions $\widehat{\mathbf{Y}}(it)$, $\widehat{\mathbf{Z}}(it)$ should be analytical for any $t \in \mathbb{R}$, from the first equation of (2.20) it would be expected that $\det \mathbf{\Phi}_*(s)$ has no zero point along the imaginary axis. For special cases of the boundary conditions and for a small number of wedges, this fact can be directly verified. Unfortunately, the author has not succeeded in proving this fact in a general case (under arbitrary geometry and the boundary conditions of the problems). Nevertheless, we shall further assume that

$$\Delta(it, \mathcal{J}^+, \mathcal{J}^-) = \det \mathbf{\Phi}_*(it) \neq 0, \qquad t \in \overline{\mathbb{R}}. \tag{4.12}$$

The reason to do this is the fact that $\Delta(s, \mathcal{J}^+, \mathcal{J}^-)$ is in turn the transcendental function which determines the eigenvalues of the solutions for the wedge-shaped media [6]. It can be calculated by applying the Mellin transform only to the similar problems under the additional assumption that the layered part of the domain is a homogeneous half-plane. In conclusion let us note that the method enabling effective calculation of zeros of this determinant is presented in [2], and it is based on the "sweep method" exposed in [8].

By $\vartheta_\infty(\mathcal{J}^+, \mathcal{J}^-) \in (0, 1)$ let us denote the real part of this zero of the function $\Delta(s, \mathcal{J}^+, \mathcal{J}^-)$ which is the nearest to the imaginary axis in half-plane $\Re s > 0$.

Using the results of Lemma 2 [15], it can be proved that $\mathbf{\Phi}_*(-s) = \mathbf{\Phi}_*(s)^\top$. Hence, matrix-function $\mathbf{\Phi}_*^{-1}(s)$ is analytical in the strip $|\Re s| < \vartheta_\infty(\mathcal{J}^+, \mathcal{J}^-)$.

Now we can rewrite the first equation of system (4.5) in an equivalent form inside the respective strip $|\Re s| < \vartheta_\infty(\mathcal{J}^+, \mathcal{J}^-)$:

$$\mathbf{\Phi}_*^{-1}(s)\left[\widehat{\mathbf{Y}} + \mathbf{L}_\infty \widehat{\mathbf{Z}}\right](s) - \mathbf{\Phi}_*^{-1}(s)\mathbf{F}(s) = \widehat{\mathbf{Z}}(s). \tag{4.13}$$

Note that the nearest pole of the left-hand side of (4.13) in the half-plane $\Re s < 0$ coincides with the first zero of the function $\Delta(s, \mathcal{J}^+, \mathcal{J}^-)$. This is because the vector-function $[\widehat{\mathbf{Y}} + \mathbf{L}_\infty \widehat{\mathbf{Z}}](s)$ is analytical in the strip $-2-\gamma_0 < \Re s < \gamma_\infty$ in view of the *a priori* estimations and (4.2), but $\mathbf{F}(s)$ is analytical in $|\Re s| < 1$, at least, except maybe the zero point (let us remind that vector-function $\mathbf{\Phi}_*^{-1}(s)\mathbf{F}(s)$ has no pole at this point). On the other hand, the vector-function $\widehat{\mathbf{Z}}(s)$ should be analytical in the strips $-\gamma_0 < \Re s < \gamma_\infty$. Consequently, we can conclude that the exponent determining the stress singularity is defined as follows:

$$\gamma_0 = \vartheta_\infty(\mathcal{J}^+, \mathcal{J}^-), \tag{4.14}$$

in the *a priori* estimations (2.13) in [15]. When this zero is simple and real, the principal asymptotic term of the solution of system (4.13) is of the form:

$$\mathbf{Z}(\lambda) = \mathbf{\Lambda}_\infty \lambda^{-\vartheta_\infty} + \mathcal{O}(\lambda^{-\vartheta_\infty^*}), \qquad \lambda \to \infty. \tag{4.15}$$

Here $\vartheta_\infty^* > \vartheta_\infty$ is the real part of the next zero of function $\Delta(s, \mathcal{J}^+, \mathcal{J}^-)$, but vector $\mathbf{\Lambda}_\infty$ can be calculated as *an integral measure* of solution $\mathbf{Z}(\lambda)$ from the relation:

$$\mathbf{\Lambda}_\infty = \lim_{s \to -\vartheta_\infty} (s + \vartheta_\infty)\mathbf{\Phi}_*^{-1}(s)\left\{[\widehat{\mathbf{Y}} + \mathbf{L}_\infty \widehat{\mathbf{Z}}](-\vartheta_\infty) - \mathbf{F}(-\vartheta_\infty)\right\}. \tag{4.16}$$

This fact is very important making it possible to calculate the constants in the principal term of the stress asymptotics near the corner tip.

Further on, we rewrite the system (4.13) taking into account the behaviour of matrix-function $\mathbf{\Phi}_*^{-1}(s)$ at infinity (4.8):

$$\mathbf{N}_3^\infty(s)\left[\widehat{\mathbf{Y}} + \mathbf{L}_\infty \widehat{\mathbf{Z}}\right](s) + \left[\mathbf{N}_1^\infty \widehat{\mathbf{Y}} + (\mathbf{N}_1^\infty \mathbf{L}_\infty - \mathbf{I})\widehat{\mathbf{Z}}\right](s) + \mathbf{N}_2^\infty \operatorname{tg}\frac{\pi s}{2}\left[\widehat{\mathbf{Y}} + \mathbf{L}_\infty \widehat{\mathbf{Z}}\right](s) = \mathbf{\Phi}_*^{-1}(s)\mathbf{F}(s).$$

Applying to this equation the inverse Mellin transform, we obtain

$$[\mathbf{N}_1^\infty \mathbf{Y} + (\mathbf{N}_1^\infty \mathbf{L}_\infty - \mathbf{I})\mathbf{Z}](\lambda) + \int_0^\infty \mathbf{\Psi}(\lambda, \xi)[\mathbf{Y} + \mathbf{L}_\infty \mathbf{Z}](\xi)d\xi - \frac{2}{\pi}\mathbf{N}_2^\infty \int_0^\infty [\mathbf{Y} + \mathbf{L}_\infty \mathbf{Z}](\xi)\frac{\lambda d\xi}{\lambda^2 - \xi^2} = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \lambda^s \mathbf{\Phi}_*^{-1}(s)\mathbf{F}(s)\, ds, \tag{4.17}$$

where homogeneous matrix-function $\boldsymbol{\Psi}(\lambda,\xi)$ of degree -1 is defined from the relation:

$$\boldsymbol{\Psi}(\lambda,\xi) = \frac{1}{2\pi i\xi}\int\limits_{-i\infty}^{i\infty} \mathbf{N}_3^{\infty}(s)\left(\frac{\lambda}{\xi}\right)^s ds.$$

4.1. Systems of integral equations for problems $(\mathcal{J}^+,\mathcal{J}^-,1)$ and $(\mathcal{J}^+,\mathcal{J}^-,5)$ $(\mathcal{J}^{\pm}=1-4)$

To analyze these problems, we can directly use the second equation of (2.20) to eliminate the vector-function $\mathbf{Y}(\lambda)$, because the matrix-function $\mathbf{L}(\lambda)$ and the vector-function $\mathbf{l}(\lambda)$ can be estimated in the following manner:

$$(4.18)\qquad \begin{array}{llll} \mathcal{J}=1: & \mathbf{L}(\lambda)=\mathcal{O}(\lambda), & \mathbf{l}(\lambda)=\mathcal{O}(\lambda), & \lambda\to 0,\\ \mathcal{J}=5: & \mathbf{L}(\lambda)=\mathcal{O}(1), & \mathbf{l}(\lambda)=\mathcal{O}(\lambda), & \lambda\to 0. \end{array}$$

Here in the case of $\mathcal{J}=5$, we take into account the fact that the value of the unknown constant vector $\mathbf{z}_+^*$ appearing in $\mathbf{l}(\lambda)$ has been defined by (3.27) of [15].

Substituting $\mathbf{Y}=\mathbf{LZ}+\mathbf{l}$ in Eq. (4.17), we obtain a system of four integral equations:

$$(4.19)\qquad \mathcal{Q}_Z(\mathcal{J}^+,\mathcal{J}^-,\mathcal{J})\mathbf{Z}=\mathcal{G}_Z,\qquad \mathcal{J}=1,5,$$

where

$$[\mathcal{Q}_Z\mathbf{u}](\lambda)=\mathbf{u}(\lambda)+\int\limits_0^{\infty}\mathbf{K}(\lambda)\boldsymbol{\Psi}(\lambda,\xi)[\mathbf{L}(\xi)+\mathbf{L}_{\infty}]\mathbf{u}(\xi)d\xi - \frac{2}{\pi}\int\limits_0^{\infty}\mathbf{K}(\lambda)\mathbf{N}_2^{\infty}[\mathbf{L}(\xi)+\mathbf{L}_{\infty}]\mathbf{u}(\xi)\frac{\lambda d\xi}{\lambda^2-\xi^2},$$

but matrix-function $\mathbf{K}(\lambda)$ and vector-function $\mathcal{G}_Z(\lambda)$ are calculated from the relations:

$$\mathbf{K}(\lambda)=\{\mathbf{N}_1^{\infty}[\mathbf{L}(\lambda)+\mathbf{L}_{\infty}]-\mathbf{I}\}^{-1},$$

$$\mathcal{G}_Z=\mathbf{K}(\lambda)\left\{\frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty}\lambda^s\boldsymbol{\Phi}_*^{-1}(s)\mathbf{F}(s)\,ds-\mathbf{N}_1^{\infty}\,\mathbf{l}(\lambda) - \int\limits_0^{\infty}\boldsymbol{\Psi}(\lambda,\xi)\mathbf{l}(\xi)d\xi+\frac{2}{\pi}\int\limits_0^{\infty}\mathbf{N}_2^{\infty}\mathbf{l}(\xi)\frac{\lambda d\xi}{\lambda^2-\xi^2}\right\}.$$

4.2. Systems of integral equations for problems $(\mathcal{J}^+, \mathcal{J}^-, 2)$ and $(\mathcal{J}^+, \mathcal{J}^-, 5)$ $(\mathcal{J}^\pm = 1-4)$

Problems $(\mathcal{J}^+, \mathcal{J}^-, 5)$ have been reduced to systems of integral equations (4.19), nevertheless we can do it in a different way together with problems $(\mathcal{J}^+, \mathcal{J}^-, 2)$. Namely, let us rewrite the second equation of (2.20) in an equivalent form:

$$\mathbf{Z}(\lambda) = \mathbf{L}^{-1}(\lambda)\mathbf{Y}(\lambda) - \mathbf{n}(\lambda), \qquad \mathbf{n}(\lambda) = \mathbf{L}^{-1}(\lambda)\mathbf{l}(\lambda); \tag{4.20}$$

then the following estimations can be proved:

$$\begin{aligned} &\mathcal{J} = 2: \qquad \mathbf{L}^{-1}(\lambda) = \mathcal{O}(\lambda), \qquad \mathbf{n}(\lambda) = \mathcal{O}(\lambda), \qquad \lambda \to 0,\\ &\mathcal{J} = 5: \qquad \mathbf{L}^{-1}(\lambda) = \mathcal{O}(1), \qquad \mathbf{n}(\lambda) = \mathcal{O}(\lambda), \qquad \lambda \to 0, \end{aligned} \tag{4.21}$$

where the unknown constant vector $\mathbf{z}_+^*$ is defined in Eq. (3.27) of [15].

Substituting (4.20) into Eq. (4.17), we obtain a system of four integral equations:

$$\mathcal{Q}_Y(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})\mathbf{Y} = \mathcal{G}_Y, \qquad \mathcal{J} = 2, 5, \tag{4.22}$$

where

$$\begin{aligned}[\mathcal{Q}_Y\mathbf{u}](\lambda) = \mathbf{u}(\lambda) + \int\limits_0^\infty \mathbf{L}(\lambda)\mathbf{K}(\lambda)\boldsymbol{\Psi}(\lambda,\xi)[\mathbf{I} + \mathbf{L}_\infty\mathbf{L}^{-1}(\xi)]\mathbf{u}(\xi)d\xi \\ - \frac{2}{\pi}\int\limits_0^\infty \mathbf{L}(\lambda)\mathbf{K}(\lambda)\mathbf{N}_2^\infty[\mathbf{I} + \mathbf{L}_\infty\mathbf{L}^{-1}(\xi)]\mathbf{u}(\xi)\frac{\lambda d\xi}{\lambda^2 - \xi^2},\end{aligned}$$

and the vector-function $\mathcal{G}_Y(\lambda)$ is calculated from the relations:

$$\begin{aligned}\mathcal{G}_Y = \mathbf{L}(\lambda)\mathbf{K}(\lambda) \times \Bigg\{ -[\mathbf{N}_1^\infty\mathbf{L}_\infty - \mathbf{I}]\mathbf{n}(\lambda) + \frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty} \lambda^s\boldsymbol{\Phi}_*^{-1}(s)\mathbf{F}(s)\,ds \\ - \int\limits_0^\infty \boldsymbol{\Psi}(\lambda,\xi)\mathbf{L}_\infty\mathbf{n}(\xi)d\xi + \frac{2}{\pi}\int\limits_0^\infty \mathbf{N}_2^\infty\mathbf{L}_\infty\mathbf{n}(\xi)\frac{\lambda d\xi}{\lambda^2 - \xi^2}\Bigg\}.\end{aligned}$$

Unfortunately, systems (4.19) and (4.22) cannot be applied to solve problems $(\mathcal{J}^+, \mathcal{J}^-, 3)$ and $(\mathcal{J}^+, \mathcal{J}^-, 4)$. This is because the matrix-function $\mathbf{L}(\lambda)$ and vector-function $\mathbf{l}(\lambda)$ are not bounded at zero point in these cases (see Lemma 1 of [15] and definition (2.20)). Moreover, $\mathbf{L}(\lambda)$ is degenerate near that point so that $\mathbf{L}^{-1}(\lambda)$ is not bounded, as well as the vector-function $\mathbf{L}^{-1}(\lambda)\mathbf{l}(\lambda)$. Consequently, the corresponding integral operators will be not bounded.

4.3. Systems of integral equations for arbitrary problems

To investigate arbitrary problems let us introduce new auxiliary vector-function as a linear combination of vector-functions $\mathbf{Y}(\lambda)$ and $\mathbf{Z}(\lambda)$:

$$\mathbf{V}(\lambda) = \mathbf{N}_1^\infty \mathbf{Y}(\lambda) + [\mathbf{N}_1^\infty \mathbf{L}_\infty - \mathbf{I}]\mathbf{Z}(\lambda). \tag{4.23}$$

Taking into account the fact that all coefficients in (4.23) are constant matrices, all the *a priori* estimations (3.27) of [15] are true for matrix-function $\mathbf{V}(\lambda)$ as well. Then the vector-functions $\mathbf{Z}(\lambda)$, $\mathbf{Y}(\lambda)$ which should be found, are calculated from the relations:

$$\begin{aligned} \mathbf{Z}(\lambda) &= \mathbf{K}(\lambda)\Big\{\mathbf{V}(\lambda) - \mathbf{N}_1^\infty \mathbf{l}(\lambda)\Big\}, \\ \mathbf{Y}(\lambda) &= \mathbf{L}(\lambda)\mathbf{K}(\lambda)\Big\{\mathbf{V}(\lambda) - [\mathbf{N}_1^\infty \mathbf{L}_\infty - \mathbf{I}]\mathbf{l}(\lambda)\Big\}, \\ \mathbf{Y}(\lambda) + \mathbf{L}_\infty \mathbf{Z}(\lambda) &= \mathbf{K}(\lambda)\Big\{[\mathbf{L}(\lambda) + \mathbf{L}_\infty]\mathbf{V}(\lambda) - \mathbf{l}(\lambda)\Big\}, \end{aligned} \tag{4.24}$$

where matrix-function $\mathbf{K}(\lambda)$ has been defined above in (4.19). Here we use the fact that matrices $\mathbf{N}_1^\infty$, $\mathbf{L}(\lambda)$ and $\mathbf{L}_\infty$ belong to a commutative algebra. This is because they are symmetrical and have nonzero components along two main diagonals only (they are of the block form as $\mathbf{N}_1^\infty$, $\mathbf{L}_\infty$, see (4.3) and (4.9)).

It can be proved that for any problems $(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ $(\mathcal{J}^\pm = 1-4,\ \mathcal{J} = 1-5)$:

$$\mathbf{K}(\lambda) = \mathcal{O}(1), \qquad \mathbf{L}(\lambda)\mathbf{K}(\lambda) = \mathcal{O}(1), \qquad \mathbf{K}(\lambda)\mathbf{l}(\lambda) = \mathcal{O}(\lambda), \qquad \lambda \to 0. \tag{4.25}$$

Then substituting (4.23) in Eq. (4.17), we obtain a system of four integral equations:

$$\mathcal{Q}_V(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})\mathbf{V} = \mathcal{G}_V, \qquad \mathcal{J} = 1-5, \tag{4.26}$$

where

$$[\mathcal{Q}_V \mathbf{u}](\lambda) = \mathbf{u}(\lambda) + \int_0^\infty \mathbf{\Psi}(\lambda,\xi)\mathbf{K}(\xi)[\mathbf{L}_\infty + \mathbf{L}(\xi)]\mathbf{u}(\xi)\,d\xi - \frac{2}{\pi}\int_0^\infty \mathbf{N}_2^\infty \mathbf{K}(\xi)[\mathbf{L}_\infty + \mathbf{L}(\xi)]\mathbf{u}(\xi)\frac{\lambda\, d\xi}{\lambda^2 - \xi^2},$$

and the vector-function $\mathcal{G}_V(\lambda)$ is calculated in the following manner:

$$\mathcal{G}_V(\lambda) = \left\{\mathbf{K}(\lambda)\mathbf{l}(\lambda) + \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \lambda^s \mathbf{\Phi}_*^{-1}(s)\mathbf{F}(s)\,ds + \int_0^\infty \mathbf{\Psi}(\lambda,\xi)\mathbf{K}(\xi)\mathbf{l}(\xi)\,d\xi - \frac{2}{\pi}\int_0^\infty \mathbf{N}_2^\infty \mathbf{K}(\xi)\mathbf{l}(\xi)\frac{\lambda\, d\xi}{\lambda^2 - \xi^2}\right\}.$$

Let us note in conclusion that the integral operators $\mathcal{Q}_Z$, $\mathcal{Q}_Y$ and $\mathcal{Q}_V$ from systems (4.19), (4.22) and (4.26) include not only fixed point singularities at zero and infinity points, but also the usual moving singularity with the kernel of the same type as $(\lambda-\xi)^{-1}$.

5. Analysis of the systems of integral equations

In this section, we investigate systems of singular integral equations (4.19), (4.22) and (4.26) obtained in the previous section. For this purpose, the results from [10, 11] are used without details.

Let $L^{p,\alpha,\beta}(\mathbb{R}_+)$ and $W^{p,\alpha,\beta}_{(l)}(\mathbb{R}_+)$ ($p \ge 1$, $\alpha,\beta \in \mathbb{R}$) be the Banach spaces of summable functions with the weight

$$\varrho_{\alpha,\beta}(\lambda) = \begin{cases} \lambda^{\alpha}, & 0<\lambda\le 1, \\ \lambda^{\beta}, & 1<\lambda<\infty, \end{cases}$$

and norms of these spaces are defined as follows (see [11]):

$$\|u\|_{L^{p,\alpha,\beta}} = \left(\int\limits_0^\infty |u(\xi)|^p \varrho^p_{\alpha,\beta}(\xi)\frac{d\xi}{\xi}\right)^{1/p}, \qquad \|u\|_{W^{p,\alpha,\beta}_{(l)}} = \sum_{j=0}^{m} \|u^{(j)}\|_{L^{p,\alpha+j,\beta+j}}.$$

Here derivatives $u^{(j)}(\xi)$ are of the distributional sense.

By $\mathbf{L}^{p,\alpha,\beta}(\mathbb{R}_+) = \left[L^{p,\alpha,\beta}(\mathbb{R}_+)\right]^4$ $\left(\mathbf{W}^{p,\alpha,\beta}_{(l)}(\mathbb{R}_+) = [W^{p,\alpha,\beta}_{(m)}(\mathbb{R}_+)]^4\right)$ we denote Banach spaces of vector-functions with any standard matrix-norm [5].

Taking into account the results of Lemma 2 from [15], one can conclude that the inclusions hold true:

$$\Psi_{ij}(\cdot,1) \in W^{1,-\vartheta_\infty+\varepsilon,\vartheta_\infty-\varepsilon}_{(m)}(\mathbb{R}_+), \tag{5.1}$$

for any $\varepsilon>0$, $1\le p<\infty$, $m\in\mathbb{N}$. Here $\mathbf{\Psi}$ is the matrix-function belonging to the kernels of the integral operators of systems (4.19), (4.22) and (4.26).

Let us note that the *a priori* estimates (3.27) of the paper [15] for solutions $\mathbf{Z}$, $\mathbf{Y}$ of the systems of integral equations under consideration can be rewritten in terms of the functional spaces in the following manner:

$$\mathbf{Z},\mathbf{Y},\mathbf{V} \in \mathbf{W}^{1,-\gamma_\infty+\varepsilon,\gamma_0-\varepsilon}_{(1)}(\mathbb{R}_+), \tag{5.2}$$

for an arbitrary $\varepsilon>0$. Here $\gamma_0=\vartheta_\infty(\mathcal{J}^+,\mathcal{J}^-)$, but $\gamma_\infty=\gamma_\infty(\mathcal{J}^+,\mathcal{J}^-,\mathcal{J})$ is the unknown constant. The inclusion for the vector-function $\mathbf{V}$ follows immediately from (4.23).

In [11] it is shown that $\mathbf{L}^{p,\alpha,\beta}(\mathbb{R}_+)$ is a natural space in which solutions of such systems can be sought. Taking this fact into account, we shall assume a weaker condition in comparison with that in (5.2)

$$\mathbf{Z}, \mathbf{Y}, \mathbf{V} \in \mathbf{L}^{p,-\gamma_\infty+\varepsilon,\vartheta_\infty-\varepsilon}(\mathbb{R}_+). \tag{5.3}$$

REMARK 2. If systems of integral equations (4.19), (4.22) and (4.26) have solutions from the spaces (5.3), then by investigating smoothness of all matrix-functions from the kernels of the corresponding integral operators $\mathcal{Q}_Z$, $\mathcal{Q}_Y$ and $\mathcal{Q}_V$, and using similar line of reasoning as in Corollary 2 from [11], we can obtain inclusion (5.2). Therefore conditions (5.2) and (5.3) are equivalent in our cases.

5.1. Symbols of operators $\mathcal{Q}_Z$

As in the previous section, let us consider problems $(\mathcal{J}^+, \mathcal{J}^-, 1)$ and $(\mathcal{J}^+, \mathcal{J}^-, 5)$ in the cases $\mathcal{J}^\pm = 1-4$. Then basing on Lemma 1 from the paper [15] it can be easily seen that the components of the matrix-functions $\mathbf{K}(\lambda)$, $\mathbf{L}(\lambda) + \mathbf{L}_\infty$ from the kernels of integral operators $\mathcal{Q}_Z$ belong to space $C^\infty(\mathbb{R}_+)$. Besides, the following estimates can be verified:

$$\begin{aligned} \mathbf{K}(\lambda) &= \mathbf{K}_Z + \mathcal{O}(\lambda), & \mathbf{L}(\lambda) + \mathbf{L}_\infty &= \mathbf{L}_Z + \mathcal{O}(\lambda), & \lambda &\to 0, \\ \mathbf{K}(\lambda) &= -\mathbf{I} + \mathcal{O}(\lambda^{-2}), & \mathbf{L}(\lambda) + \mathbf{L}_\infty &= \mathcal{O}(\lambda^{-2}), & \lambda &\to \infty. \end{aligned}$$

Here the values of the matrices $\mathbf{K}_Z = \mathbf{K}(0)$, $\mathbf{L}_Z = \mathbf{L}(0) + \mathbf{L}_\infty$ depend on the type of the boundary conditions along the exterior boundary of the layered part of the domain $(\mathcal{J} = 1, 5)$:

$$\begin{aligned} \mathcal{J} = 1: \quad & \mathbf{L}_Z = \mathbf{L}_\infty, & \mathbf{K}_Z &= \left(\mathbf{N}_1^\infty \mathbf{L}_\infty - \mathbf{I}\right)^{-1}, \\ \mathcal{J} = 5: \quad & \mathbf{L}_Z = \mathbf{L}_0 + \mathbf{L}_\infty, & \mathbf{K}_Z &= \left(\mathbf{N}_1^\infty \mathbf{L}_Z - \mathbf{I}\right)^{-1}, \end{aligned} \tag{5.4}$$

where we have introduced the notation

$$\mathbf{L}_0 = -\frac{\mu_1}{\mu_{n+1}} \mathbf{A}(1 - \nu_{n+1}, 1/2 - \nu_{n+1}), \tag{5.5}$$

with the matrix $\mathbf{A}(\xi, \eta)$ defined in (4.4). Basing on these estimations and the results from [10, 11], we can formulate the following

THEOREM 1.
Let $1 \le p < \infty$, $\beta < \vartheta_\infty(\mathcal{J}^+, \mathcal{J}^-)$, $-\min\{1, \vartheta_\infty(\mathcal{J}^+, \mathcal{J}^-)\} < \alpha$, $\beta - \alpha \ge 0$, *then operators* $\mathcal{Q}_Z : \mathbf{L}^{p,\alpha,\beta}(\mathbb{R}_+) \to \mathbf{L}^{p,\alpha,\beta}(\mathbb{R}_+)$ *are bounded and its presymbols are calculated by the relations*:

$$\operatorname{Symb}_{\mathcal{Q}_Z}(t, \theta) = \{\mathbf{I} + \mathbf{K}_Z \mathbf{\Psi}_*(\alpha - it)\mathbf{L}_Z\} \frac{1+\theta}{2} + \mathbf{I}\frac{1-\theta}{2}, \tag{5.6}$$

where the matrix-functions $\mathbf{\Psi}_*(s)$ *are of the form:*

$$\mathbf{\Psi}_*(s) = \widetilde{\mathbf{\Psi}}(s,1) + \mathbf{N}_2^{\infty}\mathrm{tg}\pi s/2 \equiv \mathbf{\Phi}_*^{-1}(s) - \mathbf{N}_1^{\infty}. \tag{5.7}$$

REMARK 3. Taking into account the form of (5.6), we can only investigate the presymbols for the value of $\theta = 1$. The corresponding matrix-functions will be denoted by

$$\mathbf{Q}_Z(t) = \mathrm{Symb}_{\mathcal{Q}_Z}(t,1) = \mathbf{I} + \mathbf{K}_Z\mathbf{\Psi}_*(\alpha - it)\mathbf{L}_Z\,.$$

They are not symbols of the operators because their limited values at the infinity point do not coincide ($\mathbf{Q}_Z(i\infty) \neq \mathbf{Q}_Z(-i\infty)$), in general.

REMARK 4. There exist cases when the presymbols represent the usual symbols of the operators. Such situations appear only if the elasticity parameters of two wedges $\Omega_{m_+}^+$, Ω_1^- (which are in contact with the layered part of domain Ω_L, see [15]) are similar:

$$\mu_{m_+}^+ = \mu_1^-\,, \qquad \nu_{m_+}^+ = \nu_1^-\,.$$

Then the matrices $\mathbf{N}_2^0$ (and consequently $\mathbf{N}_2^{\infty}$) from (4.7), (4.9) are equal to zero. In these cases, all integral operators $\mathcal{Q}_Z$, $\mathcal{Q}_Y$ and $\mathcal{Q}_V$ are singular operators with fixed point singularities only and do not contain the Cauchy-type singularities. What is interesting to note is that if the relations $\mu_1 = \mu_1^-$, $\nu_1 = \nu_1^-$ are valid additionally to those mentioned above in this Remark, then the following identities can be easily verified $\mathbf{N}_1^0 = 2(1-\nu_1)\mathbf{I}$, ($\mathbf{N}_2^0 = \mathbf{0}$, $\mathbf{N}_1^{\infty} = [2(1-\nu_1)]^{-1}\mathbf{I}$, $\mathbf{N}_2^{\infty} = \mathbf{0}$) in (4.7), (4.9).

REMARK 5. Let us note that the matrix $\mathbf{L}_Z$ is equivalent to zero if the additional conditions $\nu_1 = \nu_{n+1}$, $\mu_1 = \mu_{n+1}$ for problems $\mathcal{J} = 5$, $\mathcal{J}^{\pm} = 1-4$ are satisfied (the last layer is a half-space having the same elasticity parameters as the first layer). It means that in these cases the symbols of the corresponding operators are equal to identical matrix. Consequently, these operators are of Fredholm type (equal to the identical operator with an accuracy to compact ones). Hence, the corresponding systems of equations (4.19) have unique solutions in spaces $\mathbf{L}^{p,\alpha,\beta}(\mathbb{R}_+)$ for any $p \geq 1$, $\beta < \vartheta_{\infty}(\mathcal{J}^+,\mathcal{J}^-)$, $\beta - \alpha \geq 0$, $-\min\{1, \vartheta_{\infty}(\mathcal{J}^+,\mathcal{J}^-)\} < \alpha < 0$, and they can be calculated by projectional numerical methods, for example. Here we use the fact that the corresponding boundary value problems which are equivalent to systems of Eqs. (4.19), have unique solutions for such values of the parameters of spaces (see Remark 1 of [15]). Moreover, these solutions belong to spaces $\mathbf{W}_{(1)}^{p,\alpha,\beta}(\mathbb{R}_+)$. This fact follows from the differential properties of vector-functions $\mathbf{K}(\lambda)$, $\mathbf{L}(\lambda) + \mathbf{L}_{\infty}$ and their estimations near zero and infinity points (see Corollary 3 from [11]).

Let us note that the matrix $\mathbf{L}_Z$ is degenerate if and only if the assumptions of Remark 5 hold true. For the remaining cases, one can obtain:

$$\mathbf{Q}_Z(t) = \mathbf{K}_Z\{\mathbf{\Phi}_*^{-1}(\alpha - it) - \mathbf{L}_Z^{-1}\}\mathbf{L}_Z\,, \tag{5.8}$$

taking into account the fact that the matrix-functions $\mathbf{N}_1^\infty$, $\mathbf{L}(\lambda)$ and, consequently, $\mathbf{K}(\lambda)$ belong to the commutative algebra. Moreover, the matrix $\mathbf{K}_Z$ is nondegenerate for all the problems under consideration.

5.2. Symbols of operators $\mathcal{Q}_Y$

Now we consider problems $(\mathcal{J}^+, \mathcal{J}^-, 2)$ and $(\mathcal{J}^+, \mathcal{J}^-, 5)$ in the cases $\mathcal{J}^\pm = 1-4$. Then the components of the matrix-functions $\mathbf{L}(\lambda)\mathbf{K}(\lambda)$, $\mathbf{I}+\mathbf{L}_\infty\mathbf{L}^{-1}(\lambda)$ from the kernels of integral operators $\mathcal{Q}_Y$ belong to space $C^\infty(\mathbb{R}_+)$, and the following estimates can be verified:

$$\begin{aligned} &\mathbf{L}(\lambda)\mathbf{K}(\lambda) = \mathbf{K}_Y + \mathcal{O}(\lambda), && \mathbf{I} + \mathbf{L}_\infty\mathbf{L}^{-1}(\lambda) = \mathbf{L}_Y + \mathcal{O}(\lambda), && \lambda \to 0, \\ &\mathbf{L}(\lambda)\mathbf{K}(\lambda) = \mathbf{L}_\infty + \mathcal{O}(\lambda^{-2}), && \mathbf{I} + \mathbf{L}_\infty\mathbf{L}^{-1}(\lambda) = \mathcal{O}(\lambda^{-2}), && \lambda \to \infty. \end{aligned}$$

Here the values of the matrices $\mathbf{K}_Y$, $\mathbf{L}_Y$ depend on the type of the boundary conditions along the exterior boundary of the layered part of the domain ($\mathcal{J} = 2, 5$):

$$\begin{aligned} &\mathcal{J} = 2: \quad \mathbf{L}_Y = \mathbf{I}, && \mathbf{K}_Y = (\mathbf{N}_1^\infty)^{-1}, \\ &\mathcal{J} = 5: \quad \mathbf{L}_Y = \mathbf{L}_0^{-1}\mathbf{L}_Z, && \mathbf{K}_Y = \mathbf{L}_0\mathbf{K}_Z\,, \end{aligned} \tag{5.9}$$

where the matrices $\mathbf{L}_Z$, $\mathbf{K}_Z$ and $\mathbf{L}_0$ have been defined in (5.4), (5.5).

As above, Theorem 1 holds true for operators $\mathcal{Q}_Y$, and their presymbols are of the form:

$$\mathrm{Symb}_{\mathcal{Q}_Y}(t,\theta) = \{\mathbf{I} + \mathbf{K}_Y\mathbf{\Psi}_*(\alpha - it)\mathbf{L}_Y\}\frac{1+\theta}{2} + \mathbf{I}\frac{1-\theta}{2}\,. \tag{5.10}$$

For the cases mentioned in Remark 5, operators $\mathcal{Q}_Y$ are equal to the identical ones with an accuracy to compacts operators, and all conclusions of this Remark are valid.

For the remaining cases ($\mathbf{L}_Y$ is nondegenerate), we can obtain:

$$\mathbf{Q}_Y(t) = \mathbf{K}_Y\{\mathbf{\Phi}_*^{-1}(\alpha - it) - \mathbf{L}_*\}\mathbf{L}_Y\,, \tag{5.11}$$

using a similar line of the reasoning as that used in (5.8). Here and in the sequel, matrix $\mathbf{L}_*$ will assume the limiting value:

$$\mathbf{L}_* = \lim_{\lambda\to 0}\,[\mathbf{L}(\lambda) + \mathbf{L}_\infty]^{-1}. \tag{5.12}$$

5.3. Symbols of operators $\mathcal{Q}_V$

Now we consider problems $(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ in the cases $(\mathcal{J}^\pm = 1-4, \mathcal{J} = 1-5)$. Components of the matrix-function $\mathbf{K}(\lambda)[\mathbf{L}_\infty + \mathbf{L}(\lambda)]$ from the kernels of integral operators $\mathcal{Q}_V$ belong also to space $C^\infty(\mathbb{R}_+)$, and the following estinates can be verified:

$$\begin{aligned} \mathbf{K}(\lambda)[\mathbf{L}_\infty + \mathbf{L}(\lambda)] &= \mathbf{K}_V + \mathcal{O}(\lambda), \quad && \lambda \to 0, \\ \mathbf{K}(\lambda)[\mathbf{L}_\infty + \mathbf{L}(\lambda)] &= \mathcal{O}(\lambda^{-2}), && \lambda \to \infty. \end{aligned}$$

Here the values of the matrix $\mathbf{K}_V$ depend on the type of the boundary conditions along the exterior boundary of the layered part of the domain ($\mathcal{J} = 1 - 5$) and are calculated by the formula:

$$\mathbf{K}_V = \left(\mathbf{N}_1^\infty - \mathbf{L}_*\right)^{-1}, \tag{5.13}$$

except the cases mentioned in the Remark 5, when $\mathbf{K}_V = \mathbf{0}$. Here tle matrix $\mathbf{L}_*$ has been defined by Eq. (5.12) and can be calculated from the relatons:

$$\mathbf{L}_* = \begin{cases} \mathbf{L}_\infty^{-1}, & \mathcal{J} = 1, \\ \mathbf{0}, & \mathcal{J} = 2, \\ \mathbf{L}_3, & \mathcal{J} = 3, \\ \mathbf{L}_4, & \mathcal{J} = 4, \\ \mathbf{L}_Z^{-1}, & \mathcal{J} = 5, \end{cases} \tag{5.14}$$

where the matrices $\mathbf{L}_3, \mathbf{L}_4$ are calculated in the following manner:

$$\mathbf{L}_3 = \frac{1}{1-\nu_1}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{L}_4 = \frac{1}{1-\nu_1}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

As above, Theorem 1 holds true for operators $\mathcal{Q}_V$, and their presynbols are of the form:

$$\mathrm{Symb}_{\mathcal{Q}_V}(t,\theta) = \{\mathbf{I} + \mathbf{\Psi}_*(\alpha - it)\mathbf{K}_V\}\frac{1+\theta}{2} + \mathbf{I}\frac{1-\theta}{2}, \tag{5.15}$$

or, for all cases except those mentioned in Remark 5, we have:

$$\mathbf{Q}_V(t) = \{\mathbf{\Phi}_*^{-1}(\alpha - it) - \mathbf{L}_*\}\mathbf{K}_V. \tag{5.16}$$

Let us note that all symbols $\mathbf{Q}_Z$, $\mathbf{Q}_Y$, $\mathbf{Q}_V$ contain the common matrix-function of the form:

$$\mathbf{Q}(s, \mathcal{J}^+, \mathcal{J}^-, \mathcal{J}) = \mathbf{\Phi}_*^{-1}(s) - \mathbf{L}_*, \tag{5.17}$$

where the first term of this sum $\mathbf{\Phi}_*^{-1}(s) = \mathbf{\Phi}_*^{-1}(s, \mathcal{J}^+, \mathcal{J}^-)$ depends on the external boundary conditions along the wedge surfaces (defined by tie values of

$\mathcal{J}^+, \mathcal{J}^- = 1-4$), but the second term $\mathbf{L}_* = \mathbf{L}_*(\mathcal{J})$ depends on the external boundary conditions along the last layer ($\mathcal{J} = 1-5$) and has been given by Eq. (5.14). Asymptotics of the matrix-functions $\mathbf{\Phi}_*^{-1}(s)$ are presented in (4.8) (in the neighbourhood of the infinity point), and in the Appendix (near the zero point). Besides, the following identities can be verified:

$$\begin{aligned}
\mathbf{Q}(s,\mathcal{J}^+,\mathcal{J}^-,1) &= -\mathbf{\Phi}_*^{-1}(s)\mathbf{\Phi}(s)\mathbf{L}_*,\\
\mathbf{Q}(s,\mathcal{J}^+,\mathcal{J}^-,2) &= \mathbf{\Phi}_*^{-1}(s),\\
\mathbf{Q}(s,\mathcal{J}^+,\mathcal{J}^-,\mathcal{J}) &= \mathbf{\Phi}_*^{-1}(s)[\mathbf{I}-\mathbf{\Phi}(s)\mathbf{L}_*], \qquad \mathcal{J}=3,4,\\
\mathbf{Q}(s,\mathcal{J}^+,\mathcal{J}^-,5) &= \mathbf{\Phi}_*^{-1}(s)[\mathbf{L}(0)-\mathbf{\Phi}(s)]\mathbf{L}_*.
\end{aligned}$$

They will be useful during the investigation of the symbols. Moreover, the additional relations hold true for problems $(\mathcal{J}^+,\mathcal{J}^-,\mathcal{J})$ when $\mathcal{J}=3,4$:

$$\det \mathbf{Q}(s,\mathcal{J}^+,\mathcal{J}^-,k+2) = \det \mathbf{\Phi}_*^{-1}(s)\frac{\mu_1^2 s^2}{(1-\nu_1)^2}\mathbf{m}_{pkk}(s)\mathbf{m}_{qkk}(s), \qquad k=1,2,$$

where $\mathbf{m}_{pkk}(s)$, $\mathbf{m}_{qkk}(s)$ are the diagonal elements of the 2×2 matrix-functions $\mathbf{M}_p(s)$, $\mathbf{M}_q(s)$ defined in Lemma 2 [15].

Let us note that for the right-hand sides of systems (4.19), (4.22) and (4.26), the following inclusions hold true:

$$\mathcal{G}_Z, \mathcal{G}_Y, \mathcal{G}_V \in \mathbf{W}_{(2)}^{p,\alpha,\vartheta_\infty-\varepsilon}(\mathbb{R}_+) \tag{5.19}$$

for any $\varepsilon > 0$, $1 \le p < \infty$, $\alpha > -\min\{1, \vartheta_\infty(\mathcal{J}^+,\mathcal{J}^-\}$.

Taking into account the volume of the paper, we can not present here a complete analysis of all the problems under consideration, because there exist fifty different combinations of the external boundary conditions. Nevertheless, the results presented above make it possible to investigate arbitrary boundary conditions.

Thus, let us now outline only the main points of such analysis.

5.4. Investigation of the symbols of the operators

First of all let us remind that in the case $\mathcal{J}=5$, $\nu_1=\nu_{n+1}$, $\mu_1=\mu_{n+1}$ all systems of integral equations (4.19), (4.22) and (4.26) are of the Fredholm type with compact operators, and have unique solutions (see Remark 5). Below we do not consider these situations.

All remaining problems can be divided into two groups, depending on whether there exist zeros of the functions

$$\mathbf{q}(s,\mathcal{J}^+,\mathcal{J}^-,\mathcal{J}) = \det \mathbf{Q}(s,\mathcal{J}^+,\mathcal{J}^-,\mathcal{J}) \tag{5.20}$$

on the imaginary axis or not. We note here that these functions do not degenerate and are bounded at the infinity point ($s \to \pm\infty$). Moreover, in spite of the fact mentioned above that the limiting values of matrix-functions $\mathbf{Q}(s)$ are different for $s \to i\infty$ and $s \to -i\infty$ (Remark 3), it is easily proved that $\mathbf{q}(s)$ exhibits similar behaviour at the infinity point. This fact is a consequence of the structure of matrices $\mathbf{N}_1^\infty$, $\mathbf{N}_2^\infty$ and $\mathbf{L}_*$ (see (4.9) and (5.14)).

$$\lim_{s\to\pm i\infty} \mathbf{q}(s, \mathcal{J}^+, \mathcal{J}^-, \mathcal{J}) = \mathrm{Const}(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J}) \in \mathbb{R}.$$

Moreover, one can prove that this constant is not equal to zero. On the other hand, basing on Remark 1 and the structures of the symbols presented above, it can be proved that zeros of the symbols can appear at the point $s = 0$ only.

Let us consider such problems $(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ for which the corresponding functions $\mathbf{q}(s, \mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ have no zeros on the imaginary axis.

PROPOSITION 1. There exist such values $\vartheta_0(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J}) > 0$ that

$$\mathrm{ind}\, \mathbf{q}(\alpha - it, \mathcal{J}^+, \mathcal{J}^-, \mathcal{J}) = 0, \qquad |\alpha| < \vartheta_0(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J}).$$

It is evident that $\vartheta_0(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ are the real parts of zeros (or poles) of functions $\mathbf{q}(s, \mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ which are the nearest to the imaginary axis. Then, in order to prove this Proposition it is sufficient to note that the matrix-functions $\mathbf{Q}(s, \mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ are Hermitian ones.

THEOREM 2.

Let $-\min\{1, \vartheta_0(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J}), \vartheta_\infty(\mathcal{J}^+, \mathcal{J}^-)\} < \alpha < \vartheta_0(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$, $1 \le p < \infty$, $\beta < \vartheta_\infty(\mathcal{J}^+, \mathcal{J}^-)$, $\beta - \alpha \ge 0$, *then*

1. *Operators* $\mathcal{Q}_Z$, $\mathcal{Q}_Y$ *and* $\mathcal{Q}_V$ *in the spaces* $\mathbf{L}^{p,\alpha,\beta}(\mathbb{R}_+)$ *are normally solvable with the indices and partial indices equal to zero* ($\kappa = 0$, $\kappa_j = 0$, $j = 1,..,4$).

2. *Systems of equations* (4.19), (4.22) *and* (4.26) *have unique solutions from* $\mathbf{W}^{p,\alpha,\beta}_{(1)}(\mathbb{R}_+) \subset \mathbf{L}^{p,\alpha,\beta}(\mathbb{R}_+)$.

3. *Galerkin method for the systems of the equations with respect to the set of vector-functions* $\Theta_j^{\alpha,\beta} = \Theta_j \varrho^{-1}_{\alpha,\beta}$:

$$\Theta_j(\lambda) = \begin{cases} \sqrt{2}\,\lambda\,\Lambda_j(-2\ln\lambda), & 0 < \lambda < 1, \qquad j = 0,1,2,\ldots\,, \\ 0, & 1 < \lambda < \infty, \end{cases} \tag{5.21}$$
$$\Theta_j(\lambda) = -\Theta_{-j-1}(\lambda^{-1}), \qquad j = 0,1,2,\ldots\,,$$

is valid in the Hilbert space $\mathbf{L}^{2,\alpha,\beta}(\mathbb{R}_+)$. *Here* $\Lambda_j(t)$, ($j = 0,1,2,\ldots$) *are normed Laguerre polynomials with vector-valued constants*, [4].

4. *The solutions of the systems have asymptotic expansion in the neighbourhood of zero in the form:* $\mathbf{U}(\lambda) = \mathcal{O}(\lambda^{\gamma_\infty})$, $\lambda \to 0$, $\gamma_\infty = \min\{1, \vartheta_0(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})\}$, *and at infinity point the asymptotics are defined by the relations* (4.15), (4.16).

The remaining parameters $\mathbf{u}_*$, $\gamma_\infty(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ in the definition of the class $\mathbf{LW}(\Omega)$ (see the second section of the paper [15]) can be obtained from Theorem 2. Solving numerically the corresponding systems of the equations, we can find the approximate solutions of the corresponding boundary value problems $(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ and the asymptotics of their solutions in the neighbourhood of zero and infinity points. Note that the constants $\mathbf{u}_*$ Λ_∞, which play an important role in applications, can be calculated as *integral measure* of the approximate solutions of the systems (4.19), (4.22) and (4.26). Integral formulae for the constants in the asymptotics 4 similar to those in (4.15) and (4.16) can be obtained from functional equations (2.20) by passing to the limit $s \to \vartheta_0$.

Problems $(\mathcal{J}^+, \mathcal{J}^-, 5)$ for arbitrary $\mathcal{J}^\pm = 1 - 4$, $(1,1,2)$ and $(2,2,1)$ belong to this group of the problems under consideration. Let us note that the unknown vectors $\mathbf{z}_*^+$ are calculated by the relations $(3.27)_2$ and (4.22) in [15], so that the right-hand sides of the systems of the equations have been defined.

Now we consider situations when there exists a zero $s = 0$ of $2l$ multiplicity of the functions $\mathbf{q}(s, \mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ and investigate the corresponding group of problems $(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$. Parity of the multiplicity of this zero follows immediately from the fact that the matrix-functions $\mathbf{Q}(s, \mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ are Hermitian ones.

PROPOSITION 2. There exist such values $\vartheta_0(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J}) > 0$ that

$$\kappa = -\text{ind}\, \mathbf{q}(\alpha - it, \mathcal{J}^+, \mathcal{J}^-, \mathcal{J}) = \begin{cases} -l, & \vartheta_0 < \alpha < 0, \\ l, & 0 < \alpha < \vartheta_0. \end{cases}$$

It is evident that $\vartheta_0(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ are the real parts of the zeros (or poles) of functions $\mathbf{q}(s, \mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ nearest to the imaginary axis. Moreover, partial indices $(\kappa_j,\ j = 1 - 4)$ of the matrix-functions $\mathbf{Q}(\alpha - it, \mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ satisfy the relations: $-1 \le \kappa_j \le 0$ when $-\vartheta_0 < \alpha < 0$; and $0 \le \kappa_j \le 1$ when $0 < \alpha < \vartheta_0$.

For problems $(1,1,1)$, $(2,2,2)$, $(2,3,2)$, $(2,4,2)$, and for problems $(3,4,1)$, $(3,3,1)$, $(4,4,1)$ $(3,4,2)$, $(3,3,2)$, $(4,4,2)$ it can be shown that $l = 2$, except the last six problems when l can be equal to one under special assumptions on the geometry of the domain (see the corresponding formulas in the Appendix and Eq. (4.24) in [15]). For all remaining cases we have $l = 1$.

THEOREM 3.
Let $1 \le p < \infty$, $-\vartheta_0 < \alpha < 0$, $\beta < \vartheta_\infty$, $\beta - \alpha \ge 0$, *then*:

1. *Operators* $\mathcal{Q}_Z$, $\mathcal{Q}_Y$, $\mathcal{Q}_V$ *in spaces* $\mathbf{L}^{p,\alpha,\beta}(\mathbb{R}_+)$ *are normally solvable with the index* $\kappa = -l$.

2. *There are unique solutions of systems of the equations* (4.19), (4.22) *and* (4.26) *from* $\mathbf{W}_{(1)}^{p,\alpha,\beta}(\mathbb{R}_+)$.

3. *Asymptotic behaviour of the solutions near the zero point is* $\mathbf{U}(\lambda) = \mathcal{O}(\lambda^{\vartheta_0})$, $\lambda \to 0$, *but at the infinity point asymptotics has been defined by relation* (4.15).

Let us note that certain additional conditions are necessary for the solvability of the systems for some problems $(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$. Such conditions are presented in $(3.27)_2$, (3.29), (4.22), (4.24) of [15] depending on the problem under consideration. For some cases they represent equilibrium conditions and are discussed in

Sec. 3. For example, for problem $(2,2,2)$ the unknown vector $\mathbf{z}_*^+$ defined in (2.20) can be calculated from $(3.27)_2$ as well as from (4.22) in [15]. It leads us to the equilibrium equations (3.1).

For the other cases, when $\mathbf{z}_*^+$ can not be defined from the mentioned relations (for example (1,1,1)), the additional conditions follow from (3.29) of [15]. Thus the right-hand sides of the corresponding systems of equations can be represented in the form: $\mathcal{G}(\lambda) = \mathcal{G}_1(\lambda) + z_{*1}^+ \ \mathcal{G}_2(\lambda) + z_{*2}^+ \ \mathcal{G}_3(\lambda)$, where z_{*j}^+ $(j = 1,2)$ are the components of the vector $\mathbf{z}_*^+$, but vector-functions $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ have no singularity in the neighbourhood of the zero point. Of course, this representation is true for all the problems, but the respective vector-functions are not bounded near point $\lambda = 0$.

The systems of equations under the conditions of Theorem 3 can not be directly solved as it is shown in point 3 of Theorem 2, and a regularization of the systems is necessary (see [4]). For this purpose, a method of factorization of matrix-functions of special forms proposed in [1] could be useful.

Thus, if the value of $\mathbf{z}_*^+$ is known and the corresponding equilibrium conditions are satisfied, then Theorem 2 holds true for the corresponding regularized systems of the equations. In the opposite cases, when the value of $\mathbf{z}_*^+$ (or one of the components) can not be calculated from relations $(3.27)_2$, (4.22), (4.24) in [15], the unique solutions of regularized systems of integral equations with the right-hand sides $\mathcal{G}_1(\lambda)$, $\mathcal{G}_2(\lambda)$ and $\mathcal{G}_3(\lambda)$ can be found. Then the value of the vector $\mathbf{z}_*^+$ is calculated from relations (3.29) in [15].

REMARK 6. As it has been shown in Sec. 3, the additional torque balance condition (3.2) should be true for problems $(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ $(\mathcal{J}^\pm, \mathcal{J} = 2,4)$. However, what is interesting to note is that this condition is not necessary for solvability of the corresponding systems of integral equations. It only plays an important role when the tractions and displacements along internal boundaries Γ_j $(j = 1, 2\ldots, n)$ and $\Gamma_j^\pm$ $(j = 1, 2\ldots, m_\pm)$ (between the layers and the wedges, respectively) are calculated by the recurrent relations (3.23), (3.24) and (4.13), (4.16) shown in the previous paper [15]. Namely, if the mentioned condition is not satisfied then the tangential component of the displacements is not bounded at infinity.

THEOREM 4.

Let $1 \le p < \infty$, $0 < \alpha < \vartheta_0$, $\beta < \vartheta_\infty$, $\beta - \alpha \ge 0, m \in \mathbb{N}$; *then*

1. *Operators* $\mathcal{Q}_Z$, $\mathcal{Q}_Y$, $\mathcal{Q}_V$ *in spaces* $\mathbf{L}^{p,\alpha,\beta}(\mathbb{R}_+)$ *are normally solvable with the index* $\kappa = l$.

2. *Homogeneous systems of equations* $\mathcal{Q}_{Z(Y,V)}\mathbf{U} = 0$ *have exactly* l *non-trivial solutions* $\mathbf{U}_j \in \mathbf{L}^{p,\alpha,\beta}(\mathbb{R}_+)$ $(j = 1,\ldots,l)$ *belonging to all spaces* $\mathbf{U}_j \in \bigcap \mathbf{W}_{(m)}^{p,\alpha,\beta}(\mathbb{R}_+)$.

3. *The asymptotic expansions of the solutions in the neighbourhood of zero are:*

$$\mathbf{U}_j(\lambda) = \mathbf{A}_j \ln\lambda + \mathbf{B}_j + \mathcal{O}(\lambda^{\vartheta_0}), \qquad \lambda \to 0,$$

where $\mathbf{A}_j$, $\mathbf{B}_j$ $(j = 1, ..., l)$ *are certain vectors, but the relations* (4.15) *are satisfied at the infinity point.*

Let us note that nontrivial solutions of the respective homogeneous boundary problems play an important role in asymptotic methods [17]. However, in order to obtain them by the nontrivial solutions of the homogeneous integral equations, it is necessary to use generalized integral transforms and to justify all the obtained relations as it has been done in [16].

5.5. Analysis of the systems in the case of symmetrical domain

Now we consider such situations when the domain under consideration is symmetrical with respect to the axis $0X_2$. Besides, we assume that all mechanical parameters of the wedge parts of the domain have similar values to the symmetrical wedges. It is evident that in such cases the strain-stress state of the domain can be represented by symmetrical and antisymmetrical ones (Mode I and Mode II, respectively).

From Corollary 3 in [15] it follows that matrix-functions $\mathbf{\Phi}_*^{-1}(s)$ in the symbols of the integral operators have structures similar to those for the matrix $\mathbf{A}$ (see (4.3)) because of:

$$\mathbf{\Phi}(s) = \mu_1 s \begin{pmatrix} -f_2(s)\mathrm{tg}\dfrac{\pi s}{2} & 0 & 0 & -f_1(s) \\ 0 & -f_3(s)\mathrm{tg}\dfrac{\pi s}{2} & f_1(s) & 0 \\ 0 & f_1(s) & f_2(s)\mathrm{ctg}\dfrac{\pi s}{2} & 0 \\ -f_1(s) & 0 & 0 & f_3(s)\mathrm{ctg}\dfrac{\pi s}{2} \end{pmatrix},$$

but elements on the main diagonal are not identical. Moreover, in this case the conclusions of Remark 4 are true, so that $\mathbf{N}_2^0 = \mathbf{0}$, $\mathbf{N}_2^\infty = \mathbf{0}$ ($\chi_j^{\pm} = \chi_j^{\mp}$ in Eqs. (4.6), (4.8)), and consequently, the corresponding systems of integral equations contain only fixed point singularities.

For any 4×4 matrix $\mathbf{M}$ and vector $\mathbf{G}$ appearing in systems of the integral equations (4.19), (4.22) and (4.26), we introduce the following notations:

$$\mathbf{M}^{(1)} = \begin{pmatrix} m_{11} & m_{14} \\ m_{41} & m_{44} \end{pmatrix}, \quad \mathbf{M}^{(2)} = \begin{pmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{pmatrix},$$

$$\mathbf{G}^{(1)} = \begin{pmatrix} g_1 \\ g_4 \end{pmatrix}, \qquad \mathbf{G}^{(2)} = \begin{pmatrix} g_2 \\ g_3 \end{pmatrix}. \tag{5.22}$$

Then each of the systems of 4×4 integral equations corresponding to problems $(\mathcal{J}^+, \mathcal{J}^+, \mathcal{J})$ $(\mathcal{J}^+ = 1-4,\ \mathcal{J} = 1-5)$ is divided into two systems of 2×2 integral equations of a similar form:

$$
\begin{gathered}
\mathcal{Q}^{(j)}_{Z(Y,V)}(\mathcal{J}^+,\mathcal{J}^+,\mathcal{J})\mathbf{U}^{(j)} = \mathcal{G}^{(j)}_{Z(Y,V)}, \\
\mathcal{J}^+ = 1-4, \qquad \mathcal{J} = 1-5, \qquad j = 1,2.
\end{gathered} \tag{5.23}
$$

Symbols of the operators $\mathcal{Q}^{(j)}_{Z(Y,V)}$ in spaces $\mathbf{L}^{p,\alpha,\beta}(\mathbb{R}_+)$ of two-component vector-functions are defined by the corresponding 2×2 matrix-functions $\mathbf{Q}^{(j)}(s,\mathcal{J}^+,\mathcal{J}) = \mathbf{Q}^{(j)}(s,\mathcal{J}^+,\mathcal{J}^+,\mathcal{J})$ which are given by (5.17) and (5.22). Here superscript j is equal to 1 and 2 for the Mode I and Mode II, respectively.

Thus, we have forty different problems $(\mathcal{J}^+,\mathcal{J}^+,\mathcal{J})_j$ depending on the combinations of the external boundary problems (the values of $\mathcal{J}^+ = 1-4$, $\mathcal{J} = 1-5$) as well as the strain-stress state ($j = 1,2$ for Mode I and Mode II). As before, let us denote by $\vartheta^{(j)}_\infty(\mathcal{J}^+) > 0$ the real part of zero of the determinant of the 2×2 matrix-function $\mathbf{\Phi}^{(j)}_*(s)$ which is the nearest to the imaginary axis. But by $\vartheta^{(j)}_0(\mathcal{J}^+,\mathcal{J}) > 0$ we denote the real part of zero (pole) of the determinant of the 2×2 matrix-function $\mathbf{Q}^{(j)}(s)$ which is the nearest to the imaginary axis. Let us note that $\vartheta_0(\mathcal{J}^+,\mathcal{J}^+,\mathcal{J}) = \min\{\vartheta^{(j)}_0(\mathcal{J}^+,\mathcal{J})\}$, and $\vartheta_\infty(\mathcal{J}^+,\mathcal{J}^+) = \min\{\vartheta^{(j)}_\infty(\mathcal{J}^+)\}$.

All symmetrical problems under consideration are divided into two groups. For the first one, the indices of the determinants of matrix-functions $\mathbf{Q}^{(j)}(s)$ are equal to zero:

$$
\kappa = -\det \mathbf{Q}^{(j)}(\alpha - it, \mathcal{J}^+,\mathcal{J}) = 0, \qquad |\alpha| < \vartheta^{(j)}_0(\mathcal{J}^+,\mathcal{J}).
$$

For the second group, the relations hold true:

$$
\kappa = -\det \mathbf{Q}^{(j)}(\alpha - it, \mathcal{J}^+,\mathcal{J}) = \begin{cases} 1, & 0 < \alpha < \vartheta^{(j)}_0(\mathcal{J}^+,\mathcal{J}), \\ -1, & -\vartheta^{(j)}_0(\mathcal{J}^+,\mathcal{J}) < \alpha < 0. \end{cases}
$$

The following problems are rated to the first group:

$$
(\mathcal{J}^+,\mathcal{J}^+,5)_j,\ (2,2,1)_j,\ (1,1,2)_j, \qquad \mathcal{J}^+ = 1-4, \quad j = 1,2;
$$

$$
(1,1,3)_2,\ (1,1,4)_1,\ (2,2,3)_2,\ (2,2,4)_1,\ (3,3,3)_2,\ (3,3,4)_1,\ (4,4,3)_2,\ (4,4,4)_1.
$$

The second group consists of the problems:

$$
\begin{gathered}
(1,1,1)_j,\ (2,2,2)_j, \quad j = 1,2; \qquad (1,1,3)_1,\ (1,1,4)_2,\ (2,2,3)_1,\ (2,2,4)_2, \\
(3,3,1)_2,\ (3,3,2)_2,\ (3,3,4)_2,\ (4,4,1)_1,\ (4,4,2)_1,\ (4,4,3)_1.
\end{gathered}
$$

Finally, the following problems have symbols with nonzero indices ($|\kappa| = 1$), in general, except the special case of a crack terminating normally to the interface, when these problems are of zero indices (see (A.6) in the Appendix):

$$
(3,3,1)_1,\ (3,3,2)_1,\ (3,3,3)_1,\ (4,4,1)_2,\ (4,4,2)_2,\ (4,4,4)_2.
$$

Theorems formulated before are true for the symmetrical problems under consideration depending on the indices of the respective symbols. Moreover, for some problems for which the indices are not equal to zero in a general case, one of the two systems of integral equations can have a zero symbol! Hence, the corresponding systems of the integral equations (for Mode I or Mode II strain-stress state), should not be regularized.

6. Conclusions

We have considered all possible boundary value problems for different geometries of the domain as well as arbitrary combinations of the external boundary conditions. The corresponding problems have been reduced to systems of singular integral equations. Symbols of the corresponding operators have been presented and indices of the operators have been calculated.

Let us remember that in this paper we assume that the interfacial conditions between the first layer and the two nearest wedges (see (2.7), (2.8) in [15]) are characterized by given discontinuities of the displacements and tractions. Nevertheless, these conditions can be generalized, so that the tractions will be proportional to the jump of the displacements. Such problems can also be solved by the presented method. However, as it has been shown in [13] for the case of the Poisson equation in a similar domain, such investigation is a little different from that presented above, and the symbols of the corresponding integral operators are degenerate, in general.

Appendix

Here we present some estimations for matrix-functions $\mathbf{\Phi}_*^{-1}(s, \mathcal{J}^+, \mathcal{J}^-)$ depending on the values of $\mathcal{J}^{\pm} = 1 - 4$.

For problems $(1, 1, \mathcal{J})$, $(\mathcal{J} = 1 - 5)$ the following relations can be verified basing on the results obtained from Lemma 2 in [15]:

$$\mathbf{\Phi}_*^{-1}(s) = \mathbf{L}_\infty^{-1} + \left(\begin{array}{c|c} \mathbf{E}_2\mathbf{X} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{X} \end{array}\right) \left(\begin{array}{c|c} -\frac{b_\infty^2}{a_\infty}\mathbf{I} & b_\infty \mathbf{I} \\ \hline -b_\infty \mathbf{I} & a_\infty \mathbf{I} \end{array}\right) \left(\begin{array}{c|c} \mathbf{E}_2 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{I} \end{array}\right) + \mathcal{O}(s), \qquad s \to 0, \tag{A.1}$$

$$\mathbf{X} = \frac{1}{b_\infty^2 - a_\infty^2}\mathbf{I} + \frac{\pi}{2\mu_1 a_\infty}\left[\frac{\pi(a_\infty^2 + b_\infty^2)}{2\mu_1 a_\infty}\mathbf{I} + \mathbf{M}_-^0\right]^{-1},$$

$$a_\infty = 1 - \nu_1, \qquad b_\infty = 1/2 - \nu_1.$$

For the remaining problems $(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ $(\mathcal{J}^+\mathcal{J}^- > 1,\ \mathcal{J} = 1-5)$, under the additional assumption $\det \mathbf{M}_+^0 \neq 0$, we can obtain:

$$\mathbf{\Phi}_*^{-1}(s, \mathcal{J}^+, \mathcal{J}^-) = \left(\begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & s\mathbf{I} \end{array}\right) \left[\left(\begin{array}{c|c} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \hline \mathbf{A}_{21} & \mathbf{A}_{22} \end{array}\right) + \mathcal{O}(s) \right] \times \left(\begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & s\mathbf{I} \end{array}\right), \qquad s \to 0,$$

$$\text{(A.2)} \qquad \mathbf{A}_{11} = \left[a_\infty \mathbf{I} - \frac{\mu_1 \pi}{2} \mathbf{M}_+^0 + \frac{\mu_1 \pi}{2} \mathbf{M}_-^0 (\mathbf{M}_+^0)^{-1} \mathbf{M}_-^0 \right]^{-1},$$

$$\mathbf{A}_{12} = \frac{\pi}{2} \mathbf{A}_{11} \mathbf{M}_-^0 (\mathbf{M}_+^0)^{-1}, \qquad \mathbf{A}_{21} = -\frac{\pi}{2} (\mathbf{M}_+^0)^{-1} \mathbf{M}_-^0 \mathbf{A}_{11},$$

$$\mathbf{A}_{22} = \frac{\pi}{2\mu_1} \left[(\mathbf{M}_+^0)^{-1} - \frac{\mu_1 \pi}{2} (\mathbf{M}_+^0)^{-1} \mathbf{M}_-^0 \mathbf{A}_{11} \mathbf{M}_-^0 (\mathbf{M}_+^0)^{-1} \right].$$

In some cases these relations can be simplified. Namely, for problems $(1, 2, \mathcal{J})$, $(\mathcal{J} = 1-5)$ the identities $\mathbf{M}_-^0 = \mathbf{M}_+^0$ hold true, and the matrices are still nondegenerate. Hence we obtain:

$$\mathbf{A}_{11} = \frac{1}{a_\infty} \mathbf{I}, \qquad \mathbf{A}_{12} = -\mathbf{A}_{21} = \frac{\pi}{2a_\infty} \mathbf{I},$$

$$\mathbf{A}_{22} = \frac{\pi}{2\mu_1 a_\infty} \left[a_\infty (\mathbf{M}_+^0)^{-1} - \frac{\mu_1 \pi}{2} \mathbf{I} \right].$$

Besides, for problems $(\mathcal{J}^+, \mathcal{J}^+, \mathcal{J})$ $(\mathcal{J}^+ = 3, 4)$ when $\theta_0^+ \neq \theta_{m_-}^-$, and for problems $(3, 4, \mathcal{J})$ when $\theta_0^+ - \theta_{m_-}^- \neq \pi/2$, the following equation can be verified: $(\mathbf{M}_+^0)^{-1} \mathbf{M}_-^0 = \mathbf{M}_-^0 (\mathbf{M}_+^0)^{-1}$.

Finally, let us present the relations when matrices $\mathbf{M}_+^0$ are degenerate. Such situations appear for the problems:

a) $(1, 3, \mathcal{J})$, $(1, 4, \mathcal{J})$ $(\mathcal{J} = 1-5)$, where $\mathbf{M}_-^0 = \mathbf{M}_+^0 = \mathbf{B}_0$;

b) $(3, 3, \mathcal{J})$, $(4, 4, \mathcal{J})$ $(\mathcal{J} = 1-5)$ under the additional geometrical conditions $\theta_0^+ = \theta_{m_-}^-$, what leads to the relations: $\mathbf{M}_+^0 = y\mathbf{B}_0$, $\mathbf{M}_-^0 = \mathbf{B}_0$;

c) $(3, 4, \mathcal{J})$ $(\mathcal{J} = 1-5)$ under conditions $\theta_0^+ = \theta_{m_-}^- + \pi/2$, when the identities hold true: $\mathbf{M}_-^0 = y\mathbf{B}_0$, $\mathbf{M}_+^0 = \mathbf{B}_0$.

Here certain constants y $(|y| \leq 1)$ and the corresponding matrices $\mathbf{B}_0$ $(\mathbf{B}_0 \neq \mathbf{0}$, $\det \mathbf{B}_0 = 0)$ are calculated basing on the results of Lemma 2, so that

$$\mathbf{B}_0 = \begin{pmatrix} b_{11}^2 & b_{11} b_{12} \\ b_{11} b_{12} & b_{12}^2 \end{pmatrix}.$$

Then for the mentioned three cases, asymptotic behaviour of matrix-functions $\mathbf{\Phi}_*^{-1}(s)$ is of the form:

$$\mathbf{\Phi}_*^{-1}(s) = d \left(\begin{array}{c|c} 4a_\infty^2 \xi [b_{11}^2 + b_{12}^2]\mathbf{I} + (c - 4b_\infty^2 \xi)\mathbf{C} & 4a_\infty b_\infty \xi \mathbf{D} \\ \hline 4a_\infty b_\infty \xi \mathbf{D}^\top & (c + 4a_\infty^2 \xi)\mathbf{C} \end{array} \right) + \mathcal{O}(s), \qquad s \to 0, \tag{A.3}$$

$$\mathbf{C} = \begin{pmatrix} b_{12}^2 & -b_{11}b_{12} \\ -b_{11}b_{12} & b_{11}^2 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} b_{11}b_{12} & -b_{11}^2 \\ b_{12}^2 & -b_{11}b_{12} \end{pmatrix},$$

where

$$c = a_\infty \mu_1 \pi (b_{11}^2 + b_{12}^2)(\eta^2 - \xi^2), \qquad d = a_\infty^{-1}(b_{11}^2 + b_{12}^2)^{-1}[4\xi(a_\infty^2 - b_\infty^2) + c]^{-1}.$$

In these relations the values of parameters ξ, η are defined depending on the three situations mentioned above:

$$\textbf{a)} \leftrightarrow \xi = \eta = 1; \qquad \textbf{b)} \leftrightarrow \xi = y,\ \eta = 1; \qquad \textbf{c)} \leftrightarrow \xi = 1,\ \eta = y.$$

A.1. The cases when the domain is symmetrical with respect to the $0X_2$ axis

In this part of the Appendix the respective relations are presented for the situations investigated in the last subsection of Sec. 5.

Thus for problems $(1,1,\mathcal{J})_j$, the matrix $\mathbf{M}_+^0$ (and, consequently, the matrix $\mathbf{X}$) are diagonal matrices with elements m_1, m_2; then we can calculate from (A.1) and Lemma 2 of [15] that:

$$\left[\mathbf{\Phi}_*^{(j)}(s,1,1)\right]^{-1} = \frac{1}{a_\infty} \begin{pmatrix} 1 + y_{3-j} b_\infty^2 & (-1)^j a_\infty b_\infty y_{3-j} \\ (-1)^j a_\infty b_\infty y_{3-j} & y_{3-j} a_\infty^2 \end{pmatrix}, \tag{A.4}$$
$$y_j = \frac{\pi}{\pi(a_\infty^2 + b_\infty^2) + 2\mu_1 a_\infty m_j}, \qquad m_1,\ m_2 > 0.$$

For the remaining problems $(\mathcal{J}^+, \mathcal{J}^+, \mathcal{J})_j$ ($\mathcal{J}^+ = 2, 3, 4$, $\mathcal{J} = 1 - 5$, $j = 1, 2$) for which Eqs. (A.2) are valid (except the two cases considered in (A.3)), it can be shown that the matrices $\mathbf{M}_\pm^0$ are of the form:

$$\mathbf{M}_+^0 = \begin{pmatrix} 0 & m_1 \\ -m_1 & 0 \end{pmatrix}, \qquad \mathbf{M}_-^0 = \begin{pmatrix} m_2 & 0 \\ 0 & m_3 \end{pmatrix}, \qquad m_2, m_3 < 0.$$

Then

$$\left[\mathbf{\Phi}_*^{(1)}(s)\right]^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix} \left[\begin{pmatrix} A & \dfrac{\pi m_1}{2m_3} A \\ \dfrac{\pi m_1}{2m_3} A & \dfrac{1}{m_3} + \dfrac{\mu_1 \pi m_1^2}{2m_3^2} A \end{pmatrix} + \mathcal{O}(s) \right] \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix}, \tag{A.5}$$
$$\left[\mathbf{\Phi}_*^{(2)}(s)\right]^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix} \left[\begin{pmatrix} B & -\dfrac{\pi m_1}{2m_2} B \\ -\dfrac{\pi m_1}{2m_2} B & \dfrac{1}{m_2} + \dfrac{\mu_1 \pi m_1^2}{2m_2^2} B \end{pmatrix} + \mathcal{O}(s) \right] \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix}.$$

$$A^{-1} = a_\infty - \frac{\mu_1 \pi}{m_3}(m_2 m_3 + m_1^2), \qquad B^{-1} = a_\infty - \frac{\mu_1 \pi}{m_2}(m_2 m_3 + m_1^2).$$

Besides, for problems $(3, 3, \mathcal{J})$ and $(4, 4, \mathcal{J})$ when $\theta_0^+ \neq \theta_{m_-}^-$ it can be proved that $m_2 m_3 + m_1^2 = 0$.

Finally, only one of the three last cases, when matrices $\mathbf{M}_0^\pm$ are degenerate, can be realized. Namely, for problems $(3,3,\mathcal{J})$ and $(4,4,\mathcal{J})$ when $\theta_0^+ = \theta_{m_-}^- = -\pi/2$, it can be found that $b_{12} = 0$ and $b_{11} = 0$, respectively, and then

$$\text{(A.6)} \quad \begin{aligned}
\left[\mathbf{\Phi}_*^{(1)}(s,3,3)\right]^{-1} &= db_{11}^2 \begin{pmatrix} 4a_\infty^2 & -4a_\infty b_\infty y \\ -4a_\infty b_\infty y & c + 4a_\infty^2 y \end{pmatrix} + \mathcal{O}(s), \qquad s \to 0, \\
\left[\mathbf{\Phi}_*^{(2)}(s,3,3)\right]^{-1} &= db_{11}^2 \begin{pmatrix} 4a_\infty^2 + c - 4b_\infty^2 y & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{O}(s), \qquad s \to 0, \\
\left[\mathbf{\Phi}_*^{(1)}(s,4,4)\right]^{-1} &= db_{12}^2 \begin{pmatrix} 4a_\infty^2 + c - 4b_\infty^2 y & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{O}(s), \qquad s \to 0, \\
\left[\mathbf{\Phi}_*^{(2)}(s,4,4)\right]^{-1} &= db_{12}^2 \begin{pmatrix} 4a_\infty^2 & 4a_\infty b_\infty y \\ 4a_\infty b_\infty y & c + 4a_\infty^2 y \end{pmatrix} + \mathcal{O}(s), \qquad s \to 0.
\end{aligned}$$

In conclusion let us note that values of the matrices $\mathbf{M}_\pm^0$ in asymptotics (4.10), (4.11) as well as all other constants used in the Appendix are calculated basing on the results of Lemma 2 in [15].

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A boundary integral equations method for asymmetric Stokes flow between two parallel planes

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IN THIS PAPER we apply a direct boundary integral equations method to Stokes flow past a smooth obstacle, occurring between two parallel planes. The problem is formulated exactly as a system of linear Fredholm integral equations of the second kind, over the surface of the obstacle. It is shown that this system has a unique continuous solution when the boundary of the particle is a Lyapunov surface and the velocity distributions on the same boundary is continuous. The numerical results are obtained by a standard boundary element method.

1. Introduction

THE MOTION OF A BODY of a simple shape in a viscous fluid between parallel planar boundaries, has been the subject of many studies. For example, P. GANATOS, R. PFEFFER and S. WEINBAUM (see [7]) gave a numerical method of analysis for the motion of an asymmetric Stokes flow between parallel planes induced by the rotary or translatory motion of a sphere. Also, a special case of the flow due to the rotation of a sphere, considered by the above mentioned authors, was investigated by W.W. HACKBORN (see [8]). This author presented an analytical method for the asymmetric Stokes flow between parallel planes due to a three-dimensional rotlet whose axis is supposed to be parallel to the planes. Using the periodic Green functions, C. POZRIKIDIS (see [14]) investigated the creeping flows in two-dimensional channels. L. DRAGOŞ and A. DINU (see [2, 3]) determined a direct boundary integral method for subsonic flows with circulation in two-dimensional channels.

The aim of this paper is to give a boundary integral method for an asymmetric Stokes flow between two parallel planes and in the presence of a rigid obstacle. The corresponding Green's functions are found. By using these functions, the velocity field is determined as a sum of a single-layer potential with a double-layer potential. The properties of the double-layer and single-layer operators secure the existence and uniqueness results of the solution.

2. Mathematical formulation

The configuration of an asymmetric Stokes flow of a viscous incompressible fluid, induced by the slow motion with the velocity $\mathbf{U}_0$, of an arbitrary rigid particle Ω^1, between two parallel, rigid planes $\mathcal{P}_0$ and $\mathcal{P}_1$, is illustrated in the Fig. 1. We suppose that S, surface of the particle, is a Lyapunov surface (see [9]).

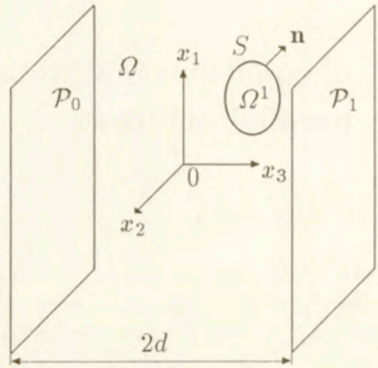

FIG. 1.

Let $\mathbf{U}_\infty$ and p_∞ be the velocity and pressure fields of an undisturbed Stokes flow in the domain with the boundaries $\mathcal{P}_0$ and $\mathcal{P}_1$. These planes $\mathcal{P}_0$, $\mathcal{P}_1$ have the dimensionless equations

$$\mathcal{P}_0 = \{(x_1, x_2, x_3) \in \mathbf{R}^3 |\ x_3 = -d\}, \qquad \mathcal{P}_1 = \{(x_1, x_2, x_3) \in \mathbf{R}^3 |\ x_3 = d\}, \tag{2.1}$$

where d is a positive constant. Also, the velocity $\mathbf{U}_\infty$ and the pressure p_∞ satisfy the following Stokes system of dimensionless equations

$$\Delta \mathbf{U}_\infty(x) - \nabla p_\infty(x) = 0, \qquad \text{for} \quad x = (x_1, x_2, x_3) \in \mathbf{R}^3, \quad |x_3| < d, \tag{2.2}$$

$$\nabla \cdot \mathbf{U}_\infty(x) = 0, \qquad \text{for} \quad x = (x_1, x_2, x_3) \in \mathbf{R}^3, \quad |x_3| < d, \tag{2.3}$$

with the nonslip boundary condition

$$\mathbf{U}_\infty(x) = 0, \qquad \text{for} \quad x \in \mathcal{P}_0 \ \text{ or } \ \mathcal{P}_1 \qquad (\text{i.e.} \quad x_3 = \pm d), \tag{2.4}$$

where Δ and ∇ are three-dimensional Laplace and the gradient operators.

Let $\mathbf{u}_1$, p_1 be the velocity field and the pressure, respectively, of the total flow, which results from the presence of the given obstacle. We denote by $\mathbf{u} = \mathbf{u}_1 - \mathbf{U}_\infty$, $p = p_1 - p_\infty$, the velocity and pressure fields of the disturbed flow. If we suppose that the Reynolds number of the flow $(\mathbf{u}_1, p_1)$ is very small, then the velocity $\mathbf{u}$ and the pressure p satisfy, as a first approximation, the following Stokes equations

$$\Delta \mathbf{u}(x) - \nabla p(x) = 0, \qquad \text{for} \quad x \in \Omega, \tag{2.5}$$

$$\nabla \cdot \mathbf{u}(x) = 0, \qquad \text{for} \quad x \in \Omega. \tag{2.6}$$

Here Ω is the domain exterior to the obstacle, with the boundaries S, $\mathcal{P}_0$ and $\mathcal{P}_1$.

The velocity field satisfies the nonslip boundary conditions

$$\mathbf{u}(x) = \mathbf{U}_0(x) - \mathbf{U}_\infty(x), \qquad \text{for} \quad x \in S, \tag{2.7}$$

$$\mathbf{u}(x) = 0, \qquad \text{for} \quad x \in \mathcal{P}_0 \ \text{ or } \ \mathcal{P}_1 \qquad (\text{i.e.} \quad x_3 = \pm d), \tag{2.8}$$

as well as the following conditions at infinity

$$\mathbf{u}(x) \to 0, \qquad p(x) \to 0, \qquad \text{as} \qquad |x| \to \infty, \tag{2.9}$$

where $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ is the usual Euclidean distance in $\mathbf{R}^3$, between the point x and the origin O of the fixed orthogonal system $Ox_1x_2x_3$.

3. Construction of Green's functions

Let $\mathbf{G}(G_{ij})$ and $\mathbf{q}(q_i)$ be the Green tensor and pressure vector of the Stokes equations (2.5), (2.6), in the infinite domain, with $\mathcal{P}_0$ and $\mathcal{P}_1$ as boundaries. Additionally, the Green function $\mathbf{G}$ becomes zero when its pole is located on any of the walls. Thus, the next equations and conditions are satisfied

$$\Delta_y G_{ij}(x,y) - \frac{\partial q_j}{\partial y_i}(x,y) = -\delta_{ij}\delta(x-y), \tag{3.1}$$

$$\sum_{i=1}^{3} \frac{\partial G_{ij}}{\partial y_i}(x,y) = 0, \qquad \text{for} \quad y = (y_1, y_2, y_3) \in \mathbf{R}^3, \quad |y_3| < d, \tag{3.2}$$

$$G_{ij}(x,y) = 0, \qquad \text{for} \quad y_3 = \pm d, \tag{3.3}$$

$$G_{ij}(x,y) \to 0, \qquad q_i(x,y) \to 0, \qquad \text{as} \quad |y| \to \infty, \tag{3.4}$$

where δ is the Dirac distribution and $\delta_{ij} = 1$, for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$.

The Green's functions $\mathbf{G}$ and $\mathbf{q}$ are determined in the following form:

$$\mathbf{G}(x,y) = \mathbf{E}(x-y) + \mathbf{D}(x,y), \qquad \mathbf{q}(x,y) = \mathbf{e}(x-y) + \mathbf{d}(x,y), \tag{3.5}$$

where $\mathbf{E}(E_{ij})$, $\mathbf{e}(e_i)$ are the fundamental solutions of the Stokes equations in the whole space and $\mathbf{D}(D_{ij})$, $\mathbf{d}(d_i)$ represent complementary functions, such that the null conditions (3.3) are satisfied.

In fact, $\mathbf{E}$ and $\mathbf{e}$ solve the following Stokes problem:

$$\Delta_y E_{ij}(x-y) - \frac{\partial e_j}{\partial y_i}(x-y) = -\delta_{ij}\delta(x-y), \tag{3.6}$$

$$\sum_{i=1}^{3} \frac{\partial E_{ij}}{\partial y_i}(x-y) = 0, \tag{3.7}$$

$$E_{ij}(x-y) \to 0, \qquad e_i(x-y) \to 0, \qquad \text{as} \quad |y| \to \infty. \tag{3.8}$$

The functions $\mathbf{D}$ and $\mathbf{d}$ are solutions of the Stokes equations and the conditions given below:

$$\Delta_y D_{ij}(x,y) - \frac{\partial d_j}{\partial y_i}(x,y) = 0, \qquad \text{for} \quad y \in \mathbf{R}^3, \quad |y_3| < d, \quad i,j = \overline{1,3}, \tag{3.9}$$

$$\text{(3.10)} \qquad \sum_{i=1}^{3} \frac{\partial D_{ij}}{\partial y_i}(x,y) = 0, \qquad \text{for} \quad y \in \mathbf{R}^3, \quad |y_3| < d,$$

$$\text{(3.11)} \qquad D_{ij}(x,y) = -E_{ij}(x-y), \qquad \text{for} \quad y \in \mathbf{R}^3, \quad y_3 = \pm d, \quad i,j = \overline{1,3},$$

$$\text{(3.12)} \qquad D_{ij}(x,y) \to 0, \qquad d_i(x,y) \to 0, \qquad \text{as} \quad |y| \to \infty.$$

The fundamental solutions $\mathbf{E}$ and $\mathbf{e}$ are given by (see [1])

$$\text{(3.13)} \quad E_{ij}(x-y) = \frac{1}{8\pi}\left[\frac{1}{|x-y|} + \frac{(x_i-y_i)(x_j-y_j)}{|x-y|^3}\right], \qquad i,j = \overline{1,3},$$

$$\text{(3.14)} \qquad e_i(x-y) = -\frac{1}{4\pi}\cdot\frac{x_i-y_i}{|x-y|^3}, \qquad i = \overline{1,3}.$$

We seek the regular parts $\mathbf{D}$ and $\mathbf{d}$ of $\mathbf{G}$ and $\mathbf{q}$ in the following form

$$\text{(3.15)} \qquad D_{ij}(x,y) = \int\limits_{z_3=-d} E_{ik}(z-y)\alpha_k^{(j)}(x,z)\,ds_z + \int\limits_{z_3=d} E_{ik}(z-y)\beta_k^{(j)}(x,z)\,ds_z\,,$$

$$\text{(3.16)} \qquad d_{ij}(x,y) = \int\limits_{z_3=-d} e_k(z-y)\alpha_k^{(j)}(x,z)\,ds_z + \int\limits_{z_3=d} e_k(z-y)\beta_k^{(j)}(x,z)ds_z\,.$$

It is easy to see that the functions $\mathbf{D}$ and $\mathbf{d}$ satisfy the Stokes equations (3.9), (3.10) and the asymptotic conditions (3.12). The unknown densities $\alpha_k^{(j)}$, $\beta_k^{(j)}$, $k,j = \overline{1,3}$ will be solutions of the following integral equations

$$\text{(3.17)} \qquad \int\limits_{\mathbf{R}^2} E_{ik}(z_1-y_1, z_2-y_2, -d-y_3)\alpha_k^{(j)}(x,z)\,dz_1\,dz_2$$
$$+ \int\limits_{\mathbf{R}^2} E_{ik}(z_1-y_1, z_2-y_2, d-y_3)\beta_k^{(j)}(x,z)\,dz_1\,dz_2 = -E_{ij}(x-y),$$
$$\text{for} \quad y \in \mathbf{R}^3, \quad y_3 = \pm d.$$

In order to solve this integral system, we apply the Fourier transform $\mathcal{F}$, with respect to the variables $y' = (y_1, y_2)$ (see [18])

$$\text{(3.18)} \qquad \mathcal{F}\phi(\xi) = \frac{1}{(2\pi)^2}\int\limits_{\mathbf{R}^2} e^{-i\xi\cdot y'}\phi(y')\,dy_1\,dy_2, \qquad \xi = (\xi_1,\xi_2) \in \mathbf{R}^2,$$

$$\text{(3.19)} \qquad \mathcal{F}^{-1}\psi(y') = \frac{1}{(2\pi)^2}\int\limits_{\mathbf{R}^2} e^{i\xi\cdot y'}\psi(\xi)\,d\xi_1\,d\xi_2, \qquad y' = (y_1,y_2) \in \mathbf{R}^2,$$

$\mathcal{F}^{-1}$ being the inverse Fourier transform.

From (3.15) and (3.16), we obtain

$$(3.20)\qquad \mathcal{F}D_{ij}(\xi;x,y_3) = \mathcal{F}E_{ik}(\xi;0,-d-y_3)\mathcal{F}\alpha_k^{(j)}(\xi;x) + \mathcal{F}E_{ik}(\xi;0,d-y_3)\mathcal{F}\beta_k^{(j)}(\xi;x), \qquad i,j=\overline{1,3},$$

$$(3.21)\qquad \mathcal{F}d_j(\xi;x,y_3) = \mathcal{F}e_k(\xi;0,-d-y_3)\mathcal{F}\alpha_k^{(j)}(\xi;x) + \mathcal{F}e_k(\xi;0,d-y_3)\mathcal{F}\beta_k^{(j)}(\xi;x), \qquad j=\overline{1,3}.$$

On the other hand, from (3.17) we obtain the following linear system with the unknowns $\mathcal{F}\alpha_k^{(j)}(\xi;x)$ and $\mathcal{F}\beta_k^{(j)}(\xi;x)$, $k,j=\overline{1,3}$:

$$(3.22)\qquad \mathcal{F}E_{ik}(\xi;0,0)\mathcal{F}\alpha_k^{(j)}(\xi;x) + \mathcal{F}E_{ik}(\xi;0,2d)\mathcal{F}\beta_k^{(j)}(\xi;x) = -\mathcal{F}E_{ij}(\xi;x',x_3+d),$$

$$(3.23)\qquad \mathcal{F}E_{ik}(\xi;0,-2d)\mathcal{F}\alpha_k^{(j)}(\xi;x) + \mathcal{F}E_{ik}(\xi;0,0)\mathcal{F}\beta_k^{(j)}(\xi;x) = -\mathcal{F}E_{ij}(\xi;x',x_3-d),$$

$$i,j=\overline{1,3}, \qquad x'=(x_1,x_2)\in\mathbf{R}^2.$$

It is convenient to write $\mathbf{E}$ and $\mathbf{e}$ as follows (see [5]):

$$(3.24)\qquad \mathbf{E}(x-y) = \frac{1}{8\pi}[\mathbf{I}\Delta_y\phi(x-y) - \nabla_y\nabla_y\phi(x-y)],$$

$$(3.25)\qquad \mathbf{e}(x-y) = -\frac{1}{8\pi}\nabla_y\Delta_y\phi(x-y),$$

where

$$(3.26)\qquad \phi(x-y) = |x-y| = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}\,.$$

By using the following properties of the Fourier transform (see [18])

$$(3.27)\qquad \nabla^{(m)}(\mathcal{F}v) = \mathcal{F}((-ix)^m v), \qquad \mathcal{F}(\nabla^{(m)}v) = (i\xi)^m\mathcal{F}v,$$

with $m=\sum_{i=1}^{2} m_i$, $x^m = x_1^{m_1}x_2^{m_2}$ and $\xi^m = \xi_1^{m_1}\xi_2^{m_2}$, we obtain

$$(3.28)\qquad \mathcal{F}\mathbf{E}(\xi;x',x_3-y_3) = -\frac{1}{4}e^{-i\xi\cdot x'}e^{-|\xi||x_3-y_3|}\cdot\mathbf{A},$$

where the matrix $\mathbf{A}$ has the form (see, also [5])

$$(3.29)\qquad \mathbf{A} = \begin{pmatrix} -\frac{2}{|\xi|}+\frac{\xi_1^2}{|\xi|^3}+|x_3-y_3|\frac{\xi_1^2}{|\xi|^2} & \frac{\xi_1\xi_2}{|\xi|^3}+|x_3-y_3|\frac{\xi_1\xi_2}{|\xi|^2} & -i(x_3-y_3)\frac{\xi_1}{|\xi|} \\ \frac{\xi_1\xi_2}{|\xi|^3}+|x_3-y_3|\frac{\xi_1\xi_2}{|\xi|^2} & -\frac{2}{|\xi|}+\frac{\xi_2^2}{|\xi|^3}+|x_3-y_3|\frac{\xi_2^2}{|\xi|^2} & -i(x_3-y_3)\frac{\xi_2}{|\xi|} \\ -i(x_3-y_3)\frac{\xi_1}{|\xi|} & -i(x_3-y_3)\frac{\xi_2}{|\xi|} & -\frac{1}{|\xi|}-|x_3-y_3| \end{pmatrix}.$$

In a similar manner, we obtain

$$\mathcal{F}\mathbf{e}(\xi; x', x_3 - y_3) = -\frac{1}{2}e^{-i\xi\cdot x'}e^{-|\xi|(x_3-y_3)}\left(\frac{i\xi_1}{|\xi|} \quad \frac{i\xi_2}{|\xi|} \quad \operatorname{sgn}(x_3-y_3)\right). \tag{3.30}$$

We denote by **B**, **C**, **D**, **H** and **Q** the following matrices:

$$\mathbf{B} = -\frac{1}{4}\begin{pmatrix} -\frac{2}{|\xi|} + \frac{\xi_1^2}{|\xi|^3} & \frac{\xi_1\xi_2}{|\xi|^3} & 0 \\ \frac{\xi_1\xi_2}{|\xi|^3} & -\frac{2}{|\xi|} + \frac{\xi_2^2}{|\xi|^3} & 0 \\ 0 & 0 & -\frac{1}{|\xi|} \end{pmatrix}, \tag{3.31}$$

$$\mathbf{C} = -\frac{1}{4}e^{-2|\xi|}\begin{pmatrix} -\frac{2}{|\xi|} + \frac{\xi_1^2}{|\xi|^3} + 2d\frac{\xi_1^2}{|\xi|^2} & \frac{\xi_1\xi_2}{|\xi|^3} + 2d\frac{\xi_1\xi_2}{|\xi|^2} & -2di\frac{\xi_1}{|\xi|} \\ \frac{\xi_1\xi_2}{|\xi|^3} + 2d\frac{\xi_1\xi_2}{|\xi|^2} & -\frac{2}{|\xi|} + \frac{\xi_2^2}{|\xi|^3} + 2d\frac{\xi_2^2}{|\xi|^2} & -2di\frac{\xi_2}{|\xi|} \\ -2di\frac{\xi_1}{|\xi|} & -2di\frac{\xi_2}{|\xi|} & -\frac{1}{|\xi|} - 2d \end{pmatrix}, \tag{3.32}$$

$$\mathbf{D} = -\frac{1}{4}e^{-2|\xi|}\begin{pmatrix} -\frac{2}{|\xi|} + \frac{\xi_1^2}{|\xi|^3} + 2d\frac{\xi_1^2}{|\xi|^2} & \frac{\xi_1\xi_2}{|\xi|^3} + 2d\frac{\xi_1\xi_2}{|\xi|^2} & 2di\frac{\xi_1}{|\xi|} \\ \frac{\xi_1\xi_2}{|\xi|^3} + 2d\frac{\xi_1\xi_2}{|\xi|^2} & -\frac{2}{|\xi|} + \frac{\xi_2^2}{|\xi|^3} + 2d\frac{\xi_2^2}{|\xi|^2} & 2di\frac{\xi_2}{|\xi|} \\ 2di\frac{\xi_1}{|\xi|} & 2di\frac{\xi_2}{|\xi|} & -\frac{1}{|\xi|} - 2d \end{pmatrix}, \tag{3.33}$$

$$\mathbf{H} = -\frac{1}{4}e^{-i\xi\cdot x'}e^{-|\xi|(x_3+1)} \cdot \begin{pmatrix} -\frac{2}{|\xi|} + \frac{\xi_1^2}{|\xi|^3} + (x_3+d)\frac{\xi_1^2}{|\xi|^2} & \frac{\xi_1\xi_2}{|\xi|^2}\left(\frac{1}{|\xi|} + x_3 + d\right) & -i(x_3+d)\frac{\xi_1}{|\xi|} \\ \frac{\xi_1\xi_2}{|\xi|^2}\left(\frac{1}{|\xi|} + x_3 + d\right) & -\frac{2}{|\xi|} + \frac{\xi_2^2}{|\xi|^2}\left(\frac{1}{|\xi|} + x_3 + d\right) & -i(x_3+d)\frac{\xi_2}{|\xi|} \\ -i(x_3+d)\frac{\xi_1}{|\xi|} & -i(x_3+d)\frac{\xi_2}{|\xi|} & -\frac{1}{|\xi|} - (x_3+d) \end{pmatrix}, \tag{3.34}$$

$$\mathbf{Q} = -\frac{1}{4}e^{-i\xi\cdot x'}e^{-|\xi|(x_3-1)} \cdot \begin{pmatrix} -\frac{2}{|\xi|} + \frac{\xi_1^2}{|\xi|^2}\left(\frac{1}{|\xi|} - x_3 + d\right) & \frac{\xi_1\xi_2}{|\xi|^2}\left(\frac{1}{|\xi|} - x_3 + d\right) & -i(x_3-d)\frac{\xi_1}{|\xi|} \\ \frac{\xi_1\xi_2}{|\xi|^2}\left(\frac{1}{|\xi|} - x_3 + d\right) & -\frac{2}{|\xi|} + \frac{\xi_2^2}{|\xi|^2}\left(\frac{1}{|\xi|} - x_3 + d\right) & -i(x_3-d)\frac{\xi_2}{|\xi|} \\ -i(x_3-d)\frac{\xi_1}{|\xi|} & -i(x_3-d)\frac{\xi_2}{|\xi|} & -\frac{1}{|\xi|} + x_3 - d \end{pmatrix}. \tag{3.35}$$

Using the above notations, the linear system of equations (3.22), (3.23), becomes

$$\mathbf{B}\cdot\mathbf{a}^i + \mathbf{C}\cdot\mathbf{b}^i = \mathbf{H}, \qquad \mathbf{D}\cdot\mathbf{a}^i + \mathbf{B}\cdot\mathbf{b}^i = \mathbf{Q}, \quad i = \overline{1,3}, \tag{3.36}$$

where

$$(3.37)\qquad \mathbf{a}^i = \begin{pmatrix} \mathcal{F}\alpha_1^{(i)}(\xi;x) \\ \mathcal{F}\alpha_2^{(i)}(\xi;x) \\ \mathcal{F}\alpha_3^{(i)}(\xi;x) \end{pmatrix}, \qquad \mathbf{b}^i = \begin{pmatrix} \mathcal{F}\beta_1^{(i)}(\xi;x) \\ \mathcal{F}\beta_2^{(i)}(\xi;x) \\ \mathcal{F}\beta_3^{(i)}(\xi;x) \end{pmatrix}, \qquad i = \overline{1,3}.$$

From (3.20) and (3.21), we obtain

$$(3.38)\qquad D_{ij}(x,y) = \int\limits_{\mathbf{R}^2} e^{i\xi\cdot y'}\Big[\mathcal{F}E_{ik}(\xi;0,-d-y_3)a_k^j + \mathcal{F}E_{ik}(\xi;0,d-y_3)b_k^j\Big]dy_1dy_2\,,$$

$$(3.39)\qquad e_j(x,y) = \int\limits_{\mathbf{R}^2} e^{i\xi\cdot y'}\Big[\mathcal{F}e_k(\xi;0,-d-y_3)a_k^j + \mathcal{F}e_k(\xi;0,d-y_3)b_k^j\Big]dy_1dy_2\,,$$

$i,j = \overline{1,3}$.

4. Integral representation of solution

The Green function $\mathbf{G}$ satisfies the condition (3.3). When the pole x approaches the point y of $\mathcal{P}_0$ or $\mathcal{P}_1$, $\mathbf{G}$ becomes singular. On the other hand, this function must vanish, due to (3.3). Thus, we obtain the following condition

$$(4.1)\qquad \mathbf{G}(x,y) = 0, \quad \text{for} \quad x \in \mathcal{P}_0 \ \text{or} \ \mathcal{P}_1 \qquad (\text{i.e. for} \quad x_3 = \pm d), \qquad \forall\, y \in \mathbf{R}^3, \quad |y_3| < d.$$

As a consequence of (4.1) and from the Stokes equations (3.1), we deduce

$$(4.2)\qquad \mathbf{q}(x,y) = 0, \quad \text{for} \quad x \in \mathcal{P}_0 \ \text{or} \ \mathcal{P}_1, \qquad \forall\, y \in \mathbf{R}^3, \quad |y_3| < d.$$

Using Green's function $\mathbf{G}$ and the pressure vector $\mathbf{q}$, we determine the stress tensor $\mathbf{T}(T_{ijk})$, given by

$$(4.3)\qquad T_{ijk}(x,y) = -q_j(x,y)\delta_{ik} + \frac{\partial G_{ij}}{\partial y_k}(x,y) + \frac{\partial G_{kj}}{\partial y_i}(x,y).$$

By using the properties (4.1) and (4.2), we obtain

$$(4.4)\qquad T_{ijk}(x,y) = 0, \quad \text{for} \quad x \in \mathcal{P}_0 \ \text{or} \ \mathcal{P}_1, \quad i,j,k = \overline{1,3}.$$

Hence, the Green function $\mathbf{G}$, the associated pressure vector $\mathbf{q}$ and the stress tensor $\mathbf{T}$, vanish when the pole of the Green function is located on the planes $\mathcal{P}_0$ and $\mathcal{P}_1$. Additionally, we have the following asymptotic properties

$$(4.5)\qquad T_{ijk}(x,y) \to 0, \qquad \text{as} \quad |x| \to \infty.$$

The pressure vector $\mathbf{q} = \mathbf{q}(y,x)$ satisfies the following continuity equation (see [10]):

$$\sum_{i=1}^{3} \frac{\partial q_i}{\partial y_i}(y,x) = 0, \qquad \text{for} \quad x \neq y, \quad |y_3| < d, \tag{4.6}$$

and, hence, it can be considered as an acceptable solution for the equations of the Stokes flow, due to a point source placed at x.

Let $P = P(x,y)$ be the pressure associated with the velocity $\mathbf{q}(y,x)$. Then we have the following equations:

$$\Delta_y q_i(y,x) - \frac{\partial P}{\partial y_i}(x,y) = 0, \qquad \text{for} \quad x \neq y, \quad |y_3| < d, \quad i = \overline{1,3}, \tag{4.7}$$

with the boundary and asymptotic conditions given below

$$q_i(y,x) = 0, \qquad \text{for} \quad y \in \mathcal{P}_0 \ \text{ or } \ \mathcal{P}_1\,, \tag{4.8}$$

$$q_i(y,x) \to 0, \qquad P(x,y) \to 0, \quad \text{as} \quad |y| \to \infty. \tag{4.9}$$

The pressure tensor $\mathbf{P}(P_{ij})$, associated to the stress tensor $\mathbf{T}$, is given by the following equalities:

$$P_{ij}(y,x) = -P(y,x)\delta_{ij} + \frac{\partial q_i}{\partial y_j}(y,x) + \frac{\partial q_j}{\partial y_i}(y,x), \qquad i,j = \overline{1,3}. \tag{4.10}$$

Now, we can consider the Stokes flow $(\mathbf{u}, p)$ written in the form

$$u_i(x) = \int\limits_S T_{jik}(x,y) n_k(y) \varphi_j(y) d\sigma_y + \int\limits_S G_{ij}(y,x) \varphi_j(y) d\sigma_y\,, \tag{4.11}$$

$$p(x) = \int\limits_S P_{jk}(y,x) n_k(y) \varphi_j(y) d\sigma_y + \int\limits_S q_j(y,x) \varphi_j(y) d\sigma_y\,, \tag{4.12}$$

where $\mathbf{n}(n_1,n_2,n_3)$ denotes the unit outward normal vector to Ω^1, and $d\sigma_y$ denotes differentiation of the surface element of S, with respect to the point y.

From (3.1) - (3.4), (4.1) - (4.10) we deduce that $\mathbf{u}$ and p, given by (4.11), (4.12) satisfy the Stokes equations (2.5), (2.6) in Ω and Ω^1, respectively. Also, the null conditions (4.9) and the asymptotic conditions (2.9) are fulfilled.

In order to determine the unknown density $\boldsymbol{\varphi} : S \to \mathbf{R}^3$, $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$, we use a set of properties specified below. For this end, we consider the following double-layer and single-layer potentials:

$$V_i^1 \boldsymbol{\varphi}(x) = \int\limits_S T_{jik}(x,y) n_k(y) \varphi_j(y)\, d\sigma_y\,, \quad i = \overline{1,3}, \quad x \in \Omega \cup \Omega^1, \tag{4.13}$$

$$V_i^2 \boldsymbol{\varphi}(x) = \int\limits_S G_{ij}(y,x) \varphi_j(y)\, d\sigma_y\,, \quad i = \overline{1,3}, \quad x \in \Omega \cup \Omega^1, \tag{4.14}$$

where the function $\boldsymbol{\varphi}$ belongs to the class of functions continuous on S.

PROPERTY 1. The single-layer potential $V_i^2\boldsymbol{\varphi}$, $i = \overline{1,3}$, is a function continuous across the surface S, i.e.

$$\lim_{\substack{x' \to x \in S \\ x' \in \Omega}} V_i^2 \boldsymbol{\varphi}(x') = \lim_{\substack{x' \to x \in S \\ x' \in \Omega^1}} V_i^2 \boldsymbol{\varphi}(x'), \quad \forall\, x \in S. \tag{4.15}$$

PROPERTY 2. (see [10, 13]). The double-layer potential $V_i^1\boldsymbol{\varphi}$, $i = \overline{1,3}$, has two different values on the two sides of S, given by

$$\lim_{x' \to x \in S} V_i^1 \boldsymbol{\varphi}(x') = \pm \frac{1}{2}\varphi_i(x) + \stackrel{PV}{\int_S} T_{jik}(x,y) n_k(y) \varphi_j(y)\, d\sigma_y\,, \quad \forall\, x \in S, \tag{4.16}$$

where the plus sign is applied for the external side of S (in the direction of the unit normal vector $\mathbf{n}$) and the minus sign is applied for the internal side of S. Symbol PV means the principal value of the integral.

We remark that the kernels $T_{jik}(x,y)n_k(y)$ and $G_{ij}(x,y)$ of the double-layer operators and single-layer operators, respectively, become singular when the point y of S approaches the point x of S. If S is a Lyapunov surface, then the kernels become weakly singular.

In fact, the kernel $K_{ij}(x,y) = T_{jik}(x,y)n_k(y)$, can be written as

$$K_{ij}(x,y) = -\frac{3}{4\pi r^2} \cdot \frac{\partial r}{\partial(x_i - y_i)} \cdot \frac{\partial r}{\partial(x_j - y_j)} \cdot \frac{\partial r}{\partial n_y} + K_{ij}^*(x,y), \tag{4.17}$$

where K_{ij}^* is a continuous function. Also,

$$\left| \frac{\partial r}{\partial n_y} \right| < \lambda r^\alpha,$$

where $\lambda > 0$, $0 < \alpha \leq 1$ is the Lyapunov constant (see [12]), and $r = |x - y|$. Hence, the first term of the right-hand side of the above equality, behaves as $r^{\alpha-2}$ for $y \to x$, and the kernel K_{ij} is weakly singular.

In an analogous way we can show that the kernel of the single layer operator is weakly singular.

In this case the operators (4.13) and (4.14) are linear and compact on the space of continuous functions on S (see [11, 12]).

The stress tensor $\mathbf{S}^0$ of a flow $(\mathbf{v}^0, p^0)$ has the following components

$$T_{ij}(x) = -p^0(x)\delta_{ij} + \frac{\partial v_i^0}{\partial x_j}(x) + \frac{\partial v_j^0}{\partial x_i}(x), \quad i = \overline{1,3}. \tag{4.18}$$

and the stress field $\mathbf{T}$ of the flow is given by $\mathbf{T} = \mathbf{S}^0\mathbf{n}$.

PROPERTY 3. (see [5]). The stress field $\mathbf{T}^1(T_1^1, T_2^1, T_3^1)$, corresponding to the single-layer potentials (4.14) and the associated pressure, on the two sides of S, has two different values. On the external side of S, we have

$$\lim_{\substack{x' \to x \in S \\ x' \in \Omega}} T_i^1(x') = -\frac{1}{2}\varphi_i(x) + n_k(x) \overset{PV}{\int_S} T_{ijk}(y,x)\varphi_j(y)\, ds_y\,, \quad i = \overline{1,3}. \tag{4.19}$$

On the internal side of S, we have

$$\lim_{\substack{x' \to x \in S \\ x' \in \Omega^1}} T_i^1(x') = \frac{1}{2}\varphi_i(x) + n_k(x) \overset{PV}{\int_S} T_{ijk}(y,x)\varphi_j(y)\, ds\,, \quad i = \overline{1,3}. \tag{4.20}$$

PROPERTY 4. The stress field $\mathbf{T}^2(T_1^2, T_2^2, T_3^2)$, corresponding to the double-layer potentials (4.13) and the associated pressure, and having well-defined limiting value at points of S, has the same value on both sides of S. Thus, we have

$$\lim_{\substack{x' \to x \in S \\ x' \in \Omega}} T_i^2(x') = \lim_{\substack{x' \to x \in S \\ x' \in \Omega^1}} T_i^2(x'), \qquad \forall\, x \in S, \quad i = \overline{1,3}. \tag{4.21}$$

P r o o f. Let $\mathbf{w}$ be the Stokes velocity defined by the single-layer potentials (4.14). Hence, we have

$$\mathbf{w}(x) = \int_S \mathbf{G}(y,x)\boldsymbol{\varphi}(y)\, ds_y, \qquad p^0(x) = \int_S \mathbf{q}(y,x)\cdot\boldsymbol{\varphi}(y)\, ds_y, \quad x \in \Omega \cup \Omega^1, \tag{4.22}$$

where p^0 denotes the corresponding pressure of $\mathbf{w}$.

Let $\mathbf{w}^1$, $\mathbf{w}^2$ be smooth and solenoidal functions in Ω (i.e. $\nabla \cdot \mathbf{w}^i(x) = 0$, $x \in \Omega$, $i \in \{1,2\}$), and p^1, p^2, be two smooth, scalar functions in Ω. By applying the Green formula, we obtain

$$\begin{aligned} \int_\Omega \left\{ w_i^1(y)\left[\Delta w_i^2(y) - \frac{\partial p^2}{\partial y_i}(y)\right] - w_i^2(y)\left[\Delta w_i^1(y) - \frac{\partial p^1}{\partial y_i}(y)\right]\right\} dy \\ = -\int_{\partial\Omega} \left\{ w_i^1(y)\left(-p^2(y)\delta_{ij} + \frac{\partial w_i^2}{\partial y_j}(y) + \frac{\partial w_j^2}{\partial y_i}(y)\right) n_j(y) \right. \\ \left. - w_i^2(y)\left(-p^1(y)\delta_{ij} + \frac{\partial w_i^1}{\partial y_j}(y) + \frac{\partial w_j^2}{\partial y_i}(y)\right) n_j(y)\right\} d\sigma_y\,, \end{aligned} \tag{4.23}$$

where $\partial\Omega$ denotes the boundary of the domain Ω and the unit normal vector $\mathbf{n}$ is directed inwardly to the domain Ω.

If we consider in the above equality $\mathbf{w}^1 = \mathbf{w}$, $p^1 = p^0$, and $\mathbf{w}^2(y) = \mathbf{G}(x,y)\boldsymbol{\alpha}$, $p^2(y) = \mathbf{q}(x,y)\boldsymbol{\alpha}$, $\boldsymbol{\alpha}$ being an arbitrary constant vector, then we obtain the following identity

$$w_i(x) = -\int\limits_S G_{ji}(x,y) T^{2+}_{jk}(\mathbf{w})(y) n_k(y)\, d\sigma_y + \int\limits_S T_{jik}(x,y) n_k(y) w^+_j(y)\, d\sigma_y\,, \qquad \forall\, x \in \Omega, \quad i = \overline{1,3}, \tag{4.24}$$

where $T^{2+}_{jk}(\mathbf{w})(y) n_k(y) = T^{2+}_j(\mathbf{w})(y)$, $j = \overline{1,3}$, represent the component of the stress field $\mathbf{T}^2$, associated with the velocity $\mathbf{w}$. The superscript $+$ denotes the limit from Ω, at the point y of S.

In an analogous manner, we obtain the following identity

$$w_i(x) = \int\limits_S G_{ji}(x,y) T^{2-}_j(\mathbf{w})(y) d\sigma_y - \int\limits_S T_{jik}(x,y) n_k(y) w^-_j(y) d\sigma_y\,, \qquad \forall\, x \in \Omega^1, \quad i = \overline{1,3}, \tag{4.25}$$

where the superscript $-$ denotes the limit at the point y, approached from Ω^1. By using the jump formulas (4.16), we deduce

$$w^+_i(x) = -2\int\limits_S G_{ji}(x,y) f^+_j(y)\, d\sigma_y + 2\overset{PV}{\int\limits_S} T_{jik}(x,y) n_k(y) w^+_j(y)\, d\sigma_y\,, \tag{4.26}$$

$$w^-_i(x) = 2\int\limits_S G_{ji}(x,y) f^-_j(y)\, d\sigma_y - 2\overset{PV}{\int\limits_S} T_{jik}(x,y) n_k(y) w^-_j(y)\, d\sigma_y\,, \tag{4.27}$$

$\forall\, x \in S$, with $f^{\pm}_j(y) = T^{2\pm}_{jk}(\mathbf{w})(y) n_k(y)$.

From (4.26) and (4.27), we obtain

$$w^+_i(x) + w^-_i(x) = -2\int\limits_S G_{ji}(x,y)(f^+_j(y) - f^-_j(y))\, d\sigma_y + 2\overset{PV}{\int\limits_S} T_{jik}(x,y) n_k(y)(w^+_j(y) - w^-_j(y))\, d\sigma_y\,, \qquad \forall\, x \in S, \quad i = \overline{1,3}. \tag{4.28}$$

The jump properties (4.16) imply

$$w^+_i(x) + w^-_i(x) = 2\overset{PV}{\int\limits_S} T_{jik}(x,y) n_k(y) \varphi_j(y)\, d\sigma_y, \qquad \forall\, x \in S, \quad i = \overline{1,3}, \tag{4.29}$$

$$w^+_j(x) - w^-_j(x) = \varphi_j(y), \qquad \forall\, x \in S, \quad j = \overline{1,3}. \tag{4.30}$$

Therefore, the above properties (4.28) – (4.30) give

$$\int_S G_{ji}(x,y)(f_j^+(y) - f_j^-(y))\, d\sigma_y = 0, \qquad \forall\, x \in S, \quad i = \overline{1,3}. \tag{4.31}$$

In order to solve the above integral system, with the functions $g_j(x) = f_j^+(x) - f_j^-(x)$, $x \in S$, $j = \overline{1,3}$, as unknowns, we consider the velocity field $\tilde{\mathbf{v}}$ and pressure $\tilde{p}$, given by

$$\tilde{v}_j(x) = \int_S G_{ji}(x,y) g_j(y)\, d\sigma_y\,, \qquad x \in \Omega \cup \Omega^1, \quad j = \overline{1,3}, \tag{4.32}$$

$$\tilde{p}(x) = \int_S q_j(x,y) g_j(y)\, d\sigma_y\,, \qquad x \in \Omega \cup \Omega^1. \tag{4.33}$$

From (3.1) – (3.4) and (4.31), we deduce that $(\tilde{\mathbf{v}}, \tilde{p})$ represent a Stokes flow in Ω and Ω^1, respectively, with zero velocity on the planes $\mathcal{P}_0$ and $\mathcal{P}_1$, on the surface S, and at infinity. Using the uniqueness result of solution for the Stokes problem (2.5) – (2.9) in Ω (see the Remark 1), we conclude that $\tilde{\mathbf{v}}(x) = 0$, $\forall\, x \in \overline{\Omega}$. Hence, $\tilde{p}$ is a constant in Ω. But, $\tilde{p}(x) \to 0$, as $|x| \to \infty$. Hence, $\tilde{p}(x) = 0$, $\forall\, x \in \Omega$. Analogously, we obtain $\tilde{\mathbf{v}}(x) = 0$, $\forall\, x \in \overline{\Omega}^1$ and $\tilde{p}(x) = c \in \mathbf{R}$, $\forall\, x \in \Omega^1$. Let $\tilde{\mathbf{T}}$ be the stress field of the flow $(\tilde{\mathbf{v}}, \tilde{p})$. From the above reasons it follows that

$$\lim_{\substack{x' \to x \in S \\ x' \in \Omega}} \tilde{T}_i(x') - \lim_{\substack{x' \to x \in S \\ x' \in \Omega^1}} \tilde{T}_i(x') = -c n_i(x), \qquad \forall\, x \in S, \quad i = \overline{1,3}. \tag{4.34}$$

On the other hand, the jump properties (4.19), (4.20) give

$$\lim_{\substack{x' \to x \in S \\ x' \in \Omega}} \tilde{T}_i(x') - \lim_{\substack{x' \to x \in S \\ x' \in \Omega^1}} \tilde{T}_i(x') = -g_i(x), \qquad \forall\, x \in S, \quad i = \overline{1,3}. \tag{4.35}$$

Thus, we have

$$g_i(x) = c n_i(x), \qquad \forall\, n \in S, \quad i = \overline{1,3}. \tag{4.36}$$

These equalities show that the function $\mathbf{g} = \mathbf{f}^+ - \mathbf{f}^-$ not depends by the distribution $\boldsymbol{\varphi}$. If $\boldsymbol{\varphi}$ is the null function, then we must have $\mathbf{g} \equiv \mathbf{0}$. We conclude that

$$\mathbf{f}^+(x) = \lim_{\substack{x' \to x \in S \\ x' \in \Omega}} \mathbf{T}^2(\mathbf{w})(x') = \lim_{\substack{x' \to x \in S \\ x' \in \Omega^1}} \mathbf{T}^2(\mathbf{w})(x') = \mathbf{f}^-(x), \qquad \forall\, x \in S. \tag{4.37}$$

By using the Properties 1 – 4, we prove the existence and the uniqueness of solution of the Stokes problem (2.5) – (2.9).

From the boundary condition (2.7), the jump property (4.16) and the continuity property of the single-layer potentials (4.15) across S, we obtain the following Fredholm integral system of the second kind:

$$\frac{1}{2}\varphi_i(x) + \int\limits_S T_{jik}(x,y)n_k(y)\varphi_j(y)\,d\sigma_y + \int\limits_S G_{ij}(y,x)\varphi_j(y)\,d\sigma_y = -U_{\infty i}(x) + U_i(x), \qquad \forall\, x \in S, \quad i = \overline{1,3}. \tag{4.38}$$

The above double-layer integrals are understood as principal values in Cauchy's sense. For convenience, we have ommitted the symbol PV.

From Fredholm's result (see [12]), we deduce that the nonhomogeneous system (4.38) has a unique continuous solution if and only if the corresponding homogeneous system has only the null solution, in the space of continuous functions on S.

Let the following homogeneous integral system be satisfied:

$$\frac{1}{2}\varphi_i^0(x) + \int\limits_S T_{jik}(x,y)n_k(y)\varphi_j^0(y)\,d\sigma_y + \int\limits_S G_{ij}(y,x)\varphi_j^0(y)\,d\sigma_y = 0, \qquad \forall\, x \in S, \quad i = \overline{1,3}. \tag{4.39}$$

With the density $\boldsymbol{\varphi}^0$, supposed to be a continuous function on S, we define the following flow $(\mathbf{v}^0, p_0)$

$$v_i^0(x) = \int\limits_S T_{jik}(x,y)n_k(y)\varphi_j^0(y)d\sigma_y + \int\limits_S G_{ij}(y,x)\varphi_j^0(y)d\sigma_y, \quad x \in \Omega \cup \Omega^1, \tag{4.40}$$

$$p_0(x) = \int\limits_S P_{jk}(y,x)n_k(y)\varphi_j^0(y)d\sigma_y + \int\limits_S q_j(y,x)\varphi_j^0(y)d\sigma_y, \quad x \in \Omega \cup \Omega^1. \tag{4.41}$$

By using the Stokes equations and conditions (3.1) – (3.4) and the properties (4.2) – (4.5), (4.6) – (4.10), (4.39), we deduce that $(\mathbf{v}^0, p_0)$ represent a Stokes flow in Ω and Ω^1, respectively, with zero velocity on the planes $\mathcal{P}_0$ and $\mathcal{P}_1$, on the surface S and at infinity. By applying the uniqueness result of the Stokes flow in Ω, we conclude that

$$\mathbf{v}^0(x) = 0, \qquad p_0(x) = 0, \quad \forall\, x \in \Omega. \tag{4.42}$$

As a consequence of the above equalities (4.42), we deduce

$$\mathbf{f}^{0+}(x) = \lim_{\substack{x' \to x \in S \\ x' \in \Omega}} \mathbf{f}^0(x') = 0, \qquad \forall\, x \in S, \tag{4.43}$$

where $\mathbf{f}^0$ is the stress field of the flow $(\mathbf{v}^0, p_0)$, defined as in (4.18).

Now, by applying the Properties 1 and 2, we obtain

$$v_i^{0+}(x) - v_i^{0-}(x) = \varphi_i^0(x), \qquad \forall\, x \in S, \quad i = \overline{1,3}, \tag{4.44}$$

where $v_i^{0+}(x) = \lim\limits_{\substack{x' \to x \in S \\ x' \in \Omega}} v_i^0(x')$ and $v_i^{0-}(x) = \lim\limits_{\substack{x' \to x \in S \\ x' \in \Omega^1}} v_i^0(x)$, respectively. But (4.42) shows that the first term of the left-hand side of (4.44) is zero. Hence, we have the following equalities

$$v_i^{0-}(x) = -\varphi_i^0(x), \qquad \forall\, x \in S, \quad i = \overline{1,3}. \tag{4.45}$$

If we use the Properties 3 and 4, then we obtain

$$f_i^{0+}(x) - f_i^{0-}(x) = -\varphi_i^0(x), \qquad \forall\, x \in S, \quad i = \overline{1,3}. \tag{4.46}$$

By combining the result (4.43) to the above relations (4.46), we obtain

$$f_i^{0-}(x) = \varphi_i^0(x), \qquad \forall\, x \in S, \quad i = \overline{1,3}. \tag{4.47}$$

Using simple computations and the Green's formula, we obtain the following identity:

$$\int\limits_S v_i^{0-}(x) f_i^{0-}(x)\, d\sigma_x = 2 \int\limits_{\Omega^1} e_{ik}^0(x) e_{ik}^0(x)\, dx, \tag{4.48}$$

where

$$e_{ik}^0(x) = \frac{1}{2}\left(\frac{\partial v_i^0}{\partial x_j}(x) + \frac{\partial v_j^0}{\partial x_i}(x)\right), \qquad i,k = \overline{1,3}, \tag{4.49}$$

define the rate of the deformation tensor.

From (4.45) and (4.47), we deduce

$$-\int\limits_C \varphi_i^0(x) \varphi_i^0(x)\, d\sigma_x = 2 \int\limits_{\Omega^1} e_{ik}^0(x) e_{ik}^0(x)\, dx. \tag{4.50}$$

Because the left-hand side of (4.50) is non-positive and the right-hand side is non-negative, we conclude that both the sides of (4.50) are zero. By using the continuity of $\boldsymbol{\varphi}^0$ on S, it follows that $\boldsymbol{\varphi}^0(x) = 0$, $\forall\, x \in S$. This argument and the Fredholm's result (see [12]) imply that the nonhomogeneous integral system (4.38) has a unique, continuous solution $\boldsymbol{\varphi}$. With this function and by the formulas (4.11), (4.12), we determine the unique solution $(\mathbf{u}, p)$ of the Stokes problem (2.5) – (2.9). Hence, the existence and uniqueness of the solution of the Stokes problem (2.5) – (2.9) is completely proved.

REMARK 1. By using Green's formula in the domain Ω, the Stokes equations (2.5), (2.6), the boundary and asymptotic conditions (2.8), (2.9), we obtain the following identity:

$$-\frac{1}{2}\int_{\Omega}\left(\frac{\partial u_i}{\partial x_j}(x)+\frac{\partial u_j}{\partial x_i}(x)\right)^2 dx=\int_{S} T_i(x)u_i(x)\,d\sigma_x\,, \tag{4.51}$$

where $\mathbf{T}(T_1,T_2,T_3)$ is the surface force on S, of the flow $(\mathbf{u},p)$.

If we suppose that the Stokes problem (2.5) – (2.9) has two solutions $(\mathbf{u}^1,p^1)$ and $(\mathbf{u}^2,p^2)$, then the function $\mathbf{u}^0=\mathbf{u}^1-\mathbf{u}^2$ satisfies null boundary conditions on $\mathcal{P}_0,\mathcal{P}_1,S$, and at infinity. From (4.51) we deduce

$$\frac{\partial u_i^0}{\partial x_j}(x)+\frac{\partial u_j^0}{\partial x_i}(x)=0, \qquad \forall\, x\in\Omega, \quad i,j=\overline{1,3}. \tag{4.52}$$

The above system has the linear independent solutions, given below

$$\mathbf{U}^i(x)=(\delta_{1i},\delta_{2i},\delta_{3i}), \qquad i=\overline{1,3}, \tag{4.53}$$

$$\mathbf{U}^4(x)=(0,x_3,-x_2), \qquad \mathbf{U}^5(x)=(-x_3,0,x_1), \qquad \mathbf{U}^6(x)=(x_2,-x_1,0). \tag{4.54}$$

Another solution of (4.52) has the following form (see [1])

$$\mathbf{u}^0(x)=\mathbf{A}_0+\omega_0\times(\mathbf{x}-\mathbf{x}_0), \qquad \forall\, x\in\Omega, \tag{4.55}$$

where $\mathbf{A}_0$ and ω_0 are constant vectors, $\mathbf{x}$ is the position vector of the point $x=(x_1,x_2,x_3)$ and $\mathbf{x}_0$ is the position vector of a point x_0 of Ω^1.

By using the null condition on the walls $\mathcal{P}_0$, $\mathcal{P}_1$, on the surface S, and at infinity, satisfied by $\mathbf{u}^0$, we conclude that $\mathbf{A}_0=\omega_0=0$. Hence $\mathbf{u}^0=0$. This proves the uniqueness of solution for the problem (2.5) – (2.9).

5. Numerical results

From the Green formula, applied in the domain Ω^1, and the Stokes equations (3.1), (3.2), we obtain the following property

$$\int_S T_{ijk}(x,y)n_k(y)\,d\sigma_y=\begin{cases} -\delta_{ij}, & \text{for} \quad x\in\Omega^1,\\ 0, & \text{for} \quad x\in\Omega,\\ -\frac{1}{2}\delta_{ij}, & \text{for} \quad x\in S. \end{cases} \tag{5.1}$$

In the last case the integral is evaluated in the sense of the principal value.

With the above property, the system (4.38), can be written in the following form

$$(5.2)\qquad \int_S T_{jik}(x,y)n_k(y)(\varphi_j(y)-\varphi_j(x))\,ds_y + \int_S G_{ij}(y,x)\varphi_j(y)\,d\sigma_y = -U_{\infty i}(x)+U_i(x), \qquad \forall\ x\in S, \quad i=\overline{1,3}.$$

Now, if we use the expression (4.17) of the kernel of the double-layer potential, then we can easily deduce that the double-layer potentials of (5.2) are proper integrals. Hence, the singularities of these integrals can be removed by considering their integrands to be equal to zero for $y=x$ (see [13]).

In order to reduce the system (5.2) to a linear system of algebraic equations, we use a boundary element method. Thus, we divide the surface S into N elements Δ_j, $j=\overline{1,N}$, and we suppose that the function $\boldsymbol{\varphi}$ is constant on each Δ_j and equal to its value at the center of this element. With these assumptions, the system (5.2) can be written approximatelly as folows,

$$(5.3)\qquad \sum_{j=1}^{N}(\varphi_l^j-\varphi_l^m)\int_{\Delta_j} T_{lik}(x^m,y)n_k(y)\,d\sigma_y + \sum_{j=1}^{N}\varphi_l^j\int_{\Delta_j} G_{il}(y,x^m)\,d\sigma_y = -U_{\infty i}(x^m)+U_i(x^m), \qquad m=\overline{1,N}, \quad i=\overline{1,3},$$

where x^m is the center of Δ_m, φ_l^m is the constant value of φ_l on Δ_m and the terms $(\varphi_l^j-\varphi_l^m)\int_{\Delta_j} T_{lik}(x^m,y)n_k(y)d\sigma_y$ are equal to zero, when $l=m$, due to the removal of singularities.

Also the integral $\int_{\Delta_j} G_{il}(y,x^m)d\sigma_y$ becomes singular when $j=m$. Then we consider the following equality:

$$\int_{\Delta_j} G_{il}(y,x^m)\,d\sigma_y = \int_{\Delta_j}[G_{il}(y,x^m)-E_{il}(x-y)]\,d\sigma_y - \int_{\Delta_j} E_{il}(x-y)\,d\sigma_y\,.$$

The first integral of the right-hand side of the above equality is proper, hence, it can be computed by a Gauss quadrature formula. By using two variables (w_1,w_2) over the element Δ_j, we have $d\sigma_y = h_{w_1}dw_1dw_2$, where $h_{w_1}=\left|\dfrac{\partial\mathbf{y}}{\partial w_1}\times\dfrac{\partial\mathbf{y}}{\partial w_2}\right|$.

The second integral of the right-hand side of the above equality can be computed exactly in the (w_1,w_2) plane (see also [6]).

The algebraic system (5.3) can be numerically solved by using some integration and matrix inversion techniques.

By using the properties of the functions $\mathbf{G}$, $\mathbf{q}$ and $\mathbf{T}$, we obtain the total force $\mathcal{F}'$ and torque $\mathbf{M}'$ on S, given by

$$(5.4)\qquad \mathcal{F}' = -\int_S \boldsymbol{\varphi}(y)\,d\sigma_y\,, \qquad \mathbf{M}' = -\int_S \mathbf{y}\times\boldsymbol{\varphi}(y)\,d\sigma_y\,.$$

After solving the system (5.3), the total force on the surface S has the following components

$$\mathcal{F}_i' = -\sum_{j=1}^{N} \varphi_i^j \int_{\Delta_j} d\sigma_y\,, \qquad i = \overline{1,3}. \tag{5.5}$$

The numerical integrations presented in this paper have been given by using the Gauss quadrature formulas, and the linear system (5.3) was solved by means of the Gaussian elimination.

The numerical results are presented in the case of the Poiseuille flow $\mathbf{U}_\infty(x) = (-U(d^2 - x_3^2), 0, 0)$, $U > 0$, past a fixed sphere Ω^1, with the radius $a < d$.

Here we use the following notation

$$s = \frac{d - |Z_0|}{2d} = \frac{1}{2} - \frac{|Z_0|}{2d}\,,$$

where (X_0, Y_0, Z_0) is the center of sphere. Hence, $|Z_0| \in (0, d]$ and $s \in (0, 1/2]$.

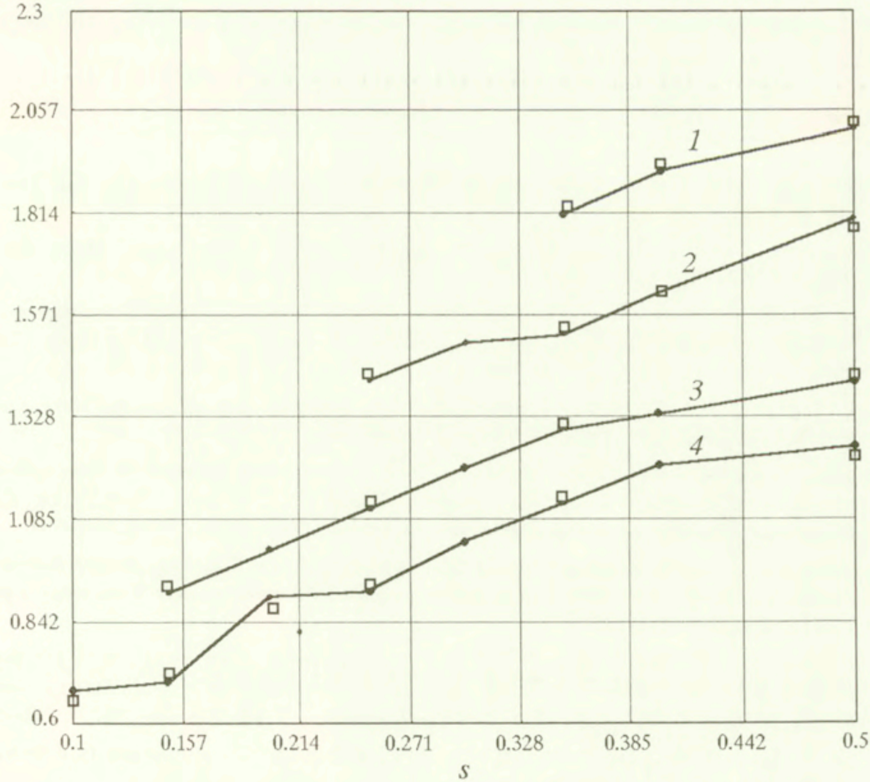

FIG. 2. —•— present method, □ Ganatos Pfeffer Weinbaum method, *1* — $d/a = 1.5$, *2* — $d/a = 2$, *3* — $d/a = 3$, *4* — $d/a = 5$.

Figure 2 gives the dependence between the modulus of the drag force $\mathcal{F} = \mathcal{F}'/(6\pi a U)$ and the parameter s, for various values of the ratio d/a. We deduce that the modulus of this force decreases when the ratio d/a increases. The maximum value is obtained when the center of the sphere is located on the plane Ox_1x_2.

Also, from Fig. 2, we conclude that the drag force $F = |\mathcal{F}|$ increases if the parameter s increases. Figure 2 shows that our results are in good agreement with similar results, obtained by P. GANATOS, R. PFEFFER and S. WEINBAUM in [7]. The partition of the sphere consists of 56 elements.

6. Conclusions

In this paper we have applied the direct boundary integral equations method to the Stokes flow past a smooth obstacle, between two plane parallel walls. Green's functions for the equations of the Stokes flow are obtained. These functions, together with the nonslip boundary condition on the surface obstacle, determine a Fredholm system of integral equations of the second kind, over the boundary of the obstacle. The integral formulation is simple and does not truncate the flow domain. This fact has the advantage of improving the accuracy of the numerical computations.

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G. Farkas, Technical University of Budapest, Hungary – *Supervision, Maintenance and Renovation of Reinforced and Prestressed Bridges*

J. Holnicki-Szulc, IFTR, Warsaw, Poland – *Adaptive Structures*

A. Jarosevic, Comenius University, Bratislava, Slovakia – *Magnetoelastic Method of Stress Measurement in Steel*

L. Jezequel, Ecol Central de Lyon, Ecully, France – *to be announced*

G.C. Lee, State University of New York at Buffalo, U.S.A. – *Development of a Bridge Monitoring System*

J. Rodellar, Universitat Politechnica de Catalunya, Barcelona, Spain – *Control Theory Sources in Active Control of Structures*

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• *Sessions* with oral presentations selected from the Call for Papers

• and *Panel Discussions* will be organized.

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