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to cutting-stock problems**

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An Inexact Bundle Approach to Cutting-Stock Problems

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We show that the LP relaxation of the cutting-stock problem can be solved efficiently by the recently proposed inexact bundle method. This method saves work by allowing inaccurate solutions to knapsack subproblems. With suitable rounding heuristics, our method solves almost all the cutting-stock instances from the literature.

Key words: nondifferentiable convex optimization, Lagrangian relaxation, integer programming, bundle methods, knapsack problems, cutting-stock

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1. Introduction

The classic Gilmore and Gomory (1961) formulation of the cutting-stock problem (CSP) is usually solved by LP-based column generation, rounding heuristics and branch-and-bound; see, e.g., (Belov and Scheithauer, 2002, 2006; Degraeve and Peeters, 2003; Degraeve and Schrage, 1999; Vance, 1998; Vanderbeck, 1999). Since column generation (CG) applied to its LP relaxation may converge slowly, there is interest in stabilized variants based on LP or QP (Ben Amor et al., 2004; Ben Amor and Valério de Carvalho, 2005; Briant et al., 2005). Alternatively, the highly efficient hybrid approach of Degraeve and Peeters (2003) generates additional columns by applying subgradient optimization to its Lagrangian relaxation.

In this paper we show that its LP relaxation can also be solved efficiently by the inexact bundle method of Kiwiel (2006). This QP-based method saves work by allowing inaccurate solutions to Lagrangian subproblems. For the CSP, each subproblem is a knapsack problem (KP). We give a simple test for *inexact KP solutions* (see §2.2 below) that works well in practice for a standard branch-and-bound KP solver of Martello and Toth (1990). Further, to avoid the difficulties arising when a bounded KP is transformed into a 0–1 KP (Vanderbeck, 2002), we use *relaxed bounds*. Next, by adapting the ideas of (Belov and Scheithauer, 2002; Holthaus, 2002; Stadler, 1990; Wäscher and Gau, 1996) to our inexact framework, we give rounding heuristics that solve *almost all* the CSP instances from the literature; in particular, they perform better than the best heuristics of Wäscher and Gau (1996). In effect, our

inexact KP solutions, bound relaxation and rounding heuristics should be of interest also for other, more traditional CG-based approaches to the CSP.

We now provide a historical perspective for our contributions. Our work was inspired by Briant et al. (2005), where (together with four other applications) the LP relaxation of the CSP was solved by several variants of CG and a standard bundle method. On some CSP instances, bundle was much slower than CG, mostly because its subproblems were more difficult for the KP solver of Vanderbeck (2002). Hence Claude Lemaréchal suggested the CSP as a testing example for our inexact bundle (Kiwiel, 2006). For technical reasons, instead of the KP solver of Vanderbeck (2002), we used the MT1R procedure of Martello and Toth (1990). Our initial quite disappointing results improved greatly once we used relaxed KP bounds and inexact solutions: our method became much faster in practice than *all* the algorithms tested in (Briant et al., 2005, §2.2) (see §5.6.2). Next, we collected more test instances and adapted some rounding heuristics from the literature. The main aim was to appraise our inexact bundle solutions: they are deemed accurate enough if the heuristics solve almost all instances.

We now summarize our findings on admissible inexactness. The relative accuracy in dual function evaluations is controlled by the tolerance ϵ_r of our KP solver (cf. §2.2). First, for $\epsilon_r = 0$ (i.e., exact bundle), the average computing times are much greater than those for $\epsilon_r = 10^{-5}$ (usually by factors of 30 or more), although the iteration numbers and the heuristic performance are almost the same. Second, the iteration numbers and timings are close for $\epsilon_r = 10^{-3}$, 10^{-4} and 10^{-5} ; however, relative to $\epsilon_r = 10^{-5}$, our heuristics perform much worse for $\epsilon_r = 10^{-3}$, and just marginally worse for $\epsilon_r = 10^{-4}$. Third, further experiments (not reported here for brevity) gave very close results for $\epsilon_r = 10^{-5}$, 10^{-6} , 10^{-7} and 10^{-8} . To sum up, $\epsilon_r = 10^{-5}$ seems to be a good borderline choice. On the other hand, since in the CSP the gap between the primal value and the relaxed dual value is usually less than 1, and either rounding heuristics or branch-and-bound should “close” this gap, it may seem more appropriate to ensure a given *absolute* accuracy $\epsilon_a < 1$ in dual function evaluations (see §5.7.3). Quite suprisingly, our results for a fairly large $\epsilon_a = 0.01$ are very close to those for $\epsilon_r = 10^{-5}$, whereas for $\epsilon_a = 0.05$ our heuristics perform slightly worse.

We thus present the first successful application of our inexact bundle method.

The paper is organized as follows. In §2 we recall the classic CSP model of Gilmore and Gomory (1961) and introduce inexact KP solutions for its Lagrangian relaxation. Our rounding heuristics are given in §3 in a general form suitable for other CSP solvers. The

inexact bundle method is reviewed in §4. Our computational results are presented in §5.

2. Lagrangian relaxation of the CSP

The one-dimensional cutting-stock problem (CSP) is to minimize the number of stock pieces of width W used to meet the demands d_i for items to be cut at their widths $w_i \in (0, W]$, for $i = 1, \dots, m$. The bin-packing problem (BPP) is a special case of the CSP with unit demands.

2.1. The Gilmore-Gomory model

This classic model is formulated as follows. Denote the set of *cutting patterns* by

$$P := \{p \in \mathbb{Z}_+^m : wp \leq W\}. \quad (1)$$

Let z_p be the number of times pattern p is used. The original model has the form

$$\min \sum_{p \in P} z_p \quad \text{s.t.} \quad \sum_{p \in P} pz_p \geq d, \quad z \in \mathbb{Z}_+^{|P|}. \quad (2a)$$

For Lagrangian relaxation we augment this model with the redundant constraint

$$\sum_{p \in P} z_p \leq N, \quad (2b)$$

where N is an upper bound on the optimal value of (2a) (e.g., $N = \sum_i d_i$); this ensures boundedness of the *ground set* $Z := \{z \in \mathbb{Z}_+^{|P|} : \sum_p z_p \leq N\}$. Relaxing the demand constraint $\sum_p pz_p \geq d$ with a price vector u yields the *Lagrangian* $L(z; u) := \sum_p z_p + u(d - \sum_p pz_p)$ and the *dual function*

$$\theta(u) := \min_{z \in Z} \left\{ L(z; u) = ud + \sum_{p \in P} (1 - up)z_p \right\}. \quad (3)$$

The *Lagrangian subproblem* above may be solved by finding a solution $p(u)$ of the KP

$$p(u) \in \text{Arg max}\{up : p \in P\} = \text{Arg max}\{up : wp \leq W, p \in \mathbb{Z}_+^m\} \quad (4)$$

and taking $z_{p(u)} = N$ and $z_p = 0$ for $p \neq p(u)$ if $up(u) > 1$, $z = 0$ otherwise, thus producing

$$\theta(u) = ud + N[1 - up(u)]_-, \quad (5)$$

where $[\cdot]_- := \min\{\cdot, 0\}$. Let v_* and v_{LP} denote the optimal values of (2) and its LP relaxation, respectively. It is well known that v_{LP} coincides with the dual optimal value

$$\theta_* := \max \left\{ \theta(u) : u \in \mathbb{R}_+^m \right\}. \quad (6)$$

Experiments show that $\tilde{u} := w/W$ is a good initial estimate of solutions to the Lagrangian dual (6) (Ben Amor and Valério de Carvalho, 2005, §4), (Briant et al., 2005, §2). In fact \tilde{u} minimizes the relaxed dual function

$$\theta_{LP}(u) := ud + N[1 - up(u)]_-, \quad (7)$$

where $p(u)$ solves the LP relaxation of (4). (Since $\theta_{LP}(\tilde{u}) = \tilde{u}d \leq v_* \leq N$, we see that $-d = -N(\tilde{u}d/N)(d/\tilde{u}d)$ is a subgradient of the second term of (7) at \tilde{u} : $0 \in \partial\theta_{LP}(\tilde{u})$.)

2.2. Inexact KP solutions

To strengthen our relaxation, we may consider only *proper* patterns p such that

$$p \leq b \quad \text{with} \quad b_i := \min \{ d_i, \lfloor W/w_i \rfloor \}, \quad i = 1:m. \quad (8)$$

Indeed, adding the bound $p \leq b$ to (1) and (4) does not change v_* , but it may raise v_{LP} (Nitsche et al., 1999). Then the CG subproblem (4) becomes a bounded KP, which can be turned into a 0–1 KP via the transformation of (Martello and Toth, 1990, §3.2). However, this transformation may duplicate solution representations, thus creating difficulties for 0–1 KP solvers (Vanderbeck, 2002). To avoid duplicates, we may use the *relaxed* bound

$$p \leq b' \quad \text{with} \quad b'_i := 2^{\lceil \log_2(b_i+1) \rceil} - 1, \quad i = 1:m, \quad (9)$$

which corresponds to replacing d_i in (8) by the smallest number of the form $2^j - 1$ with $j \geq 1$ such that $2^j - 1 \geq d_i$ ($2d_i - 1$ in the worst case); the number of transformed variables is the same. We solve the transformed KP by a double precision version of the branch-and-bound procedure MT1R of Martello and Toth (1990). To reduce its work, we allow MT1R to find an approximate solution for a given *relative accuracy tolerance* ϵ_r . Namely, the backtracking step exits if $\zeta \geq (1 - \epsilon_r)\bar{\zeta}$, where $\zeta := up$ for the incumbent p and $\bar{\zeta}$ is MT1R’s upper bound on the optimal value $up(u)$. Hence, by (5), we have the accuracy estimates

$$\underline{\theta}(u) := ud + N(1 - \bar{\zeta})_- \leq \theta(u) \leq \bar{\theta}(u) := ud + N(1 - \zeta)_-, \quad (10a)$$

$$\bar{\theta}(u) - \underline{\theta}(u) \leq N(\bar{\zeta} - \zeta) \leq N\epsilon_r\bar{\zeta}. \quad (10b)$$

For a normal exit with an optimal $p = p(u)$, we may replace $\bar{\zeta}$ by ζ and ϵ_r by 0 in (10).

As for our choice of MT1R, we add that Valério de Carvalho (2005) used MT1R as well, Belov and Scheithauer (2006) employed a similar branch-and-bound solver, whereas

Vanderbeck (1999) and Briant et al. (2005) used the more specialized branch-and-bound solver of Vanderbeck (2002). On the other hand, Degraeve and Peeters (2003) employed a similar branch-and-bound solver but with prices multiplied by 10,000 and rounded to integers, without discussing the effects of inexact KP solutions. Further, more recent KP solvers (Kellerer et al., 2004) accept integer data only; hence their use with suitable price roundings is left open for a future study. To sum up, MT1R is outdated, but we could not find anything better, and we believe that the current results will serve as a useful yardstick for future work with modern KP solvers.

3. Heuristic rounding of relaxed solutions

Typical rounding heuristics for the CSP proceed as follows; cf. (Belov and Scheithauer, 2002, 2006; Degraeve and Peeters, 2003; Holthaus, 2002; Scheithauer et al., 2001; Stadler, 1990; Wäscher and Gau, 1996). A solution \hat{z} of the LP relaxation is rounded down into an integer solution $\bar{z} := \lfloor \hat{z} \rfloor$. Next, a sequential heuristic applied to the *residual* problem (2) with d replaced by $d' := d - \sum_p p \bar{z}_p$ delivers a residual solution \tilde{z} . Then the sum $\bar{z} + \tilde{z}$ serves as a possibly inexact solution of (2) (which is exact if its value is equal to a lower bound on v_* ; e.g., $\lfloor v_{LP} \rfloor$). Since for simple rounding down ($\bar{z} = \lfloor \hat{z} \rfloor$), the residual problem may be too large to be solved optimally by a heuristic, some components of \bar{z} may be increased (Holthaus, 2002; Scheithauer et al., 2001); however, if the residual problem becomes too small to produce a solution to the original problem, some components of \bar{z} may be decreased (Belov and Scheithauer, 2002).

In §3.1 we give a general rounding procedure, which augments the ideas of Belov and Scheithauer (2002) and Holthaus (2002) with the oversupply reduction of Stadler (1990). As for sequential heuristics, in §3.2 we describe minor (but useful) modifications of the first-fit-decreasing (FFD) of Chvátal (1983) and the heuristics of Belov and Scheithauer (2004) and Holthaus (2002). Since it pays to call lighter heuristics first, useful combinations of rounding and sequential heuristics are detailed in §3.3.

We add that the rounding procedures of (Vanderbeck, 1999, §3.7) and (Wäscher and Gau, 1996, RSUC) would be difficult to implement in our context. As for sequential heuristics, we also tried the best-fit-decreasing of Chvátal (1983) and the fill bin heuristics of Vanderbeck (1999), but they did not perform significantly better than FFD in our trials.

3.1. A general rounding procedure

Numbering the patterns so that $P = \{p^j\}_{j=1}^n$, we may write (2a) as

$$\min \sum_{j=1}^n z_j \quad \text{s.t.} \quad \sum_{j=1}^n p^j z_j \geq d, \quad z \in \mathbb{Z}_+^n. \quad (11)$$

Given an incumbent solution z^* of (11) (e.g., found by FFD) and a point $\hat{z} \in \mathbb{R}_+^n$ (e.g., found by LP relaxation), the following procedure attempts to improve z^* by calling a heuristic on residual problems derived from rounded variants of \hat{z} . Let $e := (1, \dots, 1) \in \mathbb{R}^n$.

Procedure 1 (Rounding procedure).

Step 1 (*Rounding down*). Set $\bar{z} := \lfloor \hat{z} \rfloor$ and $d' := d - \sum_j p^j \bar{z}_j$. Sort the fractional parts $r_j := \hat{z}_j - \bar{z}_j$ so that $r_{j_1} \geq \dots \geq r_{j_n}$, and set $\bar{n} := |\{j : r_j > 0\}|$.

Step 2 (*Oversupply reduction*). While $d' \not\leq 0$, pick \bar{j} to maximize

$$\sum_{i: d'_i < 0} w_i \min\{p_i^{\bar{j}}, -d'_i\} \quad (12)$$

over j s.t. $\bar{z}_j > 0$, set $\bar{z}_{\bar{j}} := \bar{z}_{\bar{j}} - 1$ and $d' := d' + p^{\bar{j}}$.

Step 3 (*Partial rounding up*). Set $I := \emptyset$. For $i = 1: \bar{n}$, if $p^{j_i} \leq d'$, set $\bar{z}_{j_i} = \bar{z}_{j_i} + 1$, $d' := d' - p^{j_i}$, $I := I \cup \{j_i\}$.

Step 4 (*Heuristic improvement*). Using a heuristic, find a feasible point \tilde{z} for the residual problem (11) with d replaced by d' . If $e\tilde{z} + e\bar{z} < ez^*$, set $z^* := \bar{z} + \tilde{z}$.

Step 5 (*Residual problem extension*). If $I \neq \emptyset$, remove from I its last entry j , set $\bar{z}_j := \bar{z}_j - 1$, $d' := d' + p^j$ and return to Step 4.

If \hat{z} solves the LP relaxation of an equality-constrained CSP, our procedure reduces to the one in (Belov and Scheithauer, 2002, §2.5); otherwise Step 2 (due to (Stadtler, 1990, Fig. 3)) helps. Following (Belov and Scheithauer, 2002, §5.2), our implementation allows at most ten returns from Step 5.

One of our heuristics uses the following modification of Step 3, based on the ideas in (Holthaus, 2002, §3.2).

Step 3' (*Partial rounding up*). Set $I := \emptyset$, $K := \{j : p^j \leq d', r_j > 0\}$. While $K \neq \emptyset$, pick \bar{j} to maximize $\sum_i p_i^{\bar{j}}$ over $j \in K$, set $\bar{z}_{\bar{j}} = \bar{z}_{\bar{j}} + 1$, $d' := d' - p^{\bar{j}}$, $I := I \cup \{\bar{j}\}$, $K := \{j \in K : p^j \leq d', j \neq \bar{j}\}$.

3.2. Sequential heuristics

We now describe our heuristics for the residual problem (2a) with d replaced by $d' \geq 0$, assuming $w_1 \geq \dots \geq w_m$.

Our implementation of FFD works as follows. Set $\bar{z} := 0$, $d'' := d'$. While $d'' \neq 0$, generate the next pattern p by setting

$$p_i := \min \left\{ d''_i, \left\lceil \left(W - \sum_{j < i} w_j p_j \right) / w_i \right\rceil \right\} \quad \text{for } i = 1:m, \quad (13)$$

set $\kappa := \min\{[d''_i/p_i] : p_i > 0\}$, $\bar{z}_p := \bar{z}_p + \kappa$, $d'' := d'' - \kappa p$. The version of (Chvátal, 1983, p. 208) employs $\kappa \equiv 1$, and hence is less efficient for large demands.

Our modification of the *sequential heuristic procedure* (SHP) of (Holthaus, 2002, §3.2), given a price vector $\hat{u} \in \mathbb{R}^m$ (e.g., an approximate solution of (6)) and a *price tolerance* $u_{\text{tol}} > 0$ for rounding errors (we use $u_{\text{tol}} = 10^{-12}$), sets $\bar{u}_i := \max\{\hat{u}_i, u_{\text{tol}}\}$ for $i = 1:m$ and replaces the FFD formula (13) by the bounded KP

$$p \in \text{Arg max}\{ \bar{u}p : wp \leq W, p \leq d'', p \in \mathbb{Z}_+^m \}. \quad (14)$$

Our implementation of the *sequential value correction* (SVC) heuristic of (Belov and Scheithauer, 2004, §2) records the best solution found by calling SHP at most thirty times with \bar{u} modified as follows. Initially $\bar{u}_i := \max\{1, W\hat{u}_i\}$, $i = 1:m$. If $w d'' \not\leq W$, then after solving (14) and updating d'' , for i such that $p_i > 0$, set

$$\bar{u}_i := [\gamma_i \bar{u}_i + (W/wp)w_i^{1.04}] / (\gamma_i + 1) \quad \text{with} \quad \gamma_i := \Omega_i (d'_i + d''_i) / p_i, \quad (15)$$

for Ω_i picked randomly in $[1/\Omega'_i, \Omega'_i]$, where Ω'_i is chosen at random in $[1, 1.5]$. An early exit occurs if SHP finds \bar{z} such that $e\bar{z} + e\bar{z} = [\theta(\hat{u})]$, in which case $z^* := \bar{z} + \bar{z}$ is optimal.

3.3. Combinations of rounding and sequential heuristics

We now give more details on the five heuristics used in our experiments. The heuristics are described as if being called by a general solver for the LP relaxation of (11), which could be any variant of the CG procedure or the bundle method given in §4.

Our *initial* heuristic H0 calls FFD with $d' = d$ (i.e., on the original problem) to initialize the incumbent $z^* := \bar{z}$, the upper bound $N := ez^*$ and the lower bound $\underline{z}_1 := -\infty$.

Suppose at iteration $k \geq 1$ of the solver, the following quantities are available: z^* is an incumbent solution of (11), $\bar{z}^k \in \mathbb{R}_+^n$ and $\hat{u}^k \in \mathbb{R}_+^m$ are tentative primal and dual solutions

of the LP relaxation, and $\underline{\theta}_k$ is a lower bound on $\theta_* = v_{LP}$ (cf. (6)). If $ez^* = \lceil \underline{\theta}_k \rceil$, the solver may stop (since z^* is optimal). Otherwise, for iterations k specified below, the remaining heuristics consist in calling an extension of Procedure 1 with a copy of Step 4 inserted after Step 1; the sequential heuristics employed at these steps are listed below.

Our *periodic* heuristic H1 is called by the solver every twentieth iteration, starting from iteration $k = m + 1$ (i.e., for $k = m + 1, m + 21, \dots$), with the current relaxed solution $\hat{z} := \hat{z}^k$ and the lower bound $\underline{\theta}_k \leq \theta_*$. H1 employs FFD in Procedure 1, exiting if $ez^* = \lceil \underline{\theta}_k \rceil$.

Our *final* heuristics H2, H3 and H4 are called successively upon termination of the solver, using the final $\hat{z} := \hat{z}^k$, $\hat{u} := \hat{u}^k$ and $\underline{\theta}_k$. H2 employs both FFD and SHP, H3 just SHP and the modified Step 3', whereas H4 uses SVC. Of course, H3 and H4 (or just H4) are not called if H2 (or H3) exits with $ez^* = \lceil \underline{\theta}_k \rceil$, whereas SVC exits when $e\bar{z} + e\tilde{z} = \lceil \underline{\theta}_k \rceil$.

The impact of the various heuristics will be discussed in §5.8.

4. The inexact proximal bundle method

We now sketch the main features of the inexact bundle method of Kiwiel (2006).

Our method generates *trial points* $u^k \in \mathbb{R}_+^m$, $k = 1, 2, \dots$, at which the dual function θ is evaluated (possibly inexactly) as described in §2.2. Specifically, for each k , set p^k to the (possibly inaccurate) KP solution p satisfying the bounds of (10) for $u = u^k$, and let $\zeta_k := \zeta$, $\bar{\zeta}_k := \bar{\zeta}$. Recalling (3), define the associated *Lagrangian solution* z^k by setting

$$z_q^k := 0 \text{ for } q \neq p^k, \quad z_{p^k}^k := \begin{cases} N & \text{if } \zeta_k > 1, \\ 0 & \text{otherwise.} \end{cases} \quad (16)$$

Thus we have the lower bound $\underline{\theta}(u^k) \leq \theta(u^k)$ and $L(z^k; u^k) = \bar{\theta}(u)$ in (10); in particular,

$$L(z^k; u^k) - \theta(u^k) \leq N(\bar{\zeta}_k - \zeta_k) \leq N\epsilon_r \bar{\zeta}_k. \quad (17)$$

Further, by (3), the following *linearization* of θ at u^k majorizes $\theta(u)$ for all u :

$$\theta_k(u) := L(z^k; u) = ud + \begin{cases} N(1 - up^k) & \text{if } z^k \neq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (18)$$

Iteration k uses the polyhedral *cutting-plane model* of θ

$$\hat{\theta}_k(\cdot) := \min_{j \in J^k} \theta_j(\cdot) \quad \text{with } k \in J^k \subset \{1, \dots, k\} \quad (19)$$

for finding

$$u^{k+1} := \arg \max \left\{ \hat{\theta}_k(u) - \frac{1}{2t_k} |u - \hat{u}^k|^2 : u \in \mathbb{R}_+^m \right\}, \quad (20)$$

where $t_k > 0$ is a *stepsize* that controls the size of $|u^{k+1} - \hat{u}^k|$ and the *prox center* $\hat{u}^k := u^{k'}$ has the value $\theta_{\hat{u}}^k := \theta_{k'}(u^{k'})$ for some $k' \leq k$ (usually $\theta_{\hat{u}}^k = \max_{j=1}^k \theta_j(u^j)$). Due to evaluation errors, we may have $\theta_{\hat{u}}^k > \hat{\theta}_k(\hat{u}^k)$, in which case the *predicted increase*

$$v_k := \hat{\theta}_k(u^{k+1}) - \theta_{\hat{u}}^k \quad (21)$$

may be nonpositive; then t_k is increased and u^{k+1} is recomputed to increase $\hat{\theta}_k(u^{k+1})$ until $v_k \geq |u^{k+1} - \hat{u}^k|^2/2t_k$. An *ascent step* to $\hat{u}^{k+1} := u^{k+1}$ with $k' := k + 1$ occurs if

$$\theta_{k+1}(u^{k+1}) - \theta_{\hat{u}}^k \geq \kappa v_k \quad (22)$$

for a fixed $\kappa \in (0, 1)$ (we use $\kappa = 0.1$). Otherwise, a *null step* $\hat{u}^{k+1} := \hat{u}^k$ improves the next model $\hat{\theta}_{k+1}$ with the new linearization θ_{k+1} as stipulated in (19).

If we omitted the quadratic term in (20), the resulting cutting-plane method could generate u^{k+1} far from the previous points, and it would require storing all linearizations ($J^k = \{1, \dots, k\}$ in (19)). In contrast, the quadratic term usually keeps u^{k+1} close enough to the best point found so far, and it allows limiting the number of stored linearizations.

We solve subproblem (20) with the QP routine of Kiwiel (1994), which finds its multipliers $\{\nu_j^k\}_{j \in J^k} \subset \mathbb{R}_+$, also known as *convex weights*, such that $\sum_{j \in J^k} \nu_j^k = 1$ and the set $\hat{J}^k := \{j \in J^k : \nu_j^k \neq 0\}$ has at most $m + 1$ elements. We set $J^{k+1} := J^k \cup \{k + 1\}$ and then, if necessary, drop from J^{k+1} an index $j \in J^k \setminus \hat{J}^k$ with the largest $\theta_j(\hat{u}^k)$ to keep $|J^{k+1}| \leq M$ for a fixed $M \geq m + 2$.

Combining the accumulated Lagrangian solutions $\{z^j\}_{j \in J^k}$ with their weights $\{\nu_j^k\}_{j \in J^k}$, we may estimate solutions to the LP relaxation of (2) via the *aggregate primal solution*

$$\hat{z}^k := \sum_{j \in J^k} \nu_j^k z^j. \quad (23)$$

In other words (cf. (16)), $\hat{z}_{p^j}^k = N \nu_j^k$ for nontrivial patterns p^j indexed by $J_P^k := \{j \in \hat{J}^k : z^j \neq 0\}$ (which need not be stored, since they can be recovered from $\nabla \theta_j = d - N p^j$; see (18)). Our heuristics also use the *lower bound* $\underline{\theta}_k := \max_{j \leq k} \underline{\theta}(u^j)$ on $\theta_* = v_{LP}$ (cf. (6)).

We now point out some useful consequences of the convergence analysis in (Kiwiel, 2006, §5). The LP relaxation of (2) may be written as

$$v_{LP} := \min \bar{\psi}_0(z) := \sum_{p \in P} \hat{z}_p \quad \text{s.t.} \quad \bar{\psi}(z) := d - \sum_{p \in P} p z_p \leq 0, \quad z \in \text{conv } Z. \quad (24)$$

Let $\epsilon := \sup_k [\theta_k(u^k) - \theta(u^k)]$ be the maximum evaluation error; by (17), we have $\epsilon \leq \bar{\epsilon} := N\epsilon_r \sup_k \bar{\zeta}_k$. Consider the set of ϵ -optimal solutions of the LP relaxation (24):

$$Z_\epsilon := \{z \in \text{conv } Z : \bar{\psi}_0(z) \leq v_{LP} + \epsilon, \bar{\psi}(z) \leq 0\}. \quad (25)$$

The limits $\theta_u^\infty := \lim_k \theta_u^k$, $\underline{\theta}_\infty := \lim_k \underline{\theta}_k$ satisfy $\theta_u^\infty \in [v_{LP}, v_{LP} + \epsilon]$, $\underline{\theta}_\infty \in [\theta_u^\infty - \bar{\epsilon}, v_{LP}]$, and there exists $K \subset \{1, 2, \dots\}$ such that $\lim_{k \in K} \bar{\psi}_0(\bar{z}^k) = \theta_u^\infty$ and $\overline{\lim}_{k \in K} \max_{i=1}^n \bar{\psi}_i(\bar{z}^k) \leq 0$; in particular, the bounded sequence $\{\bar{z}^k\}_{k \in K}$ converges to the ϵ -optimal set Z_ϵ . If ϵ_r is small enough, the accuracy observed in practice corresponds to such estimates with ϵ and $\bar{\epsilon}$ determined by the maximum errors $\theta_k(u^k) - \theta(u^k)$ and $\theta(u^k) - \underline{\theta}(u^k)$ that occur for large k ; since both errors are at most $N(\bar{\zeta}_k - \zeta_k)$, where the KP gap $\bar{\zeta}_k - \zeta_k$ is usually tiny for large k , small values of ϵ and $\bar{\epsilon}$ can be attained if the algorithm runs long enough.

We stop if $\min\{v_k, |\pi^k| + \alpha_k\} \leq \epsilon_{\text{opt}}(1 + |\theta_u^k|)$, where v_k is given by (21), $\pi^k := (\bar{u}^k - u^{k+1})/t_k$, $\alpha_k := v_k - t_k|\pi^k|^2$ and $\epsilon_{\text{opt}} > 0$ is an *optimality tolerance* (cf. (Kiwiel, 2006, §4.2)). For $\epsilon_{\text{opt}} = \epsilon_r = 10^{-8}$, $\underline{\theta}_k$ usually agrees with θ_* in at least 8 digits, enough for our purposes.

5. Computational results

5.1. Data sets

In our computational experiments, for the CSP we use the 28 industrial instances of Vance (1998), the 10 industrial instances of Vanderbeck (1999), and the 20 industrial instances of Degraeve and Schrage (1999). In addition, we use the following randomly generated instances: the 4000 instances of Wäscher and Gau (1996), the 3360 instances of Degraeve and Peeters (2003) and the 120 instances of Vanderbeck (1999). For the BPP, we use the 540 randomly generated instances of Degraeve and Peeters (2003), and the 160 instances from the BINPACK collection of the OR-Library (Beasley, 1990).

The instances of Wäscher and Gau (1996) are constructed by the CUTGEN1 generator of Gau and Wäscher (1995), using the following parameter values: the number of orders $m = 10, 20, 30, 40, 50$, the width $W = 10,000$, the interval fraction $c = 0.25, 0.5, 0.75, 1$, and the average demand $\bar{d} = 10, 50$. The widths w_i are uniformly distributed integers between 1 and cW . For m uniform random numbers $R_1, \dots, R_m \in (0, 1)$, the demands $d_i := \lfloor \frac{R_i m \bar{d}}{R_1 + \dots + R_m} \rfloor$ for $i < m$, and $d_m := m\bar{d} - \sum_{i < m} d_i$ (in fact slightly more complicated formulas are used by Gau and Wäscher (1995)). Duplicate widths are aggregated by summing their demands.

Combining the different values for m , c and \bar{d} results in 40 classes; in each class, 100 instances are generated.

The *small-item-size* instances of Degraeve and Peeters (2003) are generated similarly for $m = 10, 20, 30, 40, 50, 75, 100$, $c = 0.25, 0.5, 0.75, 1$ and $\bar{d} = 10, 50, 100$, except that $R_1, \dots, R_m \in (0.1, 0.9)$ for the demand distribution. In the *medium-item-size* instances of Degraeve and Peeters (2003), only $\bar{d} = 50$ is used and the widths are uniformly distributed on $[w_{\min}, cW]$, where $w_{\min} = 500, 1000, 1500$. Both cases have 84 data classes, and 20 instances are generated in each class.

The instances of Vanderbeck (1999) comprise 6 classes with $m = 50$, and 20 instances per class. The first three classes are generated like those of Wäscher and Gau (1996) above with $c = 0.25, 0.5, 0.75$ and $\bar{d} = 50$, the next two classes have widths in $[500, 2500]$ and $[500, 5000]$ with $\bar{d} = 50$, and the sixth class has widths in $[500, 5000]$ and $\bar{d} = 100$.

In the BPP instances of Degraeve and Peeters (2003), $m = 500$ or 1000 weights are uniformly distributed in the intervals $[1, 100]$, $[20, 100]$, $[50, 100]$ as in BPPGEN (Schwerin and Wäscher, 1997), and the capacity $W = 100, 120, 150$; identical items are aggregated for the corresponding CSPs. In each of the 18 resulting classes, 20 instances are generated. The modified BPP instances of Degraeve and Peeters (2003) use $m = 500$, the weight intervals $[1, 10000]$, $[2000, 10000]$, $[5000, 10000]$, and the capacity $W = 10000, 12000, 15000$, again with 20 instances per class.

The BINPACK instances from the OR-Library (Beasley, 1990) comprise two categories. The *uniform* category has the capacity $W = 150$, m weights uniformly distributed in the interval $[20, 100]$, and 20 instances generated for each value of $m = 120, 250, 500, 1000$. (The classes with $m = 500, 1000$ also appear in the BPP category of Degraeve and Peeters (2003), but with different instances.) In the *triplet* category, each bin of capacity $W = 1000$ is filled with exactly three items (the first item w' is picked in $[380, 490]$, the second item w'' in $[250, (W - w')/2]$, and the third item equals $W - w' - w''$). There are 20 instances for each value of $m = 60, 120, 249, 501$.

5.2. Implemented variants

Our codes were programmed in Fortran 77 and run on a notebook PC (Pentium M 755 2 GHz. 1.5 GB RAM) under MS Windows XP.

For solving the dual problem (6), we used a general-purpose bundle code that treats subgradients as dense vectors in double precision. A faster code could exploit the fact that

Table 1: Small-item-size instances of Degraeve and Peeters (2003), $int = all$, $\bar{d} = all$

m	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	H2	H3	H4	n_g
10	9.99	26.77	15.14	31	0.00	0.01	113	49	70	1	0	0
20	19.95	53.13	32.51	69	0.01	0.04	120	64	64	0	0	0
30	29.91	79.76	51.90	91	0.02	0.22	130	85	57	0	0	1
40	39.85	105.55	70.41	134	0.04	0.36	134	98	53	0	0	0
50	49.75	132.16	90.20	181	0.08	0.66	134	102	55	0	0	0
75	74.36	197.32	141.82	256	0.24	2.00	149	122	43	0	0	0
100	98.92	263.36	183.88	311	0.40	2.81	165	136	34	0	1	0

each subgradient of θ has the form $\nabla\theta_k = d$ or $\nabla\theta_k = d - Np^k$ (see (18)), with a common integer part d and an integer sparse knapsack solution p^k . Ignoring sparsity, our code requires $m \times M$ memory locations for storing up to $M \geq m+3$ subgradients, and additional workspace of order M^2 for solving the QP subproblem (20) with the routine of Kiwiel (1994). We used $M = m + 3$ to test how “minimal” bundle performs.

The bounded KPs arising in column generation and SHP were solved by the modified version of MT1R (cf. §2.2) with the accuracy tolerance $\epsilon_r = 10^{-5}$ (other choices are discussed in §5.7.2); MT1R’s tolerance ϵ was set to 10^{-12} . For column generation, we used the relaxed bounds of (9), because the tighter bounds of (8) produced longer computing times. In contrast, SHP employed in (14) the natural bounds given by (8) with d replaced by d'' .

Our implementation of the rounding procedure of §3.1 is slower than necessary because the patterns are recovered as $p^j = (d - \nabla\theta_j)/N$, instead of being stored separately.

5.3. Results for the cutting-stock problem

To ease comparisons, we follow closely the presentation of Degraeve and Peeters (2003). Every data class is identified by three parameters: the number of items m , the interval in which the widths are distributed denoted by int , and the average demand \bar{d} . An indicator “all” for any of these parameters means that the reported results are aggregated over all relevant values for that particular parameter. If a parameter is constant for all instances represented in a table, its value is indicated in the table heading.

Our results for the small-item-size instances of Degraeve and Peeters (2003) with $int = all$, $\bar{d} = all$ are reported in Table 1; full details are given in Tables 17–19 in the Online Supplement to this paper on the journal’s website. The columns m_{av} and m'_{av} give the average numbers of items and variables in the associated 0–1 knapsack subproblems. The columns i_{av} and i_{mx}

Table 2: Medium-item-size instances of Degraeve and Peeters (2003), $int = all$, $\bar{d} = 50$

m	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	H2	H3	H4	n_g
10	9.98	23.09	17.52	29	0.00	0.02	54	48	112	0	1	0
20	19.95	45.58	35.05	58	0.01	0.10	68	50	114	0	1	0
30	29.84	68.47	53.71	93	0.02	0.16	73	73	105	0	0	0
40	39.78	90.65	69.94	120	0.03	0.58	70	63	110	0	0	0
50	49.64	113.69	88.76	156	0.06	0.90	74	65	118	1	1	1
75	74.08	169.10	137.04	232	0.37	8.60	82	73	105	0	0	1
100	98.45	226.07	184.45	295	1.43	62.18	73	72	117	0	4	0

report the average and maximum numbers of iterations of the bundle code. The columns t_{av} and t_{mx} give the average and maximum running times in wall-clock seconds. The column n_e lists the numbers of “early” terminations due to discovering that $ez^* = \lfloor \underline{\theta}_k \rfloor$ for the incumbent z^* delivered by H0 or H1 *before* bundle terminated on its own. Recall that H1 is called after H0, H2 after H1, etc., unless $ez^* = \lfloor \underline{\theta}_k \rfloor$ occurs earlier. The columns labelled H1 through H4 give the numbers of instances in which the corresponding heuristic found the best primal value ez^* first (for the remaining instances ez^* was found by H0); a zero entry means that heuristic was not called or did not contribute usefully. The final column n_g reports the numbers of instances with a nonzero final gap $g := ez^* - \lfloor \underline{\theta}_k \rfloor$; we stress that the final gaps never exceeded *one* unit in *all* of our instances. The averages, maxima and sums in Table 1 are taken over the 240 instances used for each value of m .

From the entries for n_e , H1 through H4 and n_g in Table 1, we see that early termination occurred on between 47% and 69% of problems, H0 and H1 solved between 70% and 85% of problems, H2 solved almost all the remaining problems, H3 and H4 helped in solving 2 problems, and just one out of the 1680 problems was not solved. Note that the best method LR of Degraeve and Peeters (2003) also could not solve one instance within 15 minutes (two instances within 6 minutes), and its FFD-based rounding heuristic solved 91.6% of problems, whereas our “lighter” heuristics H0 through H2 solved 99.8% of problems.

Our results for the medium-item-size instances of Degraeve and Peeters (2003) are presented in Table 2, where each row gives statistics over the 240 instances used for each value of m (see Tables 20 and 21 for more details). Early termination occurred on between 22% and 35% of problems, H0 and H1 solved between 49% and 56% of problems, H2 solved almost all the remaining problems, H3 solved one problem, H4 solved 7 problems, and just two out of the 1680 problems were not solved. The rounding heuristic of Degraeve and Peeters (2003)

Table 3: CSP instances of Wäscher and Gau (1996), $int = all$, $\bar{d} = all$

m	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	H2	H3	H4	n_g
10	9.99	25.37	14.27	35	0.00	0.02	449	134	192	0	0	0
20	19.96	50.46	30.73	61	0.01	8.35	485	240	183	0	2	0
30	29.90	75.72	48.18	105	0.01	0.13	503	281	161	0	1	0
40	39.84	100.10	65.06	123	0.04	3.31	502	313	160	0	2	2
50	49.73	125.22	84.75	171	0.07	0.46	526	341	138	0	4	1
<i>all</i>	29.88	75.37	48.60	171	0.03	8.35	2465	1309	834	0	9	3

Table 4: CSP instances of Vanderbeck (1999), $m = 50$

\bar{d}	int	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	H2	H3	H4	n_g
50	[1, 2500]	49.40	185.30	47.40	71	0.03	0.05	20	18	0	0	0	0
50	[1, 5000]	49.65	143.05	114.05	151	0.20	0.34	13	13	7	0	0	0
50	[1, 7500]	49.75	110.00	111.85	144	0.06	0.11	6	5	8	0	0	0
50	[500, 2500]	49.40	166.10	57.05	77	0.03	0.05	14	14	6	0	0	0
50	[500, 5000]	49.70	128.20	103.65	114	0.14	0.27	11	11	9	0	0	0
100	[500, 5000]	49.70	129.25	104.40	131	0.14	0.32	8	8	12	0	0	0

solved 69.9% of problems, whereas H0 through H2 solved 99.4% of problems.

Comparing Tables 1 and 2, we see that the average and maximum solution times are quite similar in the small- and medium-size-item cases for problem sizes m up to 50. However, for $m = 75$ and 100, in the medium-size-item case the average solution times grow significantly, and the maximum solution times jump up, most spectacularly on the instances with width interval [1500, 2500]; see Table 21. This is due to the poor performance of our knapsack solver on these instances. Similar slowdowns on this interval were reported in (Degraeve and Peeters, 2003, Tab. 4a) already for $m = 20$, i.e., even for smaller problems.

To save space, Table 3 presents only aggregate results on the instances of Wäscher and Gau (1996), with each row giving statistics over the 800 instances used for each value of m . Here our main point is that only three out of 4000 (0.075%) problems were not solved. Our “lighter” heuristics H0 through H2 solved 99.7% of problems, whereas the two best (and more complicated) heuristics RSUC and CSTAOPT of Wäscher and Gau (1996) solved 98.0% and 92.7% of problems, respectively (99.6% if they had been applied together). The fairly large maximum solution time in Tab. 3 stemmed from a single knapsack subproblem.

Table 4 gives our results for the 6 data classes of Vanderbeck (1999) with $m = 50$ and 20 instances per row. Since we used the original instances, the results are not identical to those

Table 5: BPP instances of Degraeve and Peeters (2003)

m	W	int	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	H2	H3	H4	n_g	
500	100	[1, 100]	99.35	167.20	184.10	221	0.06	0.09	12	1	1	0	0	0	
		[20, 100]	80.75	116.00	111.50	123	0.02	0.03	10	2	0	0	0	0	
		[50, 100]	51.00	52.00	56.60	63	0.00	0.01	15	0	0	0	0	0	
	120	[1, 100]	99.65	181.85	37.05	195	0.29	3.79	17	1	0	0	0	0	
		[20, 100]	80.85	131.20	132.80	146	0.03	0.04	14	6	0	0	0	0	
		[50, 100]	51.00	62.00	56.55	61	0.00	0.01	13	0	0	0	0	0	
	150	[1, 100]	99.45	201.55	1.00	1	0.00	0.00	20	0	0	0	0	0	
		[20, 100]	80.85	151.65	86.55	102	0.01	0.02	14	14	5	0	1	0	
		[50, 100]	51.00	77.00	64.80	72	0.01	0.01	12	0	0	0	0	0	
	1000	100	[1, 100]	100.00	183.65	199.20	230	0.07	0.11	12	1	1	0	0	0
			[20, 100]	81.00	117.95	114.25	133	0.02	0.02	14	4	1	0	0	0
			[50, 100]	51.00	52.00	57.35	64	0.00	0.01	9	0	0	0	0	0
[1, 100]			100.00	202.20	25.00	181	0.01	0.04	20	3	0	0	0	0	
120		[20, 100]	81.00	132.95	143.40	167	0.03	0.04	10	3	2	0	0	0	
		[50, 100]	51.00	62.00	56.90	62	0.00	0.01	11	0	0	0	0	0	
		[1, 100]	100.00	226.15	7.00	121	0.00	0.03	20	1	0	0	0	0	
150		[20, 100]	81.00	154.90	86.85	101	0.01	0.02	11	11	9	0	0	0	
		[50, 100]	51.00	77.00	67.25	77	0.01	0.01	10	0	0	0	0	0	

in Tabs. 17 and 21, but the performance of H0 through H2 is similar; in fact H0 through H2 suffice for solving *all* the CSP instances used by Vanderbeck (1999).

Quite suprisingly, all the industrial instances we could find in the literature turned out to be easy for our method: they were solved in a fraction of a second (see Tables 22–24).

5.4. Results for the bin-packing problem

Following Degraeve and Peeters (2003), in the next three tables we present our results for the BPP. Table 5 gives our results for the BPP instances of Degraeve and Peeters (2003) (20 instances per row). All the 360 instances were solved (H4 helped once).

Table 6 reports results for the BINPACK instances from the OR-Library (Beasley, 1990) (20 instances per row). The first four uniform classes were solved by calling H4 just once. However, only 19 out of the 80 triplet instances were solved (with H4 helping on one instance). The remaining instances had unit gaps; the “gap” column gives averages of *relative* gaps ($ez^* - \lfloor \underline{\ell}_k \rfloor \rfloor / \lfloor \underline{\ell}_k \rfloor$). We add that for the CSP instances of §5.3, the running times of H4 were not excessive, and H4 was called quite infrequently anyway. In contrast, on the triplet classes t249 and t501, the use of H4 increased the running times substantially, as illustrated in Table 7 (the influence of H3 could be ignored). Note that the triplet classes are quite difficult for traditional LP relaxation (Degraeve and Peeters, 2003, Tab. 12).

Table 6: BINPACK uniform and triplet instances

name	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	H2	H3	H4	gap	n_g
u120	63.20	88.75	48.60	89	0.00	0.01	20	14	0	0	0	0.0%	0
u250	77.25	129.00	86.40	122	0.01	0.03	19	19	1	0	0	0.0%	0
u500	80.80	151.05	85.90	113	0.01	0.04	16	16	3	0	1	0.0%	0
u1000	81.00	155.00	86.30	97	0.01	0.02	12	12	8	0	0	0.0%	0
t60	49.95	58.80	40.20	56	0.01	0.04	0	1	19	0	0	1.5%	6
t120	86.15	110.75	72.70	91	0.06	0.09	0	1	18	0	1	2.0%	16
t249	140.10	199.15	126.70	146	0.26	0.37	0	1	19	0	0	1.2%	20
t501	194.25	315.40	167.40	189	0.67	1.14	0	0	20	0	0	0.6%	19

Table 7: BINPACK triplet instances without H3 and H4

name	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	H2	gap	n_g
t60	49.95	58.80	40.20	56	0.00	0.01	0	1	19	1.5%	6
t120	86.15	110.75	72.70	91	0.01	0.02	0	1	19	2.1%	17
t249	140.10	199.15	126.70	146	0.04	0.06	0	1	19	1.2%	20
t501	194.25	315.40	167.40	189	0.08	0.10	0	0	20	0.6%	19

Table 8: Modified BPP instances of Degraeve and Peeters (2003)

W	int	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	H2	H3	H4	n_g
10000	[1, 10000]	488.65	494.05	1484.40	1737	34.95	48.35	14	3	0	0	0	0
	[2000, 10000]	485.15	490.20	800.70	916	7.05	9.87	15	1	0	0	0	0
	[5000, 10000]	474.75	474.80	457.70	480	1.15	1.35	16	0	0	0	0	0
12000	[1, 10000]	486.95	494.55	817.90	1732	25.89	58.02	18	7	1	0	0	0
	[2000, 10000]	484.75	492.20	1157.90	1328	15.00	21.33	18	2	0	0	0	0
	[5000, 10000]	475.95	480.35	520.75	550	2.20	2.64	15	0	0	0	0	0
15000	[1, 10000]	487.90	497.15	293.60	1171	8.00	67.00	18	6	0	0	1	1
	[2000, 10000]	482.70	494.25	805.05	1144	16.19	29.37	16	16	4	0	0	0
	[5000, 10000]	475.25	486.95	691.50	786	5.14	6.31	13	0	0	0	0	0

Table 8 presents our results for the modified BPP classes of Degraeve and Peeters (2003) (20 instances per row as described in §5.1). Just one out of the 180 problems was not solved (H4 helped on one problem). The transformation into a CSP reduced the number of items by at most 5% on average. For almost 500 variables, the large iteration numbers and running times are not too surprising.

Table 9: Small-item-size instances with tight KP bounds, $int = all$, $\bar{d} = all$

m	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	H2	H3	H4	n_g
10	9.99	26.77	15.30	31	0.00	0.03	113	50	70	0	0	0
20	19.95	53.13	32.52	68	0.01	0.12	113	59	69	0	0	0
30	29.91	79.76	52.21	97	0.04	0.55	130	82	61	0	0	0
40	39.85	105.55	70.60	141	0.10	1.37	126	93	58	0	0	0
50	49.75	132.16	90.50	171	0.20	2.20	132	101	56	0	0	0
75	74.36	197.32	141.59	249	0.61	6.29	154	127	38	0	0	0
100	98.92	263.36	183.75	319	0.88	11.15	153	125	45	0	1	0

Table 10: Medium-item-size instances with tight KP bounds, $int = all$, $\bar{d} = 50$

m	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	H2	H3	H4	n_g
10	9.98	23.09	17.55	29	0.00	0.02	57	48	112	0	1	0
20	19.95	45.58	35.08	58	0.02	0.22	74	57	108	0	0	0
30	29.84	68.47	53.63	91	0.05	0.79	78	77	100	0	1	0
40	39.78	90.65	70.06	116	0.11	3.62	70	64	109	0	0	0
50	49.64	113.69	88.91	154	0.19	4.70	83	71	111	0	3	1
75	74.08	169.10	137.08	216	1.48	53.20	80	71	106	1	0	0
100	98.45	226.07	184.88	293	7.76	410.38	74	72	119	0	2	0

5.5. Impact of tighter knapsack bounds

The results of §5.3 were obtained for the relaxed bounds of (9). Using the tighter bounds of (8) allowed us to solve just two more instances at the expense of longer running times. To save space, the following tables and remarks list only data classes on which the tightening of KP bounds mattered most, giving more details for larger problem sizes.

Concerning Tables 9–10, the good news is that tighter bounds allowed us to solve *all* the small-item-size instances of Degraeve and Peeters (2003), and *all but one* of the medium-item-size instances of Degraeve and Peeters (2003). Unfortunately the running times grew substantially relative to Tabs. 1–2. On the small-item-size instances, for $m \geq 40$ the average running times grew by about 150%; on the medium-item-size instances, the average running times grew by 200%, 217%, 303% and 446% for $m = 40, 50, 75$ and 100 (see Tabs. 25–26 for more details). The iteration numbers were about the same. The increase in running times can be attributed to the knapsack solver (which made more than two million backtrackings on some subproblems).

For the instances of Wäscher and Gau (1996), the same 3997 out of 4000 instances were solved, but relative to Tab. 3, for $m = 40$ and 50 the average running times grew by 100%

Table 11: Comparison of running times with Degraeve and Peeters (2003), $int = all$

m	Tab. 17			Tab. 18		Tabs. 20–21	
	HR	LR	BR	LR	BR	LR	BR
30	0.17	0.10	0.02	0.10	0.02	0.29	0.02
40	0.44	0.21	0.05	0.21	0.04	0.71	0.03
50	0.74	0.38	0.08	0.37	0.08	1.45	0.06
75	5.03	0.81	0.22	2.18	0.24	9.57	0.37
100	10.14	2.99	0.42	2.63	0.40	21.08	1.42

and 143%. For the instances of Vanderbeck (1999), relative to Tab. 4, the average running times grew by between 67% and 205%; their sum increased by 175%.

5.6. Comparisons with other procedures from the literature

5.6.1. Comparison with Degraeve and Peeters (2003)

In Table 11 we compare the average running times of our *bundle relaxation* code BR with the two best procedures HR and LR of Degraeve and Peeters (2003) on the instances used for Tabs. 17, 18, 20 and 21. The times for HR and LR obtained on a Pentium Pro 200 MHz were extracted from (Degraeve and Peeters, 2003, Tabs. 1–4b). Two points should be noted. First, both HR and LR employed an industrial LP solver (much more sophisticated than our dense QP solver), and LR additionally used subgradient optimization. Second, due to lacking knowledge, let's assume that the machine of Degraeve and Peeters (2003) was ten times slower than ours. Then Table 11 suggests that on the small-item-size instances BR was comparable in speed with HR (about twice slower than LR), while on the medium-item-size instances BR could perform better than LR. Similarly, in view of Tab. 3 and (Degraeve and Peeters, 2003, Tab. 10), on the instances of Wäscher and Gau (1996) BR was as fast as HR (twice slower than LR), whereas Tab. 4 and (Degraeve and Peeters, 2003, Tab. 5a) indicate that on the instances of Vanderbeck (1999) BR was comparable with HR, and sometimes faster than LR. On the industrial instances of Degraeve and Schrage (1999) (cf. Tab. 24 and (Degraeve and Peeters, 2003, Tab. 9)), BR behaved like HR (sometimes better than LR).

5.6.2. Comparison with Briant et al. (2005)

We now compare our running times with those in (Briant et al., 2005, §2.2), where the task was just to produce sufficiently accurate primal and dual solutions z^k and \hat{u}^k that satisfy the

Table 12: Industrial and random CSP instances of Briant et al. (2005), $\epsilon_r = 0$

name	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	e_{av}	e_{mx}	a_{av}	a_{mx}
ind_9	18.00	56.89	30.00	69	0.02	0.10	0.0E+00	0.0E+00	2.1E-11	1.8E-10
50b100c4	49.70	129.25	109.10	141	0.15	0.39	0.0E+00	0.0E+00	8.9E-13	4.6E-12
u120	63.20	88.75	97.10	124	0.02	0.03	0.0E+00	0.0E+00	4.0E-13	1.5E-12
u250	77.25	129.00	106.95	129	0.02	0.04	0.0E+00	0.0E+00	2.7E-12	1.5E-11
t120	86.15	110.75	76.05	93	0.01	0.02	0.0E+00	0.0E+00	1.4E-12	6.4E-12
t249	140.10	199.15	133.15	148	0.04	0.06	0.0E+00	0.0E+00	1.3E-07	2.5E-06

Table 13: Industrial and random CSP instances of Briant et al. (2005), $\epsilon_r = 10^{-5}$

name	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	e_{av}	e_{mx}	a_{av}	a_{mx}
ind_9	18.00	56.89	30.67	69	0.02	0.10	0.0E+00	0.0E+00	8.9E-11	7.9E-10
50b100c4	49.70	129.25	109.10	141	0.15	0.40	0.0E+00	0.0E+00	8.9E-13	4.6E-12
u120	63.20	88.75	97.90	124	0.02	0.03	0.0E+00	0.0E+00	5.2E-09	9.8E-08
u250	77.25	129.00	108.05	132	0.02	0.04	5.4E-16	8.8E-15	9.1E-09	8.2E-08
t120	86.15	110.75	78.00	95	0.01	0.02	0.0E+00	0.0E+00	1.8E-09	1.8E-08
t249	140.10	199.15	134.90	153	0.04	0.06	0.0E+00	0.0E+00	1.1E-09	2.2E-08

stopping criterion

$$e\hat{z}^k - \underline{\theta}_k \leq \tilde{\epsilon} \quad \text{and} \quad |\pi^k|/\sqrt{m} \leq \tilde{\epsilon} \quad (26)$$

for a given tolerance $\tilde{\epsilon} = 10^{-6}$; thus the duality gap is at most $\tilde{\epsilon}$ and (since $\sum_j p^j \hat{z}_j^k - d \geq \pi^k$) \hat{z}^k satisfies the demand constraints within $\tilde{\epsilon}$ on the average. Hence, to achieve a similar implementation context, our code was run with a tight optimality tolerance $\epsilon_{opt} = 10^{-14}$ and without early terminations due to primal heuristics. Table 12 gives our results for the instances of (Briant et al., 2005, §2.2); here “ind_9” comprises the first 9 instances from Tab. 23, “50b100c4” is the final class of Tab. 4 and the remaining classes occur in Tab. 6. The columns “ e_{av} ” and “ e_{mx} ” give average and maximum values of *relative dual errors* $|\theta_* - \underline{\theta}_k|/|\theta_*|$ (with θ_* estimated to at least 14 digits in other runs). The columns “ a_{av} ” and “ a_{mx} ” give average and maximum values of *absolute errors* a_k , with a_k being the minimum $\tilde{\epsilon}$ satisfying (26); in other words, our code might have terminated earlier if we used (26) as the stopping criterion with $\tilde{\epsilon} \geq a_{mx}$.

Table 12 was obtained for $\epsilon_r = 0$, i.e., exact KP solutions. Results for $\epsilon_r = 10^{-5}$ are given in Table 13, and for $\epsilon_r = 10^{-3}$ and 10^{-4} in Tables 27–28. The accuracy obtained was quite poor for $\epsilon_r = 10^{-3}$, a bit too weak for $\epsilon_r = 10^{-4}$, but very good for $\epsilon_r = 10^{-5}$ (the results for smaller ϵ_r were similar). These values of ϵ_r are also “representative” when our code is run

Table 14: Small-item-size instances of Degraeve and Peeters (2003), $\epsilon_r = 0$

m	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	H2	H3	H4	n_g
10	9.99	26.77	15.14	31	0.00	0.02	113	49	70	1	0	0
20	19.95	53.13	32.51	69	0.14	22.38	120	64	64	0	0	0
30	29.91	79.76	51.90	91	0.55	127.23	130	85	57	0	0	1
40	39.85	105.55	70.45	134	3.03	219.28	134	98	53	0	0	0
50	49.75	132.16	90.18	181	1.08	168.06	134	102	55	0	0	0
75	74.36	197.32	141.75	256	6.97	576.03	148	121	43	0	1	0
100	98.92	263.36	183.33	311	13.96	1035.61	165	136	33	0	2	0

Table 15: Medium-item-size instances of Degraeve and Peeters (2003), $\epsilon_r = 0$

m	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	H2	H3	H4	n_g
10	9.98	23.09	17.52	29	0.00	0.01	54	48	112	0	1	0
20	19.95	45.58	35.05	58	0.02	1.80	68	50	114	0	1	0
30	29.84	68.47	53.71	93	0.67	105.02	73	73	105	0	0	0
40	39.78	90.65	69.94	120	3.36	253.23	69	62	111	0	0	0
50	49.64	113.69	88.76	156	1.73	62.13	74	65	118	1	1	1
75	74.08	169.10	137.04	232	30.81	485.60	83	74	104	0	0	1
100	98.45	226.07	184.32	295	67.29	850.75	72	71	118	0	4	0

with the primal heuristics, as shown in §5.7.2. In view of the excellent accuracy in Tab. 13, we may compare our timings in Tab. 13 with the *best* ones of (Briant et al., 2005, Tabs. 1, 2 and 5) for various CG and bundle variants, where the machine used was about twice slower than ours, and the CG variants could stop before the first part of (26) held. Since quoting the tables of Briant et al. (2005) would take too much space, we just state the conclusion: Our running times were shorter at least 7.5 times for ind_9, 22 times for 50b100c4, 43 times for u120, 56 times for u250, 237 times for t120, and 197 times for t249.

5.7. Impact of evaluation errors

5.7.1. Comparison with exact bundle

When the dual objective evaluations happen to be exact, our BR code runs essentially like the standard bundle method used in (Feltenmark and Kiwiel, 2000). Therefore, Tables 14–16 summarize our results for exact KP solutions ($\epsilon_r = 0$) relative to Tabs. 1–3 (where $\epsilon_r = 10^{-5}$); similar features were observed on other instances. First, the iteration numbers and the performance of our heuristics did not change significantly. (In other words, the errors occurring in the inexact case were small enough to be accommodated gracefully by our code.)

Table 16: CSP instances of Wäscher and Gau (1996), $\epsilon_r = 0$

m	m_{av}	m'_{av}	t_{av}	t_{mx}	t_{av}	t_{mx}	n_e	H1	H2	H3	H4	n_g
10	9.99	25.37	14.27	35	0.00	0.24	449	134	192	0	0	0
20	19.96	50.46	30.73	61	0.13	31.45	485	240	183	0	2	0
30	29.90	75.72	48.18	105	0.43	121.99	503	281	161	0	1	0
40	39.84	100.10	65.05	123	1.60	208.44	503	314	159	0	2	2
50	49.73	125.22	84.75	171	3.13	407.79	526	341	138	0	4	1
<i>all</i>	29.88	75.37	48.60	171	1.06	407.79	2466	1310	833	0	9	3

Second, the running times increased quite dramatically. For instance, in Tab. 14 relative to Tab. 1, for $m = 30, 40, 50, 75$ and 100, the average times grew by factors of 27.5, 75.8, 13.5, 29.0 and 34.9, respectively; for Tab. 15 relative to Tab. 2, the factors are 33.5, 112.0, 28.8, 83.3 and 47.4; in Tab. 16, the “*all*” time grew by the factor of 35.3. Performance profiles (Dolan and Moré, 2002) are given in Figs. 1–3 in the supplement.

5.7.2. Other choices of the relative error tolerance

In the initial version of this paper we used the accuracy tolerance $\epsilon_r = 10^{-8}$; the results were very close to those in Tabs. 1–10 (where $\epsilon_r = 10^{-5}$). In parallel with Tabs. 14–16, Tables 29–34 give results for $\epsilon_r = 10^{-3}$ and 10^{-4} . The average iteration numbers and computing times were similar for $\epsilon_r = 10^{-3}$, 10^{-4} and 10^{-5} . However, $\epsilon_r = 10^{-3}$ was too large, causing our heuristics to fail more frequently. On the other hand, $\epsilon_r = 10^{-4}$ did not improve on our standard choice of $\epsilon_r = 10^{-5}$ (giving one more gap in Tab. 32).

Further insight may be gained as follows. By (10), the absolute error in evaluating θ is bounded by $N\epsilon_r$ once $\bar{\zeta}$ gets close to 1. The upper bound $N := ez^*$ delivered by FFD (cf. §3.3) is usually close to the optimal primal value v_* . Typical instances have the *integer round-up property* $\lceil \theta_* \rceil = v_*$, but our heuristics fail if we can’t find a lower bound $\underline{\theta}_k > v_* - 1$. Thus we may expect failures when the absolute errors get close to $N\epsilon_r > 1$. Now, in Tables 29–31 the average values of v_* and N grow linearly with m , reaching order 5000, 2875 and 1250 for the final classes, where $N\epsilon_r > 1$ for $\epsilon_r = 10^{-3}$; thus the small percentage of failures suggests that the actual errors tended to be smaller than their upper bounds.

5.7.3. Absolute error tolerances

In view of the discussion in §5.7.2, we also considered choosing ϵ_r so that the evaluation errors did not exceed a given *absolute error tolerance* $\epsilon_a < 1$ (with SHP using $\epsilon_r = 10^{-5}$

as in §5.3). Specifically, for evaluating θ we used $\epsilon_r := \epsilon_a/N$. Tables 35–40 give results for $\epsilon_a = 0.05$ and 0.01. For both values of ϵ_a , the average iteration numbers and computing times were close to those in Tabs. 1–3 (where $\epsilon_r = 10^{-5}$). However, $\epsilon_a = 0.05$ was too large, causing our heuristics to fail more frequently. On the other hand, our results for $\epsilon_a = 0.01$ were very close to those for $\epsilon_r = 10^{-5}$.

5.7.4. More inexact null steps

We now consider a modification in which our KP solver exits once at least `bkmin` backtracks have occurred, for a given parameter `bkmin`, and the incumbent value ζ satisfies

$$\zeta > 1 + (u^{k+1}d - \theta_u^k - \kappa v_k) / N, \quad (27)$$

so that $\zeta_{k+1} := \zeta$ yields a null step; cf. (22) (normally $u^{k+1}d > \theta_u^k + \kappa v_k$ and (27) holds iff (22) fails). Such “more inexact” null steps may save KP work, but shallower cuts may yield slower convergence. Tables 41–46 give results for `bkmin` = 0 and 1000 (with $\epsilon_r = 10^{-5}$). Relative to Tabs. 1–3, where `bkmin` = ∞ , for `bkmin` = 0 the average iteration numbers grew by 59–114% on the largest instances, the solution times decreased fairly mildly, and two more gaps occurred. In contrast, for `bkmin` = 1000 the average iteration numbers grew by only 5–13% on the largest instances, the solution times decreased quite mildly (although the decreases by 32% and 37% for $m = 75$ and 100 in Tab. 45 are noticeable), and three gaps disappeared. On the other hand, the maximum iteration numbers increased substantially on the larger instances, giving some cause for concern.

5.7.5. A discussion of error tolerances

Although in general one may expect tradeoffs between the accuracy of subproblem solutions and the speed of convergence, for the CSP such tradeoffs may have little practical impact, since Tables 12–34 exhibit fairly small variations in iteration numbers and computing times for “reasonable” accuracy tolerances. Therefore, we would not expect much gain from *dynamic tolerance adjustment*: loose at the beginning and progressively decreasing.

We add that dynamic handling of the accuracy may be important in general, especially if the oracle’s work depends “continuously” on the accuracy required. However, this need not be the case for our MTIR, which seems to have the following properties: (1) its work explodes on some subproblems when the accuracy required is “too high”; and (2) its work does not vary much otherwise. Thus the main point is to avoid accuracies that are “too high”, or “too

low” for the dual solver to succeed, whereas for all “intermediate” accuracies, the solution time should not vary significantly (unless smaller accuracies affect the iteration numbers “more than proportionally”). We conjecture that similar effects are likely to hold for other integer-programming applications with branch-and-bound oracles that deliver relatively good incumbents quickly.

5.8. Impact of various heuristics

For the 7,538 CSP instances reported in Tabs. 1–4 and 22–24, our heuristics H3 and H4 helped in solving 3 and 21 problems, respectively, and 6 problems were not solved. When H3 was switched off, H4 solved the three instances previously solved by H3, with the same timings. Thus H3 could be omitted, but it might become more useful on other instances.

We now consider the case where the heuristics H1 through H4 are replaced by the heuristic named H5, which consists in calling, upon bundle termination, Procedure 1 with Steps 2, 3 and 5 omitted, and Step 4 using FFD; in other words, the relaxed primal solution is rounded down and the residual problem is solved by FFD. The results for H5 (with $\epsilon_r = 10^{-5}$) given in Tables 47–49 show that H5 performs quite poorly relative to Tabs. 1–3 (and that H1 reduces the iteration numbers, and usually the computing times as well). On the other hand, we note that H5 solved 91.5% and 68.8% of problems in Tabs. 47–48, whereas the FFD-based heuristic of Degraeve and Peeters (2003) solved 91.6% and 69.9%; further, H5 solved 92.8% of problems in Tab. 49, whereas the corresponding heuristic RFFD of Wäscher and Gau (1996) solved 92.5%. Thus our bundle results with H5 are very similar to those obtained with other CG solvers.

Our next improvement on H5, named H6, consists in calling Procedure 1 with only Step 2 omitted, and Step 4 using FFD. The results for H6 given in Tables 50–52 show that H6 performs much better than H5, solving 96.4%, 91.9% and 97.2% of problems; thus the rounding procedure of Belov and Scheithauer (2002) may yield significant improvements also for FFD. Finally, we note that H2 and H4 improve on H6 by using SHP or SVC together with Step 2 of Procedure 1. Specifically, H1 and H2 solved 99.8%, 99.4% and 99.7% of problems, and together with H4 they solved 99.94%, 99.88% and 99.92% of problems. To save space, the results for H2 alone are omitted.

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Online Supplement for An Inexact Bundle Approach to Cutting-Stock Problems

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Additional tables and figures

Tables 17–19 below give details for the small-item-size instances of Degraeve and Peeters (2003). The averages, maxima and sums in Table 17 are taken over 20 instances for each interval, and thus over 80 instances for each “all” row. In Table 18, there are 60 instances per interval (i.e., 20 instances for each value of the average demand $\bar{d} = 10, 50, 100$), and each “all” row gives statistics over the 240 instances used for each value of m . Finally, each row in Table 19 reports statistics over 80 instances (obtained from the 20 instances used for each of the four width intervals).

Our detailed results for the medium-item-size instances of Degraeve and Peeters (2003) are presented in Tables 20 and 21, where each “all” row gives statistics over the 240 instances used for each value of m .

Tables 22–24 give our results for the industrial instances of Vance (1998) (as numbered in (Degraeve and Peeters, 2003, Tab. 7)), (Vanderbeck, 1999, Tab. 1) and Degraeve and Schrage (1999) (as named in (Degraeve and Peeters, 2003, Tab. 9)). The final column identifies the heuristic which delivered the optimal solution; in other words, H0 through H2 solved all these instances except for a single instance solved by H3.

Tables 25–26 complement Tables 9–10. Relative to Tabs. 18 and 21, on the small-item-size instances, for $m \geq 40$ the average running times grew by about 150% (mostly from increasing by about 200% on width interval [1, 5000]). On the medium-item-size instances, the average running times grew by 200%, 217%, 303% and 446% for $m = 40, 50, 75$ and 100, increasing by 367–531% on width interval [1500, 2500], 157–223% on [500, 5000], and 140–179% on [1000, 5000]; for $m = 100$, they went up by 67–156% on four other intervals. The iteration numbers were about the same.

Tables 27 and 28 complement Tables 12 and 13 in the paper.

Figs. 1–3 give performance profiles of exact ($\epsilon_r = 0$) vs inexact ($\epsilon_r = 10^{-5}$) bundle. Due to the poor resolution of our timer, we replaced zero running times by 0.001 and selected only instances with $m \geq 30$ (1200 instances for Fig. 1, 1200 for Fig. 2, 2400 for Fig. 3). Each figure plots the portion of instances on which a particular variant was not slower than the fastest variant by more than a given ratio. Inexact bundle had a ratio of 2.1 on 98.82%, 97.49%, 98.07% of respective instances, and was never slower more than 10 times.

Tables 29–31 and 32–34 give additional results related to Tables 14–16 in the paper.

Tables 35–37 and 38–40 support the discussion of §5.7.3 of the paper.

Tables 41–46 validate the discussion of §5.7.4 of the paper.

Tables 47–52 validate the discussion of §5.8 of the paper.

Table 17: Small-item-size instances of Degraeve and Peeters (2003), $\bar{d} = 50$

m	int	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	H2	H3	H4	n_g
10	[1, 2500]	10.00	38.15	7.95	20	0.00	0.01	19	13	0	0	0	0
	[1, 5000]	10.00	29.45	19.75	31	0.00	0.01	5	7	13	0	0	0
	[1, 7500]	9.95	22.35	19.95	29	0.00	0.01	0	0	8	0	0	0
	[1, 10000]	10.00	19.95	18.10	24	0.00	0.01	3	0	6	1	0	0
	<i>all</i>	9.99	27.47	16.44	31	0.00	0.01	27	20	27	1	0	0
20	[1, 2500]	20.00	78.65	13.00	21	0.00	0.01	20	12	0	0	0	0
	[1, 5000]	19.95	57.10	42.10	56	0.01	0.04	7	7	12	0	0	0
	[1, 7500]	20.00	45.15	42.10	59	0.01	0.01	3	2	8	0	0	0
	[1, 10000]	20.00	38.95	36.45	52	0.00	0.01	4	1	5	0	0	0
	<i>all</i>	19.99	54.96	33.41	59	0.01	0.04	34	22	25	0	0	0
30	[1, 2500]	29.90	116.85	26.40	51	0.01	0.03	20	15	0	0	0	0
	[1, 5000]	29.90	87.30	69.25	91	0.04	0.08	9	9	11	0	0	0
	[1, 7500]	30.00	68.50	65.40	91	0.01	0.02	5	3	8	0	0	0
	[1, 10000]	29.95	60.25	58.15	69	0.01	0.02	4	2	3	0	0	0
	<i>all</i>	29.94	83.22	54.80	91	0.02	0.08	38	29	22	0	0	0
40	[1, 2500]	39.80	153.20	39.65	76	0.02	0.07	19	17	1	0	0	0
	[1, 5000]	39.85	113.20	92.50	121	0.12	0.21	14	14	6	0	0	0
	[1, 7500]	39.90	89.20	92.35	121	0.03	0.05	6	6	6	0	0	0
	[1, 10000]	39.90	76.15	78.25	108	0.02	0.02	3	1	5	0	0	0
	<i>all</i>	39.86	107.94	75.69	121	0.05	0.21	42	38	18	0	0	0
50	[1, 2500]	49.60	190.70	35.75	51	0.02	0.04	20	14	0	0	0	0
	[1, 5000]	49.70	145.30	116.55	171	0.18	0.42	16	16	4	0	0	0
	[1, 7500]	49.75	113.30	124.85	151	0.07	0.13	5	3	13	0	0	0
	[1, 10000]	50.00	99.95	105.30	131	0.04	0.07	1	2	3	0	0	0
	<i>all</i>	49.76	137.31	95.61	171	0.08	0.42	42	35	20	0	0	0
75	[1, 2500]	73.85	285.85	71.75	115	0.07	0.15	19	17	1	0	0	0
	[1, 5000]	74.10	218.00	167.40	256	0.41	1.33	18	18	2	0	0	0
	[1, 7500]	74.80	170.10	196.40	226	0.29	0.61	5	5	7	0	0	0
	[1, 10000]	74.75	145.15	158.75	218	0.12	0.23	3	2	4	0	0	0
	<i>all</i>	74.38	204.78	148.57	256	0.22	1.33	45	42	14	0	0	0
100	[1, 2500]	98.50	374.00	101.50	120	0.13	0.21	20	20	0	0	0	0
	[1, 5000]	99.05	286.45	183.95	261	0.45	1.95	17	17	3	0	0	0
	[1, 7500]	99.30	227.35	272.70	311	0.78	1.27	14	13	5	0	0	0
	[1, 10000]	99.45	194.55	224.65	294	0.31	0.65	5	3	3	0	0	0
	<i>all</i>	99.08	270.59	195.70	311	0.42	1.95	56	53	11	0	0	0

Table 18: Small-item-size instances of Degraeve and Peeters (2003), $\bar{d} = all$

m	int	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	H2	H3	H4	n_g
10	[1, 2500]	10.00	36.70	8.02	28	0.00	0.01	54	30	5	0	0	0
	[1, 5000]	9.98	28.80	16.13	31	0.00	0.01	22	12	33	0	0	0
	[1, 7500]	9.98	22.10	18.17	29	0.00	0.01	19	4	18	0	0	0
	[1, 10000]	10.00	19.48	18.25	31	0.00	0.01	18	3	14	1	0	0
	<i>all</i>	9.99	26.77	15.14	31	0.00	0.01	113	49	70	1	0	0
20	[1, 2500]	19.95	74.15	14.92	61	0.00	0.02	58	31	2	0	0	0
	[1, 5000]	19.90	56.25	37.57	56	0.01	0.04	30	25	29	0	0	0
	[1, 7500]	19.97	43.53	41.33	69	0.00	0.01	16	6	21	0	0	0
	[1, 10000]	20.00	38.57	36.23	52	0.00	0.01	16	2	12	0	0	0
	<i>all</i>	19.95	53.13	32.51	69	0.01	0.04	120	64	64	0	0	0
30	[1, 2500]	29.85	110.18	22.02	51	0.01	0.03	59	36	1	0	0	0
	[1, 5000]	29.88	84.18	62.42	91	0.04	0.22	35	33	24	0	0	1
	[1, 7500]	29.95	66.50	65.53	91	0.01	0.03	16	10	19	0	0	0
	[1, 10000]	29.95	58.18	57.65	83	0.01	0.02	20	6	13	0	0	0
	<i>all</i>	29.91	79.76	51.90	91	0.02	0.22	130	85	57	0	0	1
40	[1, 2500]	39.75	146.80	31.67	76	0.02	0.07	56	36	4	0	0	0
	[1, 5000]	39.87	111.88	80.00	134	0.09	0.36	44	42	16	0	0	0
	[1, 7500]	39.92	88.75	93.35	121	0.03	0.10	20	13	19	0	0	0
	[1, 10000]	39.87	74.78	76.62	108	0.02	0.03	14	7	14	0	0	0
	<i>all</i>	39.85	105.55	70.41	134	0.04	0.36	134	98	53	0	0	0
50	[1, 2500]	49.60	182.70	32.93	71	0.02	0.06	60	37	0	0	0	0
	[1, 5000]	49.65	140.88	107.05	181	0.20	0.66	41	40	19	0	0	0
	[1, 7500]	49.82	110.80	122.42	151	0.07	0.14	19	16	24	0	0	0
	[1, 10000]	49.93	94.27	98.40	131	0.03	0.07	14	9	12	0	0	0
	<i>all</i>	49.75	132.16	90.20	181	0.08	0.66	134	102	55	0	0	0
75	[1, 2500]	73.82	271.55	57.58	115	0.06	0.15	59	43	1	0	0	0
	[1, 5000]	74.23	209.83	158.07	256	0.53	2.00	49	49	11	0	0	0
	[1, 7500]	74.67	165.52	190.80	239	0.26	0.61	31	26	16	0	0	0
	[1, 10000]	74.72	142.38	160.85	227	0.12	0.27	10	4	15	0	0	0
	<i>all</i>	74.36	197.32	141.82	256	0.24	2.00	149	122	43	0	0	0
100	[1, 2500]	98.13	359.88	75.18	174	0.10	0.21	58	41	2	0	0	0
	[1, 5000]	98.90	280.47	167.22	277	0.44	2.81	53	50	7	0	0	0
	[1, 7500]	99.18	221.73	268.80	311	0.76	1.54	37	35	14	0	1	0
	[1, 10000]	99.45	191.37	224.30	294	0.29	0.65	17	10	11	0	0	0
	<i>all</i>	98.92	263.36	183.88	311	0.40	2.81	165	136	34	0	1	0

Table 19: Small-item-size instances of Degraeve and Peeters (2003), *int = all*

m	\bar{d}	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	H2	H3	H4	n_g
10	10	10.00	24.95	10.77	26	0.00	0.01	69	14	6	0	0	0
	50	9.99	27.47	16.44	31	0.00	0.01	27	20	27	1	0	0
20	100	9.99	27.89	18.21	31	0.00	0.01	17	15	37	0	0	0
	10	19.94	48.60	25.93	58	0.00	0.02	57	16	9	0	0	0
50	19.99	54.96	33.41	59	0.01	0.04	34	22	25	0	0	0	0
	100	19.94	55.81	38.20	69	0.01	0.03	29	26	30	0	0	0
30	10	29.91	73.34	43.33	91	0.01	0.22	57	28	5	0	0	1
	50	29.94	83.22	54.80	91	0.02	0.08	38	29	22	0	0	0
100	29.88	82.73	57.59	91	0.02	0.12	35	28	30	0	0	0	0
	10	39.83	96.79	56.81	109	0.02	0.11	59	29	9	0	0	0
50	39.86	107.94	75.69	121	0.05	0.21	42	38	18	0	0	0	0
	100	39.86	111.94	78.72	134	0.05	0.36	33	31	26	0	0	0
10	49.70	121.30	75.41	151	0.05	0.36	59	35	6	0	0	0	0
	50	49.76	137.31	95.61	171	0.08	0.42	42	35	20	0	0	0
100	49.79	137.88	99.58	181	0.11	0.66	33	32	29	0	0	0	0
	10	74.41	181.36	121.49	216	0.18	1.15	61	41	7	0	0	0
50	74.38	204.78	148.57	256	0.22	1.33	45	42	14	0	0	0	0
	100	74.29	205.83	155.41	239	0.32	2.00	43	39	22	0	0	0
100	10	98.90	243.64	154.55	310	0.28	1.69	63	36	3	0	0	0
	50	99.08	270.59	195.70	311	0.42	1.95	56	53	11	0	0	0
100	98.78	275.86	201.38	303	0.49	2.81	46	47	20	0	1	0	

Table 20: Medium-item-size instances of Degraeve and Peeters (2003), $\bar{d} = 50$

m	int	m_{av}	m'_{nv}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	H2	H3	H4	n_g
10	[500, 2500]	10.00	33.80	13.65	23	0.00	0.02	13	13	7	0	0	0
	[1000, 2500]	10.00	31.40	15.45	25	0.00	0.01	9	9	11	0	0	0
	[1500, 2500]	9.90	29.70	16.40	25	0.00	0.01	4	4	16	0	0	0
	[500, 5000]	10.00	28.15	19.10	25	0.00	0.01	5	5	13	0	1	0
	[1000, 5000]	10.00	24.70	17.30	22	0.00	0.01	5	7	12	0	0	0
	[1500, 5000]	9.90	22.50	17.40	23	0.00	0.01	2	5	15	0	0	0
	[500, 7500]	10.00	20.35	20.00	29	0.00	0.01	3	1	10	0	0	0
	[1000, 7500]	10.00	19.00	18.65	24	0.00	0.01	1	0	8	0	0	0
	[1500, 7500]	10.00	17.40	19.25	25	0.00	0.01	0	0	8	0	0	0
	[500, 10000]	10.00	18.30	18.95	28	0.00	0.00	5	0	3	0	0	0
	[1000, 10000]	10.00	16.50	17.35	22	0.00	0.01	3	2	4	0	0	0
	[1500, 10000]	10.00	15.30	16.75	20	0.00	0.01	4	2	5	0	0	0
	<i>all</i>	9.98	23.09	17.52	29	0.00	0.02	54	48	112	0	1	0
	20	[500, 2500]	20.00	67.00	26.85	58	0.01	0.08	16	16	4	0	0
[1000, 2500]		19.95	63.30	33.95	58	0.02	0.10	13	13	7	0	0	0
[1500, 2500]		19.75	59.25	39.00	49	0.02	0.03	10	10	10	0	0	0
[500, 5000]		19.95	52.40	39.85	45	0.01	0.02	3	3	17	0	0	0
[1000, 5000]		19.95	48.30	36.35	41	0.01	0.02	2	2	18	0	0	0
[1500, 5000]		19.95	45.75	34.50	41	0.00	0.01	1	2	18	0	0	0
[500, 7500]		19.95	40.50	39.70	52	0.00	0.01	3	1	7	0	0	0
[1000, 7500]		20.00	37.55	37.90	44	0.00	0.01	4	2	10	0	0	0
[1500, 7500]		20.00	34.30	34.05	41	0.00	0.01	3	1	8	0	1	0
[500, 10000]		20.00	34.80	33.90	48	0.00	0.01	5	0	4	0	0	0
[1000, 10000]		19.95	32.40	32.65	45	0.00	0.01	4	0	5	0	0	0
[1500, 10000]		20.00	31.35	31.95	40	0.00	0.01	4	0	6	0	0	0
<i>all</i>		19.95	45.58	35.05	58	0.01	0.10	68	50	114	0	1	0
30		[500, 2500]	29.65	100.25	39.35	51	0.01	0.03	16	16	4	0	0
	[1000, 2500]	29.85	95.30	40.65	55	0.01	0.03	11	11	9	0	0	0
	[1500, 2500]	29.50	88.50	59.05	93	0.06	0.16	10	10	10	0	0	0
	[500, 5000]	30.00	79.65	61.00	71	0.03	0.06	6	6	14	0	0	0
	[1000, 5000]	29.80	72.95	57.75	71	0.02	0.11	9	9	11	0	0	0
	[1500, 5000]	29.60	66.70	53.80	65	0.01	0.02	6	8	12	0	0	0
	[500, 7500]	29.95	62.25	63.90	75	0.01	0.02	4	3	7	0	0	0
	[1000, 7500]	30.00	55.60	58.95	71	0.01	0.02	3	2	10	0	0	0
	[1500, 7500]	29.90	51.75	55.85	69	0.01	0.01	1	1	10	0	0	0
	[500, 10000]	29.95	52.30	52.00	67	0.00	0.01	2	2	6	0	0	0
	[1000, 10000]	29.95	49.85	53.05	64	0.00	0.01	3	3	7	0	0	0
	[1500, 10000]	29.95	46.55	49.15	61	0.00	0.01	2	2	5	0	0	0
	<i>all</i>	29.84	68.47	53.71	93	0.02	0.16	73	73	105	0	0	0
	40	[500, 2500]	39.75	135.85	46.45	61	0.02	0.04	15	15	5	0	0
[1000, 2500]		39.65	125.35	50.60	68	0.02	0.04	12	12	8	0	0	0
[1500, 2500]		39.30	117.90	66.90	101	0.15	0.58	9	9	11	0	0	0
[500, 5000]		39.85	103.30	82.05	101	0.07	0.12	9	9	11	0	0	0
[1000, 5000]		39.80	95.50	76.10	99	0.05	0.09	6	7	13	0	0	0
[1500, 5000]		39.75	90.85	71.70	84	0.03	0.06	4	4	16	0	0	0
[500, 7500]		39.80	81.05	88.40	120	0.02	0.04	1	0	11	0	0	0
[1000, 7500]		39.90	73.90	81.00	96	0.02	0.03	3	2	9	0	0	0
[1500, 7500]		39.90	71.35	76.05	89	0.01	0.03	3	2	10	0	0	0
[500, 10000]		39.85	68.80	71.15	93	0.01	0.02	2	2	6	0	0	0
[1000, 10000]		39.90	63.50	65.55	80	0.01	0.02	2	0	5	0	0	0
[1500, 10000]		39.90	60.40	63.30	82	0.01	0.02	4	1	5	0	0	0
<i>all</i>		39.78	90.65	69.94	120	0.03	0.58	70	63	110	0	0	0

Table 21: Medium-item-size instances of Degraeve and Peeters (2003), $\bar{d} = 50$

m	int	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	H2	H3	H4	n_g	
50	[500, 2500]	49.60	170.70	53.60	71	0.02	0.04	20	20	0	0	0	0	
	[1000, 2500]	49.20	155.95	64.55	83	0.03	0.05	8	8	12	0	0	0	
	[1500, 2500]	49.00	147.00	75.00	109	0.24	0.90	2	3	17	0	0	0	
	[500, 5000]	49.45	127.80	104.05	128	0.15	0.26	10	10	10	0	0	0	
	[1000, 5000]	49.80	121.15	92.75	108	0.08	0.15	8	8	12	0	0	0	
	[1500, 5000]	49.55	114.20	92.40	123	0.06	0.09	4	4	16	0	0	0	
	[500, 7500]	49.75	104.90	120.35	156	0.05	0.11	4	3	9	0	1	0	
	[1000, 7500]	49.70	93.70	106.30	125	0.04	0.05	2	2	14	1	0	0	
	[1500, 7500]	49.90	86.65	96.20	112	0.03	0.07	4	3	7	0	0	1	
	[500, 10000]	49.85	85.30	93.25	118	0.03	0.04	4	2	8	0	0	0	
	[1000, 10000]	50.00	83.05	89.65	111	0.02	0.03	4	0	9	0	0	0	
	[1500, 10000]	49.90	73.90	77.00	108	0.01	0.03	4	2	4	0	0	0	
	<i>all</i>	49.64	113.69	88.76	156	0.06	0.90	74	65	118	1	1	1	
	75	[500, 2500]	73.65	251.20	83.00	103	0.06	0.08	13	13	7	0	0	0
		[1000, 2500]	73.40	232.30	88.60	101	0.05	0.06	5	5	15	0	0	0
[1500, 2500]		71.85	215.55	102.20	164	2.73	8.60	3	3	17	0	0	0	
[500, 5000]		74.10	193.55	163.45	216	0.50	0.98	9	9	11	0	0	0	
[1000, 5000]		74.00	181.05	143.95	175	0.29	0.47	12	12	8	0	0	0	
[1500, 5000]		74.15	167.65	141.75	205	0.18	0.41	10	11	9	0	0	1	
[500, 7500]		74.70	154.65	181.65	232	0.20	0.50	7	7	11	0	0	0	
[1000, 7500]		74.50	138.80	171.25	201	0.12	0.16	6	7	5	0	0	0	
[1500, 7500]		74.50	128.95	153.60	178	0.09	0.15	6	3	7	0	0	0	
[500, 10000]		74.70	129.85	146.80	180	0.09	0.13	6	0	3	0	0	0	
[1000, 10000]		74.65	120.65	137.85	166	0.07	0.12	3	2	4	0	0	0	
[1500, 10000]		74.70	114.95	130.40	180	0.05	0.09	2	1	8	0	0	0	
<i>all</i>		74.08	169.10	137.04	232	0.37	8.60	82	73	105	0	0	1	
100		[500, 2500]	98.10	336.05	104.70	120	0.08	0.11	13	13	7	0	0	0
		[1000, 2500]	96.30	303.40	106.05	116	0.07	0.10	3	3	17	0	0	0
	[1500, 2500]	95.55	286.65	128.60	210	13.57	62.18	3	3	17	0	0	0	
	[500, 5000]	98.70	263.50	205.00	260	0.81	2.06	17	17	3	0	0	0	
	[1000, 5000]	98.65	240.15	197.50	220	0.72	1.00	9	9	11	0	0	0	
	[1500, 5000]	98.70	225.25	189.35	269	0.48	0.73	8	8	10	0	2	0	
	[500, 7500]	99.15	202.85	252.35	295	0.42	0.54	4	6	10	0	0	0	
	[1000, 7500]	99.40	188.45	240.60	295	0.31	0.39	8	8	10	0	1	0	
	[1500, 7500]	98.70	174.60	212.80	236	0.22	0.32	1	1	14	0	1	0	
	[500, 10000]	99.40	174.70	210.00	272	0.23	0.39	1	0	7	0	0	0	
	[1000, 10000]	99.45	163.90	192.15	235	0.17	0.43	3	2	4	0	0	0	
	[1500, 10000]	99.25	153.35	174.25	208	0.12	0.16	3	2	7	0	0	0	
	<i>all</i>	98.45	226.07	184.45	295	1.43	62.18	73	72	117	0	4	0	

Table 22: Industrial CSP instances of Vance (1998)

inst.	m	m'	i	t	n_e	H_i
1	2	7	1	0.00	1	H0
2	2	8	3	0.00	1	H1
3	3	11	1	0.00	1	H0
4	5	17	1	0.00	1	H0
5	14	50	15	0.01	1	H1
6	5	19	1	0.00	1	H0
7	4	14	1	0.00	1	H0
8	7	27	10	0.00	0	H2
9	11	46	12	0.00	1	H1
10	3	9	2	0.00	1	H0
11	2	7	1	0.00	1	H0
12	6	23	1	0.00	1	H0
13	2	9	1	0.00	1	H0
14	3	11	4	0.00	1	H1
15	7	20	8	0.00	1	H1
16	4	9	1	0.00	1	H0
17	12	42	24	0.00	0	H2
18	14	44	15	0.00	1	H1
19	5	15	13	0.01	0	H2
20	11	31	21	0.00	0	H2
21	9	27	16	0.00	0	H2
22	8	25	16	0.00	0	H2
23	7	20	8	0.00	1	H1
24	7	22	13	0.00	0	H2
25	12	39	13	0.01	1	H1
26	6	18	7	0.00	1	H1
27	12	40	13	0.00	1	H1
28	18	48	1	0.00	1	H0

Table 23: Industrial CSP instances of Vanderbeck (1999)

inst.	name	m	m'	i	t	n_e	H_i
1	7p18	7	22	13	0.00	0	H2
2	11p4	11	46	12	0.01	1	H1
3	12p19	12	39	13	0.00	1	H1
4	14p12	14	50	15	0.01	1	H1
5	d16p6	16	34	17	0.00	1	H1
6	25p0	25	80	66	0.06	1	H1
7	28p0	28	102	47	0.02	0	H2
8	30p0	26	86	27	0.01	1	H1
9	d33p20	23	53	24	0.06	1	H1
10	d43p21	32	74	33	0.05	1	H1

Table 24: Industrial CSP instances of Degraeve and Schrage (1999)

name	m	m'	i	t	n_e	H_i
DS01	41	90	62	0.06	1	H1
DS02	40	89	71	0.03	0	H2
DS03	26	56	47	0.01	1	H1
DS04	14	29	20	0.00	0	H3
DS05	18	33	31	0.00	0	H2
DS06	71	149	132	0.28	1	H1
DS07	14	41	15	0.01	1	H1
DS08	35	58	47	0.01	1	H1
DS09	35	86	36	0.04	1	H1
DS10	46	98	102	0.06	1	H1
DS11	42	89	43	0.01	1	H1
DS12	53	110	97	0.03	1	H1
DS13	22	47	30	0.00	1	H1
DS14	29	45	40	0.00	0	H0
DS15	43	78	55	0.01	1	H1
DS16	8	24	13	0.00	0	H2
DS17	37	111	36	0.01	0	H2
DS18	16	54	17	0.00	1	H1
DS19	23	67	43	0.02	0	H2
DS20	11	41	18	0.04	0	H2

Table 25: Small-item-size instances with tight KP bounds, $\bar{d} = all$

m	int	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	H2	H3	H4	n_g
40	[1, 5000]	39.87	111.88	80.40	141	0.28	1.37	44	42	16	0	0	0
	<i>all</i>	39.85	105.55	70.60	141	0.10	1.37	126	93	58	0	0	0
50	[1, 5000]	49.65	140.88	107.22	171	0.59	2.20	42	41	18	0	0	0
	<i>all</i>	49.75	132.16	90.50	171	0.20	2.20	132	101	56	0	0	0
75	[1, 5000]	74.23	209.83	158.78	236	1.62	6.29	53	53	7	0	0	0
	[1, 7500]	74.67	165.52	189.22	249	0.53	1.79	28	26	16	0	0	0
	[1, 10000]	74.72	142.38	160.50	231	0.19	0.48	15	6	13	0	0	0
100	<i>all</i>	74.36	197.32	141.59	249	0.61	6.29	154	127	38	0	0	0
	[1, 2500]	98.13	359.88	74.83	152	0.16	0.43	59	42	1	0	0	0
	[1, 5000]	98.90	280.47	166.87	295	1.33	11.15	50	47	10	0	0	0
	[1, 7500]	99.18	221.73	268.20	319	1.54	4.55	28	27	23	0	0	0
	[1, 10000]	99.45	191.37	225.10	301	0.51	1.68	16	9	11	0	1	0
<i>all</i>	98.92	263.36	183.75	319	0.88	11.15	153	125	45	0	1	0	

Table 26: Medium-item-size instances with tight KP bounds, $\bar{d} = 50$

m	int	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	H2	H3	H4	n_g
30	{1500, 2500}	29.50	88.50	58.50	91	0.29	0.79	12	12	8	0	0	0
	<i>all</i>	29.84	68.47	53.63	91	0.05	0.79	78	77	100	0	1	0
40	{1500, 2500}	39.30	117.90	67.50	101	0.79	3.62	9	9	11	0	0	0
	{500, 5000}	39.85	103.30	82.20	101	0.18	0.31	9	9	11	0	0	0
50	<i>all</i>	39.78	90.65	70.06	116	0.11	3.62	70	64	109	0	0	0
	{1500, 2500}	49.00	147.00	74.95	109	1.11	4.70	4	4	16	0	0	0
75	{500, 5000}	49.45	127.80	103.95	128	0.40	0.75	11	11	9	0	0	0
	{1000, 5000}	49.80	121.15	93.15	110	0.20	0.35	8	8	12	0	0	0
100	<i>all</i>	49.64	113.69	88.91	154	0.19	4.70	83	71	111	0	3	1
	{1500, 2500}	71.85	215.55	101.80	159	13.87	53.20	3	3	17	0	0	0
150	{500, 5000}	74.10	193.55	164.75	216	1.42	3.15	10	10	9	1	0	0
	{1000, 5000}	74.00	181.05	144.05	175	0.80	1.33	12	12	8	0	0	0
200	{1500, 5000}	74.15	167.65	142.70	205	0.43	0.67	9	9	11	0	0	0
	{500, 7500}	74.70	154.65	180.45	216	0.36	0.74	11	11	7	0	0	0
300	<i>all</i>	74.08	169.10	137.08	216	1.48	53.20	80	71	106	1	0	0
	{1500, 2500}	95.55	286.65	128.85	199	84.63	410.38	3	3	17	0	0	0
450	{500, 5000}	98.70	263.50	206.20	260	2.61	7.86	17	17	3	0	0	0
	{1000, 5000}	98.65	240.15	197.00	220	1.96	2.75	9	9	11	0	0	0
600	{1500, 5000}	98.70	225.25	189.80	263	1.22	1.76	7	7	12	0	1	0
	{500, 7500}	99.15	202.85	255.25	288	0.75	1.21	4	4	12	0	0	0
900	{1000, 7500}	99.40	188.45	238.95	293	0.52	0.82	7	9	10	0	0	0
	{1000, 10000}	99.45	163.90	193.25	246	0.31	1.93	2	0	6	0	0	0
1200	<i>all</i>	98.45	226.07	184.88	293	7.76	410.38	74	72	119	0	2	0

Table 27: Industrial and random CSP instances of Briant et al. (2005), $\epsilon_r = 10^{-3}$

name	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	e_{av}	e_{mx}	a_{av}	a_{mx}
ind_9	18.00	56.89	28.67	54	0.02	0.10	3.7E-07	3.3E-06	9.8E-03	4.2E-02
50b100c4	49.70	129.25	97.70	110	0.08	0.15	2.5E-05	1.4E-04	6.8E-02	2.8E-01
u120	63.20	88.75	103.45	124	0.02	0.03	2.8E-15	2.1E-14	4.8E-04	4.4E-03
u250	77.25	129.00	118.45	147	0.02	0.04	4.0E-06	4.8E-05	4.7E-03	1.5E-02
t120	86.15	110.75	87.85	98	0.01	0.02	0.0E+00	0.0E+00	1.4E-03	5.1E-03
t249	140.10	199.15	148.70	160	0.05	0.07	0.0E+00	0.0E+00	2.0E-03	6.0E-03

Table 28: Industrial and random CSP instances of Briant et al. (2005), $\epsilon_r = 10^{-4}$

name	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	e_{av}	e_{mx}	a_{av}	a_{mx}
ind_9	18.00	56.89	31.67	67	0.02	0.10	0.0E+00	0.0E+00	2.6E-05	2.0E-04
50b100c4	49.70	129.25	109.10	141	0.15	0.39	0.0E+00	0.0E+00	8.9E-13	4.6E-12
u120	63.20	88.75	96.60	124	0.02	0.03	1.3E-16	1.8E-15	4.5E-07	8.1E-06
u250	77.25	129.00	110.45	141	0.02	0.04	2.3E-15	3.4E-14	1.9E-05	1.7E-04
t120	86.15	110.75	76.85	98	0.01	0.02	0.0E+00	0.0E+00	2.8E-06	2.8E-05
t249	140.10	199.15	135.00	154	0.04	0.06	0.0E+00	0.0E+00	9.2E-06	5.4E-05

Figure 1: Performance profile for Degraeve and Peeters small-item-size instances, $m \geq 30$

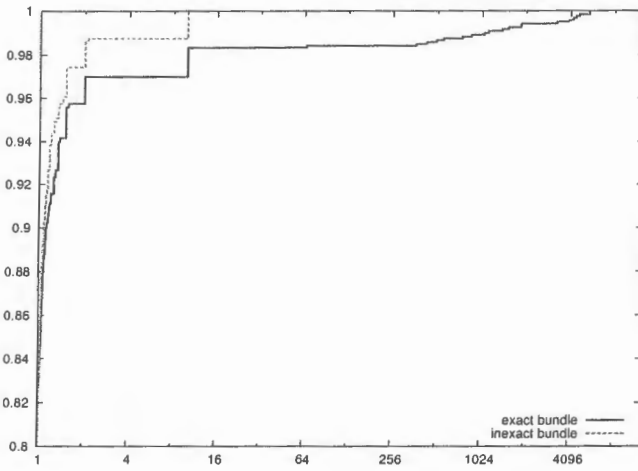


Figure 2: Performance profile for Degraeve and Peeters medium-item-size instances, $m \geq 30$

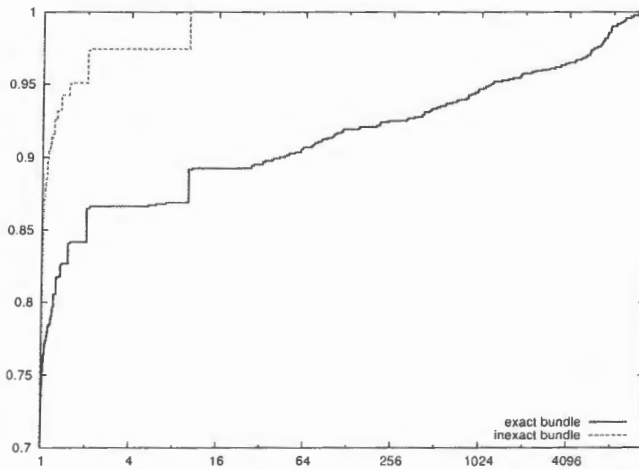


Figure 3: Performance profile for Wäscher and Gau instances, $m \geq 30$

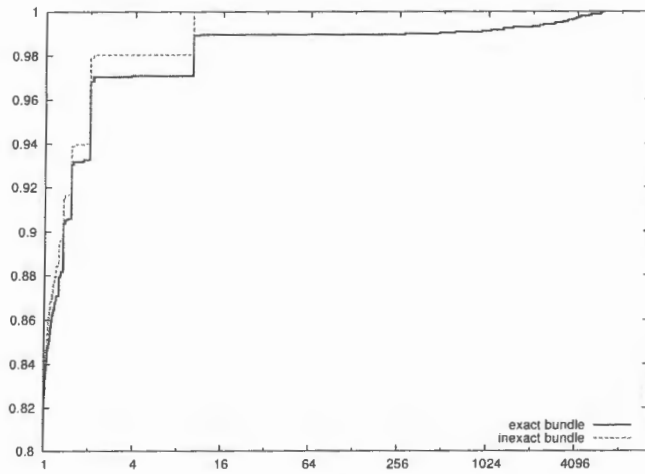


Table 29: Small-item-size instances of Degraeve and Peeters (2003), $\bar{d} = all$, $\epsilon_r = 10^{-3}$

m	int	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	H2	H3	H4	n_g
10	<i>all</i>	9.99	26.77	15.17	31	0.00	0.01	112	48	71	1	0	0
20	<i>all</i>	19.95	53.13	32.22	69	0.00	0.05	111	57	71	0	0	3
30	<i>all</i>	29.91	79.76	51.18	91	0.01	0.26	126	80	62	0	0	3
40	<i>all</i>	39.85	105.55	69.08	131	0.03	1.05	120	91	60	0	0	5
50	<i>all</i>	49.75	132.16	88.54	181	0.05	0.46	122	93	62	0	0	5
75	<i>all</i>	74.36	197.32	141.58	242	0.21	5.32	142	119	42	0	3	15
100	<i>all</i>	98.92	263.36	196.09	416	0.65	14.45	136	120	48	0	1	22

Table 30: Medium-item-size instances of Degraeve and Peeters (2003), $\bar{d} = 50$, $\epsilon_r = 10^{-3}$

m	int	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	H2	H3	H4	n_g
10	<i>all</i>	9.98	23.09	17.55	29	0.00	0.01	52	46	114	0	1	0
20	<i>all</i>	19.95	45.58	33.68	52	0.00	0.08	59	42	122	0	1	1
30	<i>all</i>	29.84	68.47	51.97	93	0.02	0.33	76	77	101	0	0	1
40	<i>all</i>	39.78	90.65	68.33	114	0.03	0.58	74	66	106	0	1	0
50	<i>all</i>	49.64	113.69	86.85	178	0.08	2.13	74	69	114	1	1	5
75	<i>all</i>	74.08	169.10	133.93	216	0.81	49.70	84	86	92	0	0	13
100	<i>all</i>	98.45	226.07	180.35	297	2.79	102.61	68	80	110	0	3	17

Table 31: CSP instances of Wäscher and Gau (1996), $int = all$, $\bar{d} = all$, $\epsilon_r = 10^{-3}$

m	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	H2	H3	H4	n_g
10	9.99	25.37	14.24	31	0.00	0.02	448	133	193	0	0	0
20	19.96	50.46	30.58	61	0.00	0.08	457	212	208	0	2	3
30	29.90	75.72	47.95	111	0.01	0.26	480	268	170	0	2	6
40	39.84	100.10	64.21	120	0.03	2.70	479	300	167	0	3	17
50	49.73	125.22	82.78	169	0.04	1.27	524	349	126	1	3	16
<i>all</i>	29.88	75.37	47.95	169	0.02	2.70	2388	1262	864	1	10	42

Table 32: Small-item-size instances of Degraeve and Peeters (2003), $\bar{d} = all$, $\epsilon_r = 10^{-4}$

m	int	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	H2	H3	H4	n_g
10	<i>all</i>	9.99	26.77	15.14	31	0.00	0.01	113	49	70	1	0	0
20	<i>all</i>	19.95	53.13	32.58	70	0.01	0.04	118	62	66	0	0	0
30	<i>all</i>	29.91	79.76	52.19	91	0.02	0.23	129	84	58	0	0	1
40	<i>all</i>	39.85	105.55	70.91	139	0.04	0.36	133	97	54	0	0	0
50	<i>all</i>	49.75	132.16	90.89	211	0.08	0.66	136	104	53	0	0	0
75	<i>all</i>	74.36	197.32	143.78	239	0.23	2.00	145	118	47	0	0	0
100	<i>all</i>	98.92	263.36	188.89	311	0.38	2.74	163	135	35	0	1	1

Table 33: Medium-item-size instances of Degraeve and Peeters (2003), $\bar{d} = 50$, $\epsilon_r = 10^{-4}$

m	int	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	H2	H3	H4	n_g
10	<i>all</i>	9.98	23.09	17.52	29	0.00	0.01	54	48	112	0	1	0
20	<i>all</i>	19.95	45.58	35.04	58	0.01	0.09	68	50	114	0	1	0
30	<i>all</i>	29.84	68.47	53.83	93	0.02	0.16	75	75	103	0	0	0
40	<i>all</i>	39.78	90.65	70.22	120	0.03	0.57	76	69	104	0	0	0
50	<i>all</i>	49.64	113.69	89.36	156	0.06	0.88	79	69	114	1	1	1
75	<i>all</i>	74.08	169.10	137.87	232	0.36	8.53	84	75	103	0	0	1
100	<i>all</i>	98.45	226.07	186.87	295	1.44	61.50	76	75	114	0	4	0

Table 34: CSP instances of Wäscher and Gau (1996), $int = all$, $\bar{d} = all$, $\epsilon_r = 10^{-4}$

m	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	H2	H3	H4	n_g
10	9.99	25.37	14.27	35	0.00	0.02	449	134	192	0	0	0
20	19.96	50.46	30.79	61	0.01	0.06	487	242	181	0	2	0
30	29.90	75.72	48.37	105	0.01	0.13	502	280	162	0	1	0
40	39.84	100.10	65.26	136	0.03	0.31	507	318	155	0	2	2
50	49.73	125.22	84.93	171	0.14	60.70	529	344	135	0	4	1
<i>all</i>	29.88	75.37	48.72	171	0.04	60.70	2474	1318	825	0	9	3

Table 35: Small-item-size instances of Degraeve and Peeters (2003), $\bar{d} = all$, $\epsilon_a = 0.05$

m	int	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	H2	H3	H4	n_g
10	<i>all</i>	9.99	26.77	15.18	31	0.00	0.01	112	48	71	1	0	0
20	<i>all</i>	19.95	53.13	32.50	69	0.00	0.04	116	60	68	0	0	0
30	<i>all</i>	29.91	79.76	52.21	91	0.02	0.23	127	82	60	0	0	1
40	<i>all</i>	39.85	105.55	71.10	139	0.04	0.35	133	97	54	0	0	0
50	<i>all</i>	49.75	132.16	90.96	181	0.08	0.66	135	103	54	0	0	0
75	<i>all</i>	74.36	197.32	143.24	256	0.24	2.00	144	117	48	0	0	0
100	<i>all</i>	98.92	263.36	184.83	311	0.38	2.81	164	135	35	0	1	0

Table 36: Medium-item-size instances of Degraeve and Peeters (2003), $\bar{d} = 50$, $\epsilon_a = 0.05$

m	int	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	H2	H3	H4	n_g
10	<i>all</i>	9.98	23.09	17.54	29	0.00	0.01	53	47	113	0	1	0
20	<i>all</i>	19.95	45.58	34.66	55	0.01	0.14	64	47	117	0	1	1
30	<i>all</i>	29.84	68.47	53.82	93	0.02	0.16	73	73	105	0	0	0
40	<i>all</i>	39.78	90.65	70.21	120	0.04	0.59	73	66	107	0	0	0
50	<i>all</i>	49.64	113.69	89.41	156	0.06	0.89	75	66	117	1	1	1
75	<i>all</i>	74.08	169.10	137.33	232	0.37	8.60	86	77	101	0	0	1
100	<i>all</i>	98.45	226.07	185.37	295	1.44	62.23	74	73	116	0	4	0

Table 37: CSP instances of Wäscher and Gau (1996), $int = all$, $\bar{d} = all$, $\epsilon_a = 0.05$

m	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	H2	H3	H4	n_g
10	9.99	25.37	14.24	31	0.00	0.02	446	130	195	0	0	1
20	19.96	50.46	30.58	61	0.01	0.14	476	230	191	0	2	2
30	29.90	75.72	48.23	105	0.01	0.60	496	274	167	0	1	1
40	39.84	100.10	65.58	123	0.03	0.78	499	309	163	0	2	3
50	49.73	125.22	84.91	171	0.07	0.46	531	346	132	0	4	2
<i>all</i>	29.88	75.37	48.71	171	0.02	0.78	2448	1289	848	0	9	9

Table 38: Small-item-size instances of Degraeve and Peeters (2003), $\bar{d} = all$, $\epsilon_a = 0.01$

m	int	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	H2	H3	H4	n_g
10	<i>all</i>	9.99	26.77	15.14	31	0.00	0.01	113	49	70	1	0	0
20	<i>all</i>	19.95	53.13	32.55	69	0.01	0.05	118	62	66	0	0	0
30	<i>all</i>	29.91	79.76	51.95	91	0.02	0.23	129	84	58	0	0	1
40	<i>all</i>	39.85	105.55	70.32	134	0.04	0.35	134	98	53	0	0	0
50	<i>all</i>	49.75	132.16	90.23	181	0.08	0.66	135	103	54	0	0	0
75	<i>all</i>	74.36	197.32	141.99	256	0.24	2.00	146	119	45	0	1	0
100	<i>all</i>	98.92	263.36	184.07	311	0.40	2.81	167	138	31	0	2	0

Table 39: Medium-item-size instances of Degraeve and Peeters (2003), $\bar{d} = 50$, $\epsilon_a = 0.01$

m	int	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	H2	H3	H4	n_g
10	<i>all</i>	9.98	23.09	17.52	29	0.00	0.02	54	48	112	0	1	0
20	<i>all</i>	19.95	45.58	35.07	58	0.01	0.10	68	50	114	0	1	0
30	<i>all</i>	29.84	68.47	53.72	93	0.02	0.16	73	73	105	0	0	0
40	<i>all</i>	39.78	90.65	69.97	120	0.03	0.58	74	67	106	0	0	0
50	<i>all</i>	49.64	113.69	88.76	156	0.06	0.90	74	65	118	1	1	1
75	<i>all</i>	74.08	169.10	137.06	232	0.37	8.60	82	73	105	0	0	1
100	<i>all</i>	98.45	226.07	184.43	295	1.44	62.23	71	70	119	0	4	0

Table 40: CSP instances of Wäscher and Gau (1996), $int = all$, $\bar{d} = all$, $\epsilon_a = 0.01$

m	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	H2	H3	H4	n_g
10	9.99	25.37	14.27	35	0.00	0.02	449	134	192	0	0	0
20	19.96	50.46	30.73	61	0.01	0.05	487	242	181	0	2	0
30	29.90	75.72	48.23	105	0.01	0.14	502	280	162	0	1	0
40	39.84	100.10	65.06	123	0.03	0.32	509	320	153	0	2	2
50	49.73	125.22	84.78	171	0.07	0.46	529	344	135	0	4	1
<i>all</i>	29.88	75.37	48.61	171	0.02	0.46	2476	1320	823	0	9	3

Table 41: Small-item-size instances of Degraeve and Peeters (2003), $\bar{d} = all$, $bkmin = 0$

m	int	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	H2	H3	H4	n_g
10	<i>all</i>	9.99	26.77	19.32	51	0.00	0.01	116	52	67	1	0	0
20	<i>all</i>	19.95	53.13	43.90	161	0.00	0.04	120	69	59	0	0	0
30	<i>all</i>	29.91	79.76	77.51	271	0.01	0.28	126	83	58	0	1	1
40	<i>all</i>	39.85	105.55	113.00	430	0.03	0.26	136	103	46	0	2	0
50	<i>all</i>	49.75	132.16	156.08	530	0.07	0.56	142	112	44	0	1	0
75	<i>all</i>	74.36	197.32	285.02	780	0.22	2.26	150	123	39	0	3	0
100	<i>all</i>	98.92	263.36	394.15	1030	0.39	3.65	154	128	40	0	3	2

Table 42: Medium-item-size instances of Degraeve and Peeters (2003), $\bar{d} = 50$, $bkmin = 0$

m	int	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	H2	H3	H4	n_g
10	<i>all</i>	9.98	23.09	19.58	50	0.00	0.02	50	42	118	0	1	0
20	<i>all</i>	19.95	45.58	44.02	128	0.01	0.08	86	71	93	0	1	0
30	<i>all</i>	29.84	68.47	73.53	167	0.01	0.11	96	87	91	0	0	0
40	<i>all</i>	39.78	90.65	100.27	261	0.02	0.37	91	87	86	0	0	0
50	<i>all</i>	49.64	113.69	135.05	405	0.05	0.56	95	86	94	0	5	1
75	<i>all</i>	74.08	169.10	233.36	770	0.36	26.26	100	87	90	0	1	2
100	<i>all</i>	98.45	226.07	330.16	990	1.38	104.51	91	91	97	0	5	1

Table 43: CSP instances of Wäscher and Gau (1996), $int = all$, $\bar{d} = all$, $bkmin = 0$

m	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	H2	H3	H4	n_g
10	9.99	25.37	16.75	51	0.00	0.01	439	129	197	0	0	0
20	19.96	50.46	40.19	190	0.00	0.07	487	261	163	0	1	0
30	29.90	75.72	66.79	251	0.01	0.13	498	293	148	0	2	0
40	39.84	100.10	97.90	420	0.02	0.24	534	341	132	0	1	2
50	49.73	125.22	134.75	511	0.05	0.63	542	358	120	0	5	1
<i>all</i>	29.88	75.37	71.28	511	0.02	0.63	2500	1382	760	0	9	3

Table 44: Small-item-size instances of Degraeve and Peeters (2003), $\bar{d} = all$, $bkmin = 1000$

m	int	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	H2	H3	H4	n_g
10	<i>all</i>	9.99	26.77	15.13	31	0.00	0.01	113	49	70	1	0	0
20	<i>all</i>	19.95	53.13	32.77	77	0.00	0.04	123	66	62	0	0	0
30	<i>all</i>	29.91	79.76	53.28	131	0.02	0.15	132	88	55	0	0	0
40	<i>all</i>	39.85	105.55	73.24	221	0.03	0.26	138	103	48	0	0	0
50	<i>all</i>	49.75	132.16	97.07	300	0.07	0.52	143	110	47	0	0	0
75	<i>all</i>	74.36	197.32	155.75	376	0.20	1.89	158	125	40	0	0	0
100	<i>all</i>	98.92	263.36	202.22	781	0.34	3.62	157	127	42	0	2	0

Table 45: Medium-item-size instances of Degraeve and Peeters (2003), $\bar{d} = 50$, $bkmin = 1000$

m	int	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	H2	H3	H4	n_g
10	<i>all</i>	9.98	23.09	17.49	29	0.00	0.01	55	49	111	0	1	0
20	<i>all</i>	19.95	45.58	35.26	57	0.01	0.07	77	59	105	0	1	0
30	<i>all</i>	29.84	68.47	54.77	101	0.01	0.10	79	79	99	0	0	0
40	<i>all</i>	39.78	90.65	71.24	121	0.03	0.38	79	71	102	0	0	0
50	<i>all</i>	49.64	113.69	90.22	173	0.05	0.70	86	76	106	1	2	1
75	<i>all</i>	74.08	169.10	150.49	533	0.25	6.27	92	84	94	0	0	1
100	<i>all</i>	98.45	226.07	208.19	858	0.89	33.81	83	79	110	0	4	0

Table 46: CSP instances of Wäscher and Gau (1996), $int = all$, $\bar{d} = all$, $bkmin = 1000$

m	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	H2	H3	H4	n_g
10	9.99	25.37	14.36	35	0.00	0.02	449	134	192	0	0	0
20	19.96	50.46	30.88	65	0.00	0.07	483	240	183	0	2	0
30	29.90	75.72	48.66	111	0.01	0.11	509	291	151	0	1	0
40	39.84	100.10	66.64	171	0.03	0.28	514	323	149	0	3	1
50	49.73	125.22	89.11	306	0.06	0.72	538	351	127	0	5	0
<i>all</i>	29.88	75.37	49.93	306	0.02	0.72	2493	1339	802	0	11	1

Table 47: Small-item-size instances of Degraeve and Peeters (2003), $\bar{d} = all$, H5

m	int	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H5	n_g
10	<i>all</i>	9.99	26.77	17.60	39	0.00	0.03	68	119	6
20	<i>all</i>	19.95	53.13	36.16	75	0.01	0.03	56	123	14
30	<i>all</i>	29.91	79.76	58.37	111	0.02	0.33	49	141	14
40	<i>all</i>	39.85	105.55	79.28	143	0.05	0.35	37	150	17
50	<i>all</i>	49.75	132.16	102.44	189	0.10	0.64	34	154	20
75	<i>all</i>	74.36	197.32	158.43	272	0.27	1.61	31	161	32
100	<i>all</i>	98.92	263.36	202.92	311	0.40	2.96	32	169	40

Table 48: Medium-item-size instances of Degraeve and Peeters (2003), $\bar{d} = 50$, H5

m	int	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H5	n_g
10	<i>all</i>	9.98	23.09	18.76	29	0.00	0.02	15	161	17
20	<i>all</i>	19.95	45.58	37.65	58	0.01	0.07	19	165	36
30	<i>all</i>	29.84	68.47	56.24	93	0.02	0.17	8	178	63
40	<i>all</i>	39.78	90.65	72.87	120	0.04	0.68	9	173	82
50	<i>all</i>	49.64	113.69	92.10	156	0.06	0.75	13	185	88
75	<i>all</i>	74.08	169.10	140.06	235	0.30	8.56	15	177	111
100	<i>all</i>	98.45	226.07	187.75	295	1.24	49.64	5	193	127

Table 49: CSP instances of Wäscher and Gau (1996), $int = all$, $\bar{d} = all$, H5

m	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	n_g
10	9.99	25.37	15.84	39	0.00	0.02	325	314	26
20	19.96	50.46	34.31	77	0.02	8.36	252	405	48
30	29.90	75.72	54.34	111	0.04	13.00	230	428	56
40	39.84	100.10	73.75	136	0.07	14.42	204	464	69
50	49.73	125.22	95.60	181	0.09	0.55	192	466	90
<i>all</i>	29.88	75.37	54.77	181	0.04	14.42	1203	2077	289

Table 50: Small-item-size instances of Degraeve and Peeters (2003), $\bar{d} = all$, H6

m	int	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H5	n_g
10	<i>all</i>	9.99	26.77	17.60	39	0.00	0.03	68	120	2
20	<i>all</i>	19.95	53.13	36.16	75	0.01	0.04	56	128	3
30	<i>all</i>	29.91	79.76	58.37	111	0.02	0.33	49	141	7
40	<i>all</i>	39.85	105.55	79.28	143	0.05	0.35	37	151	7
50	<i>all</i>	49.75	132.16	102.44	189	0.10	0.64	34	155	7
75	<i>all</i>	74.36	197.32	158.43	272	0.27	1.61	31	163	15
100	<i>all</i>	98.92	263.36	202.92	311	0.40	2.96	32	171	20

Table 51: Medium-item-size instances of Degraeve and Peeters (2003), $\bar{d} = 50$, H6

m	int	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H5	n_g
10	<i>all</i>	9.98	23.09	18.76	29	0.00	0.01	15	161	7
20	<i>all</i>	19.95	45.58	37.65	58	0.01	0.06	19	165	4
30	<i>all</i>	29.84	68.47	56.24	93	0.02	0.17	8	178	9
40	<i>all</i>	39.78	90.65	72.87	120	0.04	0.68	9	173	14
50	<i>all</i>	49.64	113.69	92.10	156	0.06	0.75	13	185	25
75	<i>all</i>	74.08	169.10	140.06	235	0.30	8.56	15	178	32
100	<i>all</i>	98.45	226.07	187.75	295	1.24	49.63	5	193	45

Table 52: CSP instances of Wäscher and Gau (1996), $int = all$, $\bar{d} = all$, H6

m	m_{av}	m'_{av}	i_{av}	i_{mx}	t_{av}	t_{mx}	n_e	H1	n_g
10	9.99	25.37	15.84	39	0.00	0.02	325	325	3
20	19.96	50.46	34.31	77	0.02	8.40	252	416	18
30	29.90	75.72	54.34	111	0.04	12.93	230	435	18
40	39.84	100.10	73.75	136	0.07	14.48	204	471	30
50	49.73	125.22	95.60	181	0.09	0.55	192	476	41
<i>all</i>	29.88	75.37	54.77	181	0.04	14.48	1203	2123	110

the 1990s, the number of publications on the topic of the present paper has increased steadily. This is a reflection of the fact that the topic has become more and more important in the eyes of the scientific community. The present paper is a contribution to this research.

The paper is organized as follows. In Section 2, the basic concepts of the theory of the firm are reviewed. In Section 3, the theory of the firm is extended to include the possibility of multiple ownership. In Section 4, the theory of the firm is extended to include the possibility of multiple ownership and multiple ownership is shown to be a natural extension of the theory of the firm. In Section 5, the theory of the firm is extended to include the possibility of multiple ownership and multiple ownership is shown to be a natural extension of the theory of the firm.

In Section 6, the theory of the firm is extended to include the possibility of multiple ownership and multiple ownership is shown to be a natural extension of the theory of the firm. In Section 7, the theory of the firm is extended to include the possibility of multiple ownership and multiple ownership is shown to be a natural extension of the theory of the firm. In Section 8, the theory of the firm is extended to include the possibility of multiple ownership and multiple ownership is shown to be a natural extension of the theory of the firm.

In Section 9, the theory of the firm is extended to include the possibility of multiple ownership and multiple ownership is shown to be a natural extension of the theory of the firm. In Section 10, the theory of the firm is extended to include the possibility of multiple ownership and multiple ownership is shown to be a natural extension of the theory of the firm. In Section 11, the theory of the firm is extended to include the possibility of multiple ownership and multiple ownership is shown to be a natural extension of the theory of the firm.

In Section 12, the theory of the firm is extended to include the possibility of multiple ownership and multiple ownership is shown to be a natural extension of the theory of the firm. In Section 13, the theory of the firm is extended to include the possibility of multiple ownership and multiple ownership is shown to be a natural extension of the theory of the firm. In Section 14, the theory of the firm is extended to include the possibility of multiple ownership and multiple ownership is shown to be a natural extension of the theory of the firm.

In Section 15, the theory of the firm is extended to include the possibility of multiple ownership and multiple ownership is shown to be a natural extension of the theory of the firm. In Section 16, the theory of the firm is extended to include the possibility of multiple ownership and multiple ownership is shown to be a natural extension of the theory of the firm. In Section 17, the theory of the firm is extended to include the possibility of multiple ownership and multiple ownership is shown to be a natural extension of the theory of the firm.

In Section 18, the theory of the firm is extended to include the possibility of multiple ownership and multiple ownership is shown to be a natural extension of the theory of the firm. In Section 19, the theory of the firm is extended to include the possibility of multiple ownership and multiple ownership is shown to be a natural extension of the theory of the firm. In Section 20, the theory of the firm is extended to include the possibility of multiple ownership and multiple ownership is shown to be a natural extension of the theory of the firm.

In Section 21, the theory of the firm is extended to include the possibility of multiple ownership and multiple ownership is shown to be a natural extension of the theory of the firm. In Section 22, the theory of the firm is extended to include the possibility of multiple ownership and multiple ownership is shown to be a natural extension of the theory of the firm. In Section 23, the theory of the firm is extended to include the possibility of multiple ownership and multiple ownership is shown to be a natural extension of the theory of the firm.

