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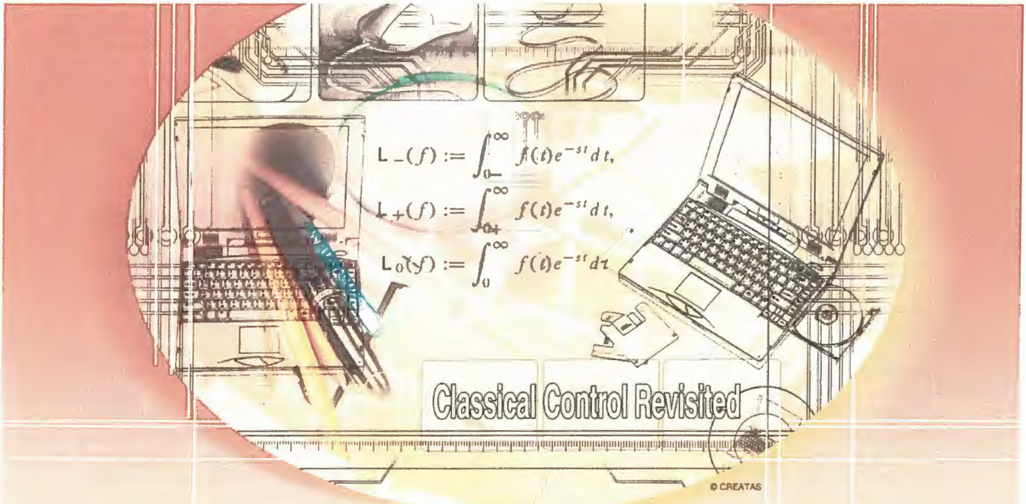

Root–Locus Invariance

EXPLOITING ALTERNATIVE ARRIVAL AND DEPARTURE POINTS

WIESLAW KRAJEWSKI and UMBERTO VIARO

Since its introduction by Walter H. Evans in the 1950s [1], the root-locus method has continued to be of interest to control engineers [2], [3]. Indeed, this graphic tool is invaluable for developing s-plane intuition with respect to both analysis and synthesis problems. Consequently, the root locus still forms an integral part of most undergraduate control courses, with rules for its construction given in all introductory textbooks on feedback systems. Here we focus attention on an invariance property of root-locus diagrams. Specifically, we show that the same root locus can be described by different equations, and that the rules for constructing a root locus can be expressed in terms of each of these representations. This property is exploited to find the locus asymptotes when the usual rule for their determination cannot be applied and to show how the locus behaves at points where two or more locus branches intersect. The invariance property is then used to provide insight into a stabilization procedure leading to a circle-shaped loop Nyquist diagram around the critical point $-1 + j0$.

Let $d(s)$ and $n(s)$ be coprime real monic polynomials with $\mu := \deg[n(s)] \leq \nu := \deg[d(s)]$, and consider the roots of the equation

$$d(s) + Kn(s) = 0, \tag{1}$$

where the real parameter K, called the *varying parameter*, varies from $-\infty$ to ∞. The roots of (1) coincide with the poles of the closed-loop transfer function

$$T(s) = \frac{Y(s)}{R(s)} = \frac{K\,n(s)}{d(s) + K\,n(s)}$$

of the unity-feedback system shown in Figure 1 whose loop transfer function is

$$L(s) = K\frac{n(s)}{d(s)}. \tag{2}$$

It is customary to distinguish between the part of the locus corresponding to positive values of K, called the *positive root locus*, and the part corresponding to negative values of K, called the *negative root locus*. The union of the two

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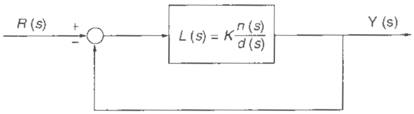

FIGURE 1 Unity-feedback system. The rules for constructing a root locus are usually given with reference to this structure, where K is the varying parameter.

is the *complete root locus*. The negative root locus is considered less frequently because, in most cases, K in (2) must be positive to ensure stability of the closed-loop system. Sometimes, however, stability is possible only with negative values of K. For example, this is the case for the unstable, nonminimum-phase loop transfer function $L(s) = K(s-1)/(s-2)$.

From (1), it follows that all points s of the positive locus satisfy the odd phase condition

$$\arg[d(s)] - \arg[n(s)] = (2k+1)\pi, \qquad (3)$$

while all points s of the negative locus satisfy the even phase condition

$$\arg[d(s)] - \arg[n(s)] = 2k\pi, \qquad (4)$$

where k is an integer. Relationships (3) and (4) are critical for the development that follows.

The root locus (1) is usually constructed starting from the roots of $d(s)$ and $n(s)$. The roots of $d(s)$ are called the *departure points* because they coincide with the roots of (1) for $K = 0$. For $K \neq 0$, the roots of (1) coincide with those of

$$\frac{d(s)}{K} + n(s) = 0. \qquad (5)$$

As $K \to \pm\infty$, μ roots of (5) tend to the μ roots of $n(s)$, which are called the *arrival points*, while the remaining $\nu - \mu$ roots, if any, tend asymptotically to infinity. As shown in the section "Asymptotes of the Complete Root Locus," the $\nu - \mu$ asymptotes approached by the roots of (5) as $K \to +\infty$ differ from the $\nu - \mu$ asymptotes approached as $K \to -\infty$.

Most of the rules for plotting a root locus concern the sets of departure and arrival points. Evans [1] showed that (1) can be solved by plotting the locus of points s that have a simple relationship with the departure and arrival points. The dependence of the locus plot on the arrival and departure points might suggest that changing these points necessarily changes the locus shape. For example, consider $0 = d(s) + Kn(s) = d(s) + K_P n(s) + (K - K_P)n(s) = d'(s) + K'n(s)$, where $d'(s) := d(s) + K_P n(s)$ and $K' := K - K_P$. In this case, the complete root locus remains unchanged if the original departure points are replaced by the roots of $d'(s)$. This simple property is illustrated in Figure 2. The next section investigates more general sets of departure and arrival points that lead to the same complete root locus.

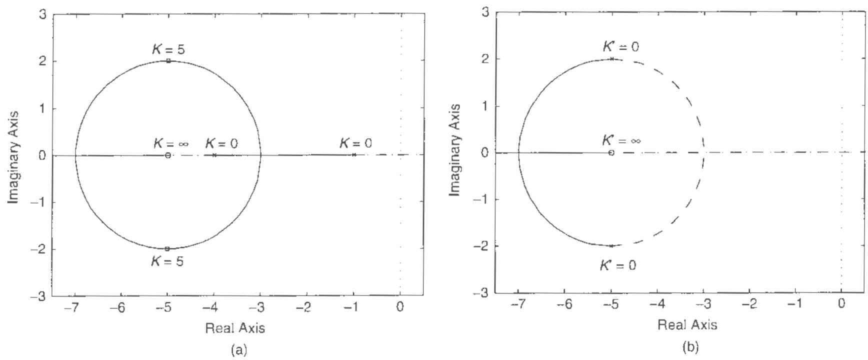

FIGURE 2 Moving the departure points along the root locus. When the departure points are moved from their original positions denoted by small crosses in a) to the positions denoted by small squares in a) corresponding to an arbitrary nonzero value of the varying parameter K, the complete root locus remains unchanged. Plot a) shows the root locus for $(s+1)(s+4) + K(s+5) = 0$, whose departure points are -1 and -4. The positive locus is represented by a solid line, and the negative locus by a dashed line. The roots corresponding to $K = 5$ are $P_{1,2} = -5 \pm j2$. Plot b) shows the locus for $(s - P_1)(s - P_2) + K'(s+5)$, whose departure points are $P_{1,2}$ instead of -1 and -4.

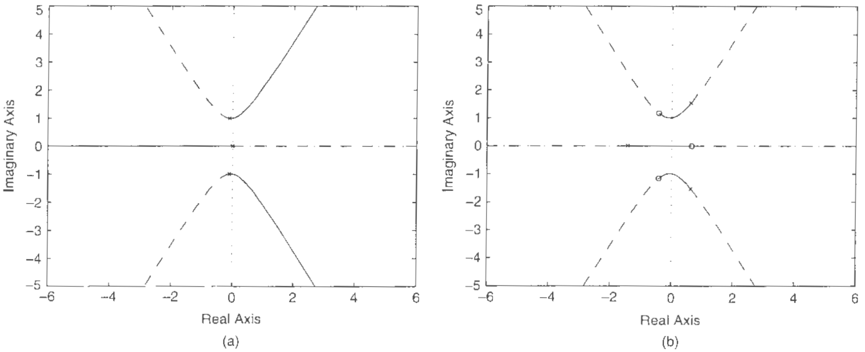

FIGURE 3 Invariance of the complete root locus. Each loop transfer function whose numerator and denominator are linear combinations of the polynomials $d(s)$ and $n(s)$ gives rise to the same complete root locus. a) The complete root locus constructed from the poles of the all-pole function $L(s) = Kn(s)/d(s)$, where $n(s) = 1$ and $d(s) = s^3 + 0.2s^2 + s = s(s + 0.1 + j\,0.995)(s + 0.1 - j\,0.995)$. b) The same complete root locus constructed from the poles and zeros of $L'(s) = K'p_n(s)/p_d(s)$, where $p_n(s) = d(s) - n(s) = (s - 0.6464)(s + 0.4232 + j\,1.1696)\quad(s + 0.4232 - j1.1696)$, $p_d(s) = d(s) + 4n(s) = (s + 1.4383)(s - 0.6192 + j\,1.5485)(s - 0.6192 - j\,1.5485)$, and $K' = (4 - K/1 + K)$.

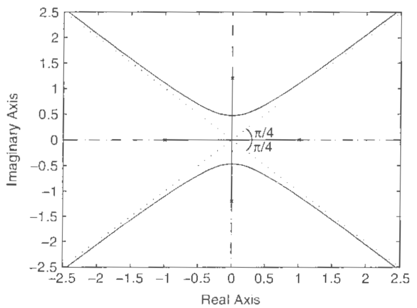

FIGURE 4 Angles formed by the asymptotes. The complete root locus is shown for the strictly proper loop transfer function $L(s) = Kn(s)/d(s)$, where $n(s) = 1$ and $d(s) = (s^2 - 1)(s^2 + 1.44)$. Each of the four negative–locus asymptotes, which overlap the four coordinate semi–axes and are denoted by dashed lines, bisects the angle of $\pi/2$ rad formed by two adjacent asymptotes, denoted by dotted lines, of the positive root locus.

ROOT-LOCUS INVARIANCE

Consider a real variable $K' \neq -1$ and the linear fractional transformation

$$K = f(K') = \frac{\alpha + \beta K'}{1 + K'}, \tag{6}$$

where α, β are distinct real numbers. Substituting (6) for K in (1) and rearranging, we obtain

$$[d(s) + \alpha\, n(s)] + K'\,[d(s) + \beta\, n(s)] = 0. \tag{7}$$

Since (6) defines a one-to-one mapping from the extended real axis $\overline{\mathbb{R}} := [-\infty, \infty]$ onto itself [4, p. 26] the locus described by the roots of (1) as K varies over $\overline{\mathbb{R}}$ is exactly the same as the locus described by the roots of (7) as K' varies over $\overline{\mathbb{R}}$. In particular, if $\mu < \nu$ and $\alpha - \beta > 0$, the asymptotes approached by the roots of (1) as $K \to +\infty$ and $K \to -\infty$ coincide with the asymptotes approached by the roots of (7) as $K' \to -1$ from the right and left, respectively. The left and right limits as $K' \to -1$ are reversed if $\alpha - \beta < 0$.

Equation (7) can be regarded as the equation of the locus describing the poles of the unity–feedback system whose loop transfer function is

$$L'(s) = K'\frac{p_n(s)}{p_d(s)}, \tag{8}$$

where

$$p_d(s) := d(s) + \alpha\, n(s), \quad p_n(s) := d(s) + \beta\, n(s). \tag{9}$$

The complete root locus for (2) is therefore equal to the complete root locus for (8), whose branches depart for $K' = 0$ from the roots of $p_d(s)$, where $K = \alpha$, and arrive for $K' \to +\infty$ at the roots of $p_n(s)$, where $K = \beta$. Depending on the values of α and β, the zeros and poles of (8) can belong to either the positive or negative locus for (2).

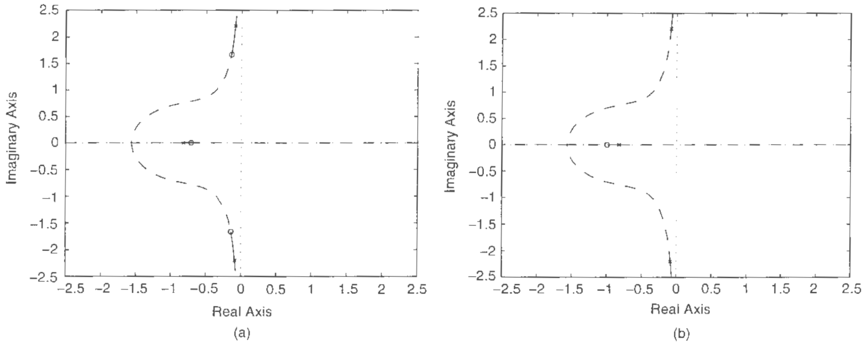

FIGURE 5 Determining the asymptotes of the complete root locus. The positive root locus for an exactly proper loop transfer function $L(s) = Kn(s)/d(s)$ has no asymptotes, whereas at least one branch of negative locus for $L(s)$ goes to infinity. According to the invariance property, to find the asymptotes of the complete root locus when $\mu = \nu$ it suffices to consider a strictly proper loop transfer function $L'(s)$ that gives rise to the same complete root locus as the exactly proper $L(s)$. The simplest transfer function of this kind is $L'(s) = K'[d(s) - n(s)]/d(s)$. a) The complete root locus for $L(s) = Kn(s)/d(s)$, where $n(s) = s^3 + s^2 + 3s + 2 = (s + 0.7152)(s + 0.1424 + j1.6661)(s + 0.1424 - j1.6661)$ and $d(s) = s^3 + s^2 + 5s + 4 = (s + 0.8239)(s + 0.0880 + j2.2016)(s + 0.0880 - j2.2016)$. b) The same complete root locus is generated by the strictly proper loop transfer function $L'(s) = K'[d(s) - n(s)]/d(s) = K'(2s + 2)/(s^3 + s^2 + 5s + 4)$ with the same poles as $L(s)$ and one zero at -1.

Figure 3 shows that the complete root locus for the loop transfer function $L(s) = Kn(s)/d(s)$, where $n(s) = 1$ and $d(s) = s^3 + 0.2s^2 + s$, is equal to the complete root locus for the loop transfer function $L'(s) = K'p_n(s)/p_d(s)$, where $p_n(s)$ and $p_d(s)$ are given by (9) with $\alpha = 4$ and $\beta = -1$.

ASYMPTOTES OF THE COMPLETE ROOT LOCUS

If (2) is strictly proper with pole-zero excess $\lambda := \nu - \mu > 0$, the number of asymptotes of the complete root locus (1) is 2λ, where λ asymptotes pertain to the positive root locus and λ to the negative root locus. If $\lambda \geq 2$, then all of the 2λ asymptotes intersect at a single real point, called the *center of the asymptotes* and located at $\tau = [\sum_i p_i - \sum_i z_i]/\lambda$, where p_i and z_i are the poles and zeros of $L(s)$, respectively [5]. Each asymptote of the negative locus bisects the angle formed by two adjacent asymptotes of the positive locus, as shown in Figure 4.

If $L(s)$ is exactly proper, that is, $\mu = \nu$, then no branch of the positive locus approaches infinity. However, the entire real axis necessarily belongs to the complete root locus because to every real point s, there corresponds a real value of K satisfying (1), that is, $K = -d(s)/n(s)$ [5]. Therefore, the complete root locus has at least two asymptotes. To find the asymptotes of the complete root locus when $\mu = \nu$, according to the invariance property, it suffices to consider a strictly proper loop transfer function $L'(s)$ that gives rise to the same complete root locus as the exactly proper $L(s)$. The simplest $L'(s)$ of this kind is

$$L'(s) = K' \frac{p_n(s)}{p_d(s)} = K' \frac{d(s) - n(s)}{d(s)}. \tag{10}$$

where $p_n(s)$ and $p_d(s)$ are given by (9) with $\alpha = 0$ and $\beta = -1$. The pole-zero excess of (10) is $\lambda' = \deg[p_n(s)] - \deg[d(s)] > 0$ and, consequently, the complete root locus for both $L'(s)$ and $L(s)$ has $2\lambda'$ asymptotes. For instance, if $n(s) = s^3 + s^2 + 3s + 2$ and $d(s) = s^3 + s^2 + 5s + 4$, then $p_n(s) = d(s) - n(s) = 2s + 2$ and the pole-zero excess λ' of $L'(s)$ is two. It follows that two branches of the complete root locus for both $L'(s)$ and $L(s)$ go to infinity and return from there, as shown in Figure 5. To find the center τ of the $2\lambda'$ asymptotes of the complete root locus for $L'(s)$ and thus for the exactly proper $L(s)$, we denote the numerator and denominator polynomials $n(s)$ and $d(s)$ of the exactly proper $L(s)$ by

$$n(s) = s^\nu + b_{\nu-1}s^{\nu-1} + \cdots + b_0,$$
$$d(s) = s^\nu + a_{\nu-1}s^{\nu-1} + \cdots + a_0.$$

In addition, let $\lambda' \geq 2$ and assume

$$b_{\nu-i} = a_{\nu-i}, \; i = 1, 2, \ldots, \lambda' - 1$$

and

$$b_{\nu-\lambda'} \neq a_{\nu-\lambda'}.$$

Since the sum of the poles of (10) is equal to $-a_{\nu-1}$ and the

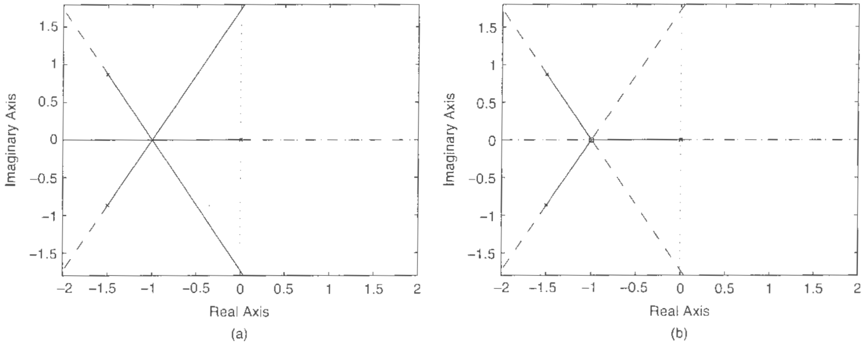

FIGURE 6 Configuration of the root locus near the breakaway points. At a breakaway point of multiplicity σ, the tangents to the branches of the root locus entering and leaving the breakaway point form 2σ angles of equal measure. This configuration can be explained by viewing the breakaway point as a multiple point of arrival for a complete root locus of the same shape. Plot a) shows the complete root locus for $L(s) = Kn(s)/d(s)$, where $n(s) = 1$ and $d(s) = s(s^2 + 3s + 3) = s(s + 1.5 - j\,0.866)(s + 1.5 - j\,0.866)$. Plot b) shows the complete root locus for $L'(s) = K'(s+1)^3/d(s)$ with a zero at the triple point -1 of the locus for $L(s)$.

sum of its zeros is equal to $-(a_{\nu-\lambda'-1} - b_{\nu-\lambda'-1})/(a_{\nu-\lambda'} - b_{\nu-\lambda'})$, the center of the asymptotes is

$$\tau = \frac{1}{\lambda'}\left[\frac{a_{\nu-\lambda'-1} - b_{\nu-\lambda'-1}}{a_{\nu-\lambda'} - b_{\nu-\lambda'}} - a_{\nu-1}\right].$$

In the example of Figure 5, the center is $\tau = 0$.

LOCUS CONFIGURATION AT THE BREAKAWAY POINTS

The invariance property can be exploited to show how the locus behaves at a *breakaway point*, that is, a point where two or more locus branches intersect. The approach involves making the breakaway point a multiple departure or arrival point for a complete root locus of identical shape.

Let us assume that a breakaway point B of multiplicity σ corresponds to the value $K_B \neq 0$ of the varying parameter K in (1) so that

$$p_B(s) := d(s) + K_B\, n(s) = (s - B)^{\sigma}\, \bar{p}_B(s), \tag{11}$$

where $\bar{p}_B(s)$ is a polynomial of degree $\nu - \sigma$. The locus configuration in the neighborhood of B can be analyzed by considering the loop transfer function

$$L'_B(s) = K'\frac{p_B(s)}{d(s)} \tag{12}$$

whose zeros are the roots of (11). Therefore, the point B is an arrival point of multiplicity σ on the locus for (12).

Evaluating the odd phase condition (3) at a point arbitrarily close to B on the positive root locus for (12), we obtain

$$\begin{aligned}\arg[d(B)] - \arg[p_B(B)] &= \arg[d(B)] - \sigma\phi_B - \arg[\bar{p}_B(B)] \\ &= (2k+1)\pi.\end{aligned} \tag{13}$$

where ϕ_B is the angle of the tangent to a locus branch approaching B along the positive locus. From (13), it follows that

$$\phi_B = -\frac{(2k+1)\pi}{\sigma} + \Theta_B, \tag{14}$$

where

$$\Theta_B := \frac{\arg[d(B)] - \arg[\bar{p}_B(B)]}{\sigma}.$$

As is known [5], (14) provides σ different angles (one for each entering branch) obtainable by setting $k = 0, \ldots, \sigma - 1$.

Similarly, evaluating the even phase condition (4) at a point arbitrarily close to B on the negative root locus for (12), the angles ψ_B formed by the tangents to the σ locus branches approaching B along the negative locus are given by

$$\psi_B = -\frac{2k\pi}{\sigma} + \Theta_B.$$

Therefore, as shown in Figure 6, each of the σ branches arriving at B bisects, at least locally, the angle of $2\pi/\sigma$ rad formed by two adjacent branches departing from B.

ROOT–LOCUS CHARACTERIZATION OF THE ALL-PASS STABILIZATION PROCEDURE

In this section, we use the root-locus invariance property to characterize a stabilization procedure leading to a circle-shaped Nyquist plot around the critical point $-1 + j0$. After motivating the stabilization procedure, we note that, if the Nyquist diagram of the loop transfer function $L(s) := G_c(s)G_p(s)$, where $G_c(s)$ and $G_p(s)$ are the controller and plant transfer functions, travels along a circle centered at the critical point, then $E(s) := 1 + L(s)$ is all-pass. Next, we point out that the complete root locus constructed from the poles and zeros of $L(s)$ is the same as the complete root locus constructed from the poles and zeros of $E(s)$. Finally, an algorithm for synthesizing the stabilizing controller $G_c(s)$ is provided and then illustrated with the aid of two examples.

Since open-left-half-plane (OLHP) zeros and poles can safely be canceled, we assume that all of the zeros and poles of $G_p(s)$ are in the open right half-plane (ORHP). We also assume, for convenience, that $G_p(s)$ is exactly proper. When the pole-zero excess λ of $G_p(s)$ is greater than zero, the stabilization procedure can be applied to the fictitious exactly proper plant transfer function

$$\hat{G}_p(s) := (-z)^{-\lambda}(s-z)^{\lambda}\, G_p(s), \qquad (15)$$

where z is chosen to be positive and sufficiently large so that the zero at infinity of multiplicity λ of $G_p(s)$ is converted into a large but finite ORHP zero of the same multiplicity. For instance, if $G_p(s) = (s-2)/[(s-1)(s-3)]$, the procedure can be applied to

$$\hat{G}_p(s) = -\frac{(s-2)(s-1000)}{1000(s-1)(s-3)},$$

as shown at the end of this section.

Motivation

Gain and phase margins have traditionally been important measures of stability robustness since if either is small, the system is close to instability. However, the gain and phase margins can be large and yet the Nyquist diagram of the loop transfer function can pass close to the critical point $-1 + j0$. A better measure of stability robustness is provided by the distance δ_c from the critical point to the nearest point on the Nyquist plot of the loop transfer function $L(s)$. This distance is given by [6]

$$\delta_c = \frac{1}{\sup_\omega |S(j\,\omega)|},$$

that is, the reciprocal of the infinity norm of the sensitivity function $S(s) = 1/[1 + L(s)]$.

When the plant $G_p(s)$ is unstable and nonminimum phase, the maximum attainable stability margins can be tiny (upper bounds on their values are given, for instance, in [6]) and the robust stabilization problem is more

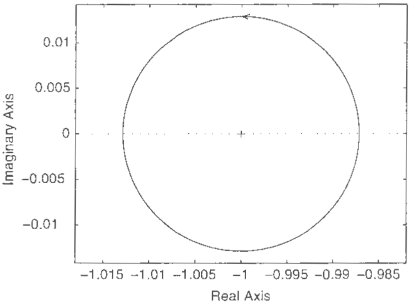

FIGURE 7 Circle–shaped loop Nyquist diagram produced by the all-pass stabilization procedure. When the plant transfer function $G_p(s)$ is nonminimum phase, the maximum attainable stability margins can be very small. The stabilization procedure ensures that all of the points on the loop Nyquist diagram are at a uniform distance from the critical point $-1 + j\,0$. This figure shows the Nyquist diagram of $L(s) = G_c(s)G_p(s)$, where $G_p(s) = [(s-1)(s-3)]/[(s-2)(s-4)]$ is the exactly proper plant transfer function and $G_c(s) = -1.01286(s - 2.95619)/(s - 1.13746)$ is the controller transfer function leading to the desired circle–shaped Nyquist diagram. As required by the Nyquist criterion, this diagram encircles the critical point three times in the counterclockwise direction. The radius of the circle–shaped diagram is $\delta_c = 0.01286$, which is equal to the reciprocal of the minimum infinity norm of the all-pass sensitivity $S(s)$.

delicate. Here we consider a stabilization procedure that ensures an approximately circle-shaped Nyquist loop around the critical point so that all of the points of $L(j\,\omega)$ are about the same distance δ_c from $-1 + j\,0$. Note, however, that δ_c may be small. The intent of this procedure is consistent with the fact that minimizing the maximum of $|S(j\,\omega)|$ leads to an all-pass sensitivity transfer function $S(s)$ [7], that is, with $|S(j\,\omega)|$ constant.

Nyquist Diagrams of $L(s)$ and $1 + L(s)$

If the Nyquist diagram of $L(s)$ travels along a circle centered at $-1 + j\,0$, then the Nyquist diagram of $E(s) = 1 + L(s)$, whose poles are the same as those of $L(s)$, travels along a circle of the same radius centered at the origin. Therefore, $E(s)$ is an all-pass function, that is, $|E(j\,\omega)|$ is constant, which implies that the zeros of $E(s)$ are the negatives of its poles.

Note, by the way, that the denominator of $E(s)$ is the product of the denominators $d_c(s)$ and $d_p(s)$ of $G_c(s)$ and $G_p(s)$, respectively, because internal stability does not permit cancelations of ORHP poles and zeros [6]. This fact is used in the subsection dealing with the controller synthesis.

Root Locus for $L(s)$ and $1 + L(s)$

According to the root-locus invariance property, the zeros of

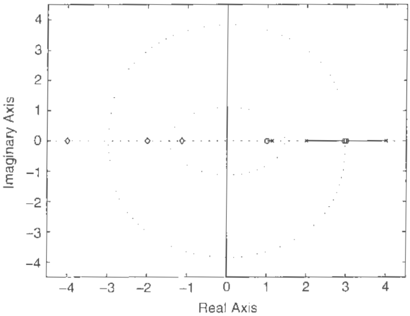

FIGURE 8 Root-locus characterization of the stabilization technique. The complete root locus for the exactly proper loop transfer function $L(s)$ is symmetric with respect to the imaginary axis because its branches must pass simultaneously through the points, denoted by small diamonds, that are symmetric to the poles of $L(s)$ with respect to the imaginary axis. The figure shows the complete root locus constructed from the poles and zeros of the transfer function $L(s) = G_c(s)G_p(s)$, where $G_p(s)$ and $G_c(s)$ are as in Figure 7. The positive root locus consists of the segments (1, 1.13746), (2, 2.95619) and (3, 4). The gain −1.01286 of $L(s)$ is the value taken by the varying parameter K in $d(s) + Kn(s) = 0$, where $d(s) = (s - 1.13746)(s - 2)(s - 4)$ and $n(s) = (s - 2.95619)(s - 1)(s - 3)$, at the negatives of the poles of $L(s)$, that is, $-1.01286 = -d(-2)/n(-2) = -d(-4)/n(-4) = -d(-1.13746)$

$$E(s) = 1 + L(s) = \frac{d(s) + K\,n(s)}{d(s)}, \tag{16}$$

which lie on the complete root locus associated with $L(s)$, can be viewed as a new set of arrival points for the same locus. To ensure that the zeros of (16) are the negatives of the roots of $d(s)$, the polynomials $d(s) + K\,n(s)$ and $d(-s)$, both of degree ν, must coincide except for a constant proportionality factor r, that is,

$$d(s) + K\,n(s) = r\,d(-s). \tag{17}$$

Since the leading coefficients of $d(s) + K\,n(s)$ and $d(-s)$ are, respectively, $1 + K$ and $(-1)^\nu$, the proportionality factor r is equal to

$$r = (1 + K)(-1)^\nu. \tag{18}$$

Equation (17) implies that the poles and zeros of the desired controller transfer function $G_c(s)$ must be chosen such that the complete root locus passes through the negatives of the poles of $L(s) = G_c(s)G_p(s)$. The complete root locus for the desired loop transfer function can therefore be plotted from two sets of departure and arrival points that are symmetric about the imaginary axis and, consequently, the complete root locus itself is symmetric about this axis.

Controller Synthesis

The polynomial identity (17) can be used to find the parameters of

$$G_c(s) = \frac{\hat{n}_c(s)}{d_c(s)} = \frac{K_c s^{\nu_c} + b_{\nu_c-1}s^{\nu_c-1} + b_{\nu_c-2}s^{\nu_c-2} + \cdots + b_0}{s^{\nu_c} + a_{\nu_c-1}s^{\nu_c-1} + a_{\nu_c-2}s^{\nu_c-2} + \cdots + a_0}. \tag{19}$$

Letting ν_p denote the degree of both the numerator and the denominator of the exactly proper plant transfer function

$$G_p(s) = K_p\frac{n_p(s)}{d_p(s)},$$

it follows from (17) and (18) that

$$d_c(s)d_p(s) + K_p\hat{n}_c(s)n_p(s) = (1 + K_cK_p)(-1)^{\nu_c+\nu_p}d_c(-s)d_p(-s). \tag{20}$$

By equating the coefficients of s^i, $i = 0, 1, \ldots, \nu_c + \nu_p - 1$, on both sides of (20) (the coefficients of $s^{\nu_c+\nu_p}$ are necessarily equal), we can form a set of $\nu_c + \nu_p$ equations in the $2\nu_c + 1$ parameters of (19). If we consider K_c as a fixed parameter, these equations are linear in the remaining $2\nu_c$ coefficients of (19). For $\nu_c = \nu_p - 1$, so that $\nu_p = \nu_c + 1$, the identity (20) yields a system of $2\nu_c + 1$ equations in the $2\nu_c$ unknown coefficients of $G_c(s)$. This system admits a solution only if K_c is such that the determinant Δ of the $(2\nu_c + 1) \times (2\nu_c + 1)$ system matrix is zero. To determine K_c, it is therefore necessary to solve the algebraic equation $\Delta = 0$. The remaining controller parameters are then found by inserting the resulting value of K_c into a set of $2\nu_c$ equations obtained by deleting one equation from the overdetermined system of $2\nu_c + 1$ equations. Further details can be found in [8].

Exactly Proper Example

Consider the second-order plant transfer function

$$G_p(s) = \frac{(s-1)(s-3)}{(s-2)(s-4)}. \tag{21}$$

Applying the above synthesis procedure to (21) with $\nu_c = \nu_p - 1 = 1$, we find $K_c = -1.01286$ and

$$G_c(s) = -1.01286\,\frac{s - 2.95619}{s - 1.13746}.$$

The circle-shaped Nyquist diagram of $L(s) = G_c(s)G_p(s)$ is shown in Figure 7. Its radius and, therefore, the minimum distance δ_c from $-1 + j\,0$ are equal to the absolute value of r in (18), that is,

$$\delta_c = |r| = |1 + K_cK_p| = 0.01286, \tag{22}$$

which is equal to the reciprocal of the minimum infinity norm of the all-pass sensitivity $S(s)$ [6]. The complete root locus for $L(s) = G_c(s)G_p(s)$ is shown in Figure 8. This locus is symmetric with respect to the imaginary axis because it can be plotted from the zeros and poles of the all-pass function $E(s) = 1 + L(s)$. As (17) indicates, the loop gain $K = K_cK_p$ is the value of the varying parameter corresponding to the negatives of the poles of $L(s)$ along the locus constructed from the poles and zeros of $L(s)$, that is,

$$K = -1.01286 = \frac{d(-2)}{n(-2)} = -\frac{d(-4)}{n(-4)} = \frac{-d(-1.13746)}{n(-1.13746)}.$$

Observe that the radius of the circle-shaped Nyquist plots for both $S(s) = 1/E(s)$ and $T(s) = 1 - S(s)$ is $1/|1+K|$. However, these Nyquist diagrams travel ν times in the clockwise direction around their respective centers at (0, 0) and (1, 0). This property implies, in particular, that $S(s)$ is a stable Blaschke product of order ν. Obviously, the poles and zeros of both $S(s)$ and $T(s)$ can play the role of departure and arrival points for the same complete locus.

Strictly Proper Example

Finally, consider the strictly proper plant transfer function

$$G_p(s) = \frac{s - 2}{(s-1)(s-3)}.$$

According to (15), the controller can be designed with reference to the fictitious plant transfer function

$$\hat{G}_p(s) = -\frac{(s-2)(s-1000)}{1000(s-1)(s-3)},$$

leading to

$$G_c(s) = 933.77\frac{s - 1.4993}{s - 874.8336}.$$

The Nyquist diagram for $\hat{L}(s) := G_c(s)\hat{G}_p(s)$ is exactly circular, whereas the diagram for $L(s) = G_c(s)G_p(s)$, shown in Figure 9, exhibits an almost circular shape at the lower frequencies and tends to the origin as $\omega \to \infty$. The minimum distance of the Nyquist plot of $L(s)$ from $-1 + j0$ is $\delta_c = 0.06$.

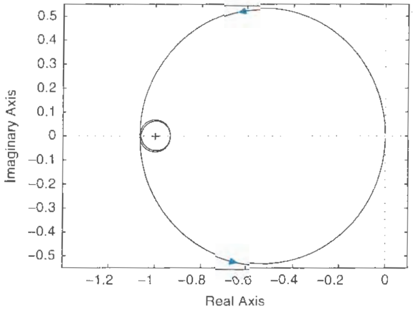

FIGURE 9 Loop Nyquist diagram resulting from the stabilization of a strictly proper plant. The stabilization procedure, which is applicable to an exactly proper plant with all of its poles and zeros in the ORHP, can be adapted to the case of a plant $G_p(s)$ with a pole–zero excess $\lambda > 0$. The controller can be determined with reference to the fictitious exactly proper plant transfer function $\hat{G}_p(s) = (-z)^{-\lambda}(s-z)^{\lambda}G_p(s)$ with z positive and sufficiently large. In this way, the zero at infinity of multiplicity λ of $G_p(s)$ is converted into a large but finite ORHP zero of $\hat{G}_p(s)$ of the same multiplicity. The figure shows the Nyquist diagram for $L(s) = G_c(s)G_p(s)$, where $G_p(s) = (s-2)/[(s-1)(s-3)]$ is the strictly proper plant transfer function and $G_c(s) = 933.77(s-1.4993)/(s-874.8336)$ is the controller transfer function determined with reference to the fictitious plant transfer function $\hat{G}_p(s) = -[(s-2)(s-1000)]/[1000(s-1)(s-3)]$. The minimum distance of the Nyquist plot of $L(s)$ from $-1 + j0$ is $\delta_c = 0.06$.

CONCLUSIONS

The same complete root locus is generated by all loop transfer functions whose numerator and denominator are linear combinations of the polynomials $n(s)$ and $d(s)$. Therefore, a complete root locus can be constructed starting from different sets of arrival and departure points. This property can be used to find the asymptotes of the negative locus for an exactly proper loop transfer function and to determine the configuration of the locus branches around the breakaway points. The root-locus invariance property can also be exploited to characterize a stabilization procedure leading to an exactly proper loop transfer function whose Nyquist diagram travels along a circle centered at $-1 + j0$.

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Callouts

Since its introduction by Walter H. Evans in the 1950s, the root-locus method has continued to be of interest to control engineers.

Most of the rules for plotting a root locus concern the sets of departure and arrival points.

A measure of stability robustness is provided by the distance from $-1 + j0$ to the nearest point on the Nyquist plot of the loop transfer function.