4/2006

Raport Badawczy
Research Report

RB/9/2006

Quasilinear thermoelasticity system arising in shape memory materials

S. Yoshikawa, I. Pawłow,
W.M. Zajączkowski

Instytut Badań Systemowych
Polska Akademia Nauk

Systems Research Institute
Polish Academy of Sciences

POLSKA AKADEMIA NAUK

Instytut Badań Systemowych

ul. Newelska 6

01-447 Warszawa

tel.: (+48) (22) 8373578

fax: (+48) (22) 8372772

Kierownik Pracowni zgłaszający pracę:
prof. dr hab. inż. Kazimierz Malanowski

Warszawa 2006

Quasilinear Thermoelasticity System Arising in Shape Memory Materials *

Shuji Yoshikawa

Mathematical Institute

Tohoku University,

Aoba, 980-8578 Sendai, Japan

E-mail: yosikawa@math.kyoto-u.ac.jp

Irena Pawłow †

Systems Research Institute

Polish Academy of Sciences,

Newelska 6, 01-447 Warsaw, Poland

E-mail: pawlow@ibspan.waw.pl

Wojciech M. Zajączkowski †

Institute of Mathematics

Polish Academy of Sciences,

Śniadeckich 8, 00-956 Warsaw, Poland

E-mail: wz@impan.gov.pl

Abstract

In this paper we establish the global existence and uniqueness of solution for the three-dimensional and two-dimensional quasilinear thermoelasticity system which arises as a mathematical model of shape memory alloys. The system represents a multi-dimensional version with viscosity and capillarity of the well-known Falk model for one-dimensional martensitic phase transitions. In the set-up considered by Pawlow and Zajączkowski ([21], [22] and [23]) some conditions have been required for

*Correspondence to: Shuji Yoshikawa, Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-8502, Japan (current address).

†Institute of Mathematics and Cryptology, Cybernetics Faculty, Military University of Technology, S. Kaliskiego 2, 00-908 Warsaw, Poland.

the nonlinear term. In the present paper we improve the result by imposing less restrictive assumptions.

1 Introduction

We consider the following initial-boundary value problem in quasi-linear thermoelasticity:

$$(TE)_d \begin{cases} u_{tt} + \kappa QQu - \nu Qu_t = \nabla \cdot F_{,\varepsilon}(\varepsilon, \theta), \\ [c_v - F_{,\theta\theta}(\varepsilon,\theta)\theta]\theta_t - k\Delta\theta = \theta F_{,\theta\varepsilon}(\varepsilon,\theta) : \varepsilon_t + \nu(A\varepsilon_t) : \varepsilon_t & \text{in } \Omega_T = (0,T] \times \Omega, \\ u = Qu = \nabla\theta \cdot n = 0 & \text{on } S_T = [0,T] \times \partial\Omega, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad \theta(0,x) = \theta_0(x) \geq 0 & \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ is a bounded domain with a smooth boundary $\partial\Omega$. Let $u = (u_i) \in \mathbb{R}^d$ denote the displacement vector, $\varepsilon = (\varepsilon_{ij})$ with $\varepsilon_{ij}(u) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ the linearized strain tensor, θ the absolute temperature and $F \in \mathbb{R}$ is called the elastic energy density. The capillarity term QQu with constant coefficient $\kappa > 0$ corresponds to interaction effects on phase interfaces. The coefficients ν, c_v and k are positive constants corresponding to the viscosity coefficient, caloric specific heat and the heat conductivity, respectively.

We use the notations $F_{,\varepsilon} = (\frac{\partial F}{\partial \varepsilon_{ij}})$, $F_{,\theta} = \frac{\partial F}{\partial \theta}$ and $\tilde{\varepsilon} : \varepsilon = \sum_{i,j=1}^{d} \tilde{\varepsilon}_{ij}\varepsilon_{ij}$. We define the linearized elasticity operator Q by the following second order differential operator

$$Qu = \mu\Delta u + (\lambda + \mu)\nabla(\nabla \cdot u),$$

where λ and μ are the Lamé constants such that

$$\mu > 0 \quad \text{and} \quad d\lambda + 2\mu > 0. \tag{1.1}$$

The fourth order tensor A represents linear isotropic Hooke's law, defining by

$$A_{ijkl} := \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

We note that the tensor has the following symmetry properties

$$A_{ijkl} = A_{klij}, \quad A_{ijkl} = A_{jikl}, \quad A_{ijkl} = A_{ijlk},$$

and the relation $Qu = \nabla \cdot \varepsilon(u)A$ holds. The assumption (1.1) assures the strong ellipticity of the operator Q and the following inequality

$$a_*|\varepsilon|^2 \leq (A\varepsilon) : \varepsilon \leq a^*|\varepsilon|^2,$$

where $a_* = \min\{d\lambda + 2\mu, 2\mu\}$ and $a^* = \max\{d\lambda + 2\mu, 2\mu\}$. In this article, we consider the following structure of the elastic energy density:

(A) $F(\varepsilon, \theta) = G(\theta)H(\varepsilon) + \overline{H}(\varepsilon)$ such that

(i) $G \in C^3(\mathbb{R}, \mathbb{R})$ is as follows:

$$G(\theta) = \begin{cases} C_1\theta & \text{if } \theta \in [0, \theta_1] \\ \varphi(\theta) & \text{if } \theta \in [\theta_1, \theta_2] \\ C_2\theta^r & \text{if } \theta \in [\theta_2, \infty), \end{cases}$$

where $\varphi \in C^3(\mathbb{R}, \mathbb{R})$, $\varphi'' \leq 0$ and C_1 and C_2 are positive constants for some fixed θ_1, θ_2 satisfying $0 < \theta_1 < \theta_2 < \infty$. We extend G defined on $\mathbb{R}$ as an odd function.

(ii) $H \in C^3(\mathbb{S}^2, \mathbb{R})$ satisfies the condition $H(\varepsilon) \geq 0$, where $\mathbb{S}^2$ denotes the set of symmetric second order tensors in $\mathbb{R}^d$.

(iii) $\overline{H} \in C^3(\mathbb{S}^2, \mathbb{R})$ satisfies $\overline{H}(\varepsilon) \geq -C_3$, where C_3 is some real number.

(iv) $H(\varepsilon)$ and $\overline{H}(\varepsilon)$ satisfy the following growth conditions:

$$|H_{,\varepsilon}(\varepsilon)| \leq C|\varepsilon|^{K_1-1}, \qquad |H_{,\varepsilon\varepsilon}(\varepsilon)| \leq C|\varepsilon|^{K_1-2}, \qquad |H_{,\varepsilon\varepsilon\varepsilon}(\varepsilon)| \leq C|\varepsilon|^{K_1-3},$$

$$|\overline{H}_{,\varepsilon}(\varepsilon)| \leq C|\varepsilon|^{K_2-1}, \qquad |\overline{H}_{,\varepsilon\varepsilon}(\varepsilon)| \leq C|\varepsilon|^{K_2-2}, \qquad |\overline{H}_{,\varepsilon\varepsilon\varepsilon}(\varepsilon)| \leq C|\varepsilon|^{K_2-3}$$

for large $|\varepsilon|$.

Here we note that the regularity assumption for $H(\varepsilon)$ and $\overline{H}(\varepsilon)$ assures that there exists a positive constant M such that

$$|H_{,\varepsilon}(\varepsilon)| + |H_{,\varepsilon\varepsilon}(\varepsilon)| + |H_{,\varepsilon\varepsilon\varepsilon}(\varepsilon)| + |\overline{H}_{,\varepsilon}(\varepsilon)| + |\overline{H}_{,\varepsilon\varepsilon}(\varepsilon)| + |\overline{H}_{,\varepsilon\varepsilon\varepsilon}(\varepsilon)| \leq M$$

for small $|\varepsilon|$. Under the above structure of nonlinearity the system $(TE)_d$ can be rewritten as follows:

$$u_{tt} + \kappa QQu - \nu Qu_t = \nabla \cdot [G(\theta)H_{,\varepsilon}(\varepsilon) + \overline{H}_{,\varepsilon}(\varepsilon)], \tag{1.2}$$

$$c_v\theta_t - k\Delta\theta = \theta G''(\theta)\theta_t H(\varepsilon) + \theta G'(\theta)\partial_t H(\varepsilon) + \nu(A\varepsilon_t) : \varepsilon_t \qquad \text{in } \Omega_T, \tag{1.3}$$

$$u = Qu = \nabla\theta \cdot n = 0 \qquad \text{on } S_T, \tag{1.4}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad \theta(0,x) = \theta_0(x) \geq 0 \qquad \text{in } \Omega. \tag{1.5}$$

In this paper we show the unique global existence of a solution for (1.2)–(1.5) under the following power of nonlinearity:

$$0 \leq r < \frac{5}{6}, \qquad 0 \leq K_1, K_2 < 6, \qquad 6r + K_1 < 6 \tag{1.6}$$

in the 3-D case, and

$$0 \leq r < 1, \qquad 0 \leq K_1, K_2 < \infty \tag{1.7}$$

in the 2-D case.

Before discussing the result of this paper more precisely we shall explain the related results and the physical background of this model. In [10], Falk presents the Landau-Ginzburg type theory using the shear strain $\varepsilon := \partial_x u$ as an order parameter to describe the martensitic-austenitic phase transitions occurring in 1-D SMA. There are many papers related to 1-D SMA (e.g. [2], [3], [6], [12], [16], [17] and [24]). The system $(TE)_d$ is a generalization of the 1-D Falk model with internal viscosity to the 3-D case. The Helmholtz free energy density takes the following form

$$\widetilde{F}(\varepsilon, \nabla\varepsilon, \theta) = F_0(\theta) + F(\varepsilon, \theta) + |Qu|^2,$$

$$F_0(\theta) = -c_v\theta\log(\theta/\theta_3) + c_v\theta + \tilde{c}$$

and the stress tensor is given by

$$\sigma = \frac{\delta\widetilde{F}}{\delta\varepsilon} + \nu A\varepsilon_t,$$

where $\tilde{c}$ and θ_3 denote the positive physical constants. System $(TE)_d$ can be derived by an argument similar to that in the 1-D case (see [5]). For more details on the derivation of this system, we refer to [19]. In [11], Falk and Konopka give the form of the elastic energy density F as follows:

$$F(\varepsilon,\theta) = \sum_{i=1}^{3} \alpha_i^2(\theta - \theta_c)J_i^2(\varepsilon) + \sum_{i=1}^{5} \alpha_i^4(\theta - \theta_c)J_i^4(\varepsilon) + \sum_{i=1}^{2} \alpha_i^6 J_i^6(\varepsilon), \tag{1.8}$$

proved under no conditions between κ and ν, and the class of nonlinearities is generalized to $K_2 < 6$. The first two assumptions in (1.10) are present due to the semilinearization which causes the lack of energy conservation law (Lemma 4.1 below). Recently, Pawlow and Zajączkowski [21] have proved the unique global existence for the quasilinear system (1.2)–(1.5) under the assumptions

$$0 < r < \frac{2}{3}, \quad 0 < K_1 < \frac{15}{4} \text{ and } 15r + 4K_1 = 15 \text{ if } K_1 > 1, \quad 0 < K_2 \le \frac{9}{2}, \quad 0 < 2\sqrt{\kappa} \le \nu. \tag{1.12}$$

The latter, restrictive condition between viscosity and capillarity has been removed by the above mentioned authors in [23]. The aim of the present paper is to prove the unique global existence of a solution to system (1.2)–(1.5) under weaker assumptions than (1.12). More precisely, we admit the nonlinearity specified in (1.6), (1.7), and arbitrary positive coefficients of capillarity $\kappa > 0$ and viscosity $\nu > 0$. Unfortunately, our result still does not cover the physically realistic case (1.8).

Here we add some remarks on the 2-D case. The results of [20] include the 2-D case of the semilinearized problem $(SLTE)_2$. The unique global existence for the 2-D quasilinear system $(TE)_2$ is established in [22] under the assumption:

$$0 \le r < \frac{7}{8}, \qquad 0 \le K_1 < \infty, \qquad 0 \le K_2 < \infty. \tag{1.13}$$

In [26] the unique global existence for $r = 1$ is proved under other strong assumptions. Roughly speaking, the restrictions in [26] are such that $K_1 = 0$ and that the energy of initial data $\|u_0\|_{H^2} + \|u_1\|_{L^2} + \|\theta_0\|_{L^1}$ is sufficiently small. We note that if we take $r = 1$ then the quasilinear term $\theta G''(\theta)H(\varepsilon)\theta_t$ of (1.3) does not appear. We also describe the result for the 2-D case in Section 5 of this paper. We show that the system $(TE)_2$ has a unique global solution under the assumptions (1.7). Comparing these assumptions with (1.13), we see that the restriction for r is weaker, nevertheless we cannot admit $r = 1$.

We now introduce some notations and function spaces. Throughout this paper C and Λ are positive constants independent of time T and depending on time T, respectively. In particular, we may use Λ instead of $\Lambda(\|(u_0, u_1, \theta_0)\|_X)$ for some X if there is no danger of confusion.

- $L^p(\Omega_T) = L^p_T L^p = L^p(0, T; L^p(\Omega))$ is the standard Lebesgue space. We often use the notation $L^p(\Omega_I) = L^p_I L^p$ for some interval I.

where α_i^k, θ_c are constants and J_i^k denote certain k-th order monomials with respect to (ε_{ij}). Here we remark that in the 1-D case the elastic energy density takes the following form:

$$F_{1D}(\varepsilon,\theta) = \alpha_1\varepsilon^2(\theta-\theta_c) - \alpha_2\varepsilon^4 + \alpha_3\varepsilon^6, \tag{1.9}$$

where $\varepsilon := \partial_x u$ and α_i, θ_c are positive constants. Comparing (1.8) with the 1-D form (1.9), we see that in the 3-D case $H(\varepsilon)$ must be the fourth order with respect to ε. This causes some difficulties in the mathematical treatment of the system (1.2)–(1.5). Moreover, the difficulties arise also from the fact that the useful embedding $H^1 \hookrightarrow L^\infty$ does not hold in the multi-dimensional case. There had been no papers on the solvability of this system with the Falk-Konopka elastic energy density (1.8), $r = 1$, $K_1 = 4$ and $K_2 = 6$. Then Pawlow and Żochowski [20] studied the energy density F under several stronger assumptions than (1.8), namely, lower order powers of nonlinearity. Moreover, for the simplification of treatments they first considered the semilinearized equations of the quasilinear system $(TE)_d$:

$$(SLTE)_d \begin{cases} u_{tt} + \kappa QQu - \nu Qu_t = \nabla\cdot F_{,\varepsilon}(\varepsilon,\theta), & \\ c_v\theta_t - k\Delta\theta = \theta F_{,\theta\varepsilon}(\varepsilon,\theta) : \varepsilon_t + \nu(A\varepsilon_t) : \varepsilon_t & \text{in } \Omega_T, \\ u = Qu = \nabla\theta\cdot n = 0 & \text{on } S_T, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad \theta(0,x) = \theta_0(x) \geq 0 & \text{in } \Omega, \end{cases}$$

which is the model $(TE)_d$ with removed quasilinear term $\theta G''(\theta)\theta_t H(\varepsilon)$. They showed the unique global existence of a sufficiently smooth solution for $(SLTE)_d$ under the following assumptions on the nonlinearity:

$$0 \leq r < \frac{1}{2}, \qquad 0 \leq K_1 \leq \left(\frac{1}{2} - r\right)K_2 + 1, \qquad 0 \leq K_2 \leq \frac{7}{2} \tag{1.10}$$

in the 3-D case, and

$$0 \leq r < \frac{1}{2}, \qquad 0 \leq K_1 \leq \left(\frac{1}{2} - r\right)K_2 + 1, \qquad 0 \leq K_2 < \infty \tag{1.11}$$

in the 2-D case. In addition, due to the applied parabolic decomposition of elasticity system, they assumed the condition $0 < 2\sqrt{\kappa} \leq \nu$ between viscosity and capillarity. Such assumption, however, does not seem realistic for SMA viscosity effects which are negligibly small. In [25] the unique global existence of the solution to $(SLTE)_3$ in a larger class is proved by using the contraction mapping principle. The result is

- $W_p^{2l,l}(\Omega_T)$ is the Sobolev space equipped with the norm

$$\|u\|_{W_p^{2l,l}(\Omega_T)} := \sum_{j=0}^{2l} \sum_{2r+|\alpha|=j} \|D_t^r D_x^\alpha u\|_{L^p(\Omega_T)},$$

  where $D_t := i\frac{\partial}{\partial t}$, $D_x^\alpha = \prod_{\alpha=\alpha_1+\alpha_2+\alpha_3} D_k^{\alpha_k}$ and $D_k := i\frac{\partial}{\partial x_k}$ for multi index $\alpha = (\alpha_i)_{i=1}^n$.

- $H^j(\Omega) := W_2^j(\Omega)$, where W_p^j is the Sobolev space equipped with the norm $\|u\|_{W_p^j(\Omega)} := \sum_{|\alpha|\le j} \|D_x^\alpha u\|_{L^p(\Omega)}$.

- $B_{p,q}^s = B_{p,q}^s(\Omega)$ is the Besov space. Namely, $B_{p,q}^s := [L^p(\Omega), W_p^j(\Omega)]_{s/j,q}$, where $[X,Y]_{s/j,q}$ is the real interpolation space. For more details we refer to [1] by Adams and Fournier.

- $C^{\alpha,\alpha/2}(\Omega_T)$ is the Hölder space: the set of all continuous functions in Ω_T satisfying Hölder condition in x with exponent α and in t with exponent $\alpha/2$.

We now state the main result of this paper.

Theorem 1.1. *Let the positive physical constants κ, ν, c_v and k be fixed arbitrarily. Assume that $\min_\Omega \theta_0 \ge 0$ and (1.6) holds. Then, given $5 < p \le q < \infty$, for any $T > 0$ and $(u_0, u_1, \theta_0) \in B_{p,p}^{4-2/p} \times B_{p,p}^{2-2/p} \times B_{q,q}^{2-2/q} =: U(p,q)$, there exists at least one solution (u,θ) to (1.2)–(1.5) satisfying*

$$(u,\theta) \in W_p^{4,2}(\Omega_T) \times W_q^{2,1}(\Omega_T) =: V_T(p,q).$$

Moreover, if we assume $\min_\Omega \theta_0 = \theta_* > 0$ then there exists a positive constant ω such that

$$\theta \ge \theta_* \exp(-\omega t) \qquad \text{in } \Omega_T.$$

For completeness we recall also the uniqueness result which follows by repeating the arguments of the corresponding result in [22, Section 6].

Theorem 1.2. *In addition to assumptions of Theorem 1.1, suppose that $F(\epsilon,\theta) \in C^4(\mathbb{S}^2 \times \mathbb{R}^+, \mathbb{R})$. Then the solution $(u,\theta) \in V_T(p,q)$ to (1.2)–(1.5) constructed above is unique.*

We prove Theorem 1.1 by using the Leray-Schauder fixed point principle. The key estimates are the maximal regularity estimate for (1.2), and the classical energy estimate and the parabolic De Giorgi method for (1.3). In general, the derivative of a solution is less regular than the right-hand side of the corresponding equation. However, for parabolic equations such a loss of regularity does not occur, as in the case of elliptic equations. The estimate ensuring this regularity is called the maximal regularity. For more precise information on the maximal regularity we refer to [4], and for more recent topics of the maximal L^p-regularity we refer to [9]. Since the maximal regularity theory is limited to linear parabolic equations, we cannot use it directly for the quasilinear equation (1.3). To obtain the higher order a priori estimates we apply the classical energy methods and the parabolic De Giorgi method (see [14], [15]). Using these methods we can show the Hölder continuity of θ. By virtue of such regularity, we arrive at the estimate in higher Sobolev norm.

In Section 2 we list several preliminary results which are used in the paper. In Section 3 we prove the unique global existence of the solution for certain truncated version of problem (1.2)–(1.5). To this purpose we use the Leray-Schauder fixed point principle. In Section 4 we show that the solution of $(TE)_3$ coincides with the solution of the truncated problem constructed in Section 3 for a sufficiently large truncation level L. In Section 5 we consider the 2-D system $(TE)_2$.

2 Preliminaries

In this section, we present some auxiliary results which will be used in the subsequent sections.

Lemma 2.1 (Maximal Regularity). (i) *Let $p \in (1,\infty)$. Denote by u the solution of the linear problem*

$$\begin{cases} u_{tt} + \kappa QQu - \nu Qu_t = \nabla\cdot f & \text{in } \Omega_T, \\ u = Qu = 0 & \text{on } S_T, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) & \text{in } \Omega. \end{cases}$$

Then the following estimates hold

$$\|u\|_{W_p^{4,2}(\Omega_T)} \le C(\|u_0\|_{B_{p,p}^{4-\frac{2}{p}}} + \|u_1\|_{B_{p,p}^{2-\frac{2}{p}}} + \|\nabla\cdot f\|_{L^p(\Omega_T)}) \tag{2.1}$$

for any $(u_0,u_1) \in B_{p,p}^{4-2/p} \times B_{p,p}^{2-2/p}$ and $\nabla\cdot f \in L^p(\Omega_T)$, and

$$\|\nabla u\|_{W_p^{2,1}(\Omega_T)} \le C(\|u_0\|_{B_{p,p}^{3-\frac{2}{p}}} + \|u_1\|_{B_{p,p}^{1-\frac{2}{p}}} + \|f\|_{L^p(\Omega_T)}) \tag{2.2}$$

for any $(u_0,u_1) \in B_{p,p}^{3-2/p} \times B_{p,p}^{1-2/p}$ and $f \in L^p(\Omega_T)$.

(ii) *Let $q \in (1,\infty)$. Assume that $\rho(x)$ is Hölder continuous in Ω such that $\inf_\Omega \rho > 0$. Denote by θ the solution of the linear problem*

$$\begin{cases} \theta_t - \rho\Delta\theta = g & \text{in } \Omega_T, \\ n\cdot\nabla\theta = 0 & \text{on } S_T, \\ \theta(0,x) = \theta_0(x) & \text{in } \Omega. \end{cases}$$

Then the following estimate holds

$$\|\theta\|_{W_q^{2,1}(\Omega_T)} \leq C(\|\theta_0\|_{B_{q,q}^{2-\frac{2}{q}}} + \|g\|_{L^q(\Omega)}) \tag{2.3}$$

for any $\theta_0 \in B_{q,q}^{2-2/q}$, where C depends on $\inf_\Omega \rho$.

For the proof of (*i*) we refer to [25, Lemma 2.1, Proposition 2.4], and (*ii*) is the particular case of [13, 3.2 Examples A), 2)]. Next, we recall the useful space-time embedding lemma.

Lemma 2.2 (Embedding [14, Lemma II.3.3]). *Let $f \in W_p^{2l,l}(\Omega_T)$. Then, for $l \in \mathbb{Z}^+$ and multi index α, it follows that*

$$\|D_t^r D_x^\alpha f\|_{L^q(\Omega_T)} \leq C\delta^{l-\psi}\|f\|_{W_p^{2l,l}(\Omega_T)} + C\delta^{-\psi}\|f\|_{L^p(\Omega_T)}, \tag{2.4}$$

provided $q \geq p$ and $\psi := r + \frac{|\alpha|}{2} + \frac{d+2}{2}\left(\frac{1}{p} - \frac{1}{q}\right) \leq l$. If $\varphi := r + \frac{|\alpha|}{2} + \frac{d+2}{2p} < l$, then

$$\|D_t^r D_x^\alpha f\|_{L^\infty(\Omega_T)} \leq C\delta^{l-\varphi}\|f\|_{W_p^{2l,l}(\Omega_T)} + C\delta^{-\varphi}\|f\|_{L^p(\Omega_T)}, \tag{2.5}$$

moreover, $D_t^r D_x^\alpha f$ is Hölder continuous. Here, $\delta \in (0, \min(T,\zeta^2)]$, ζ is the altitude of the cone in the statement of the cone condition satisfied by Ω.

Lemma 2.3. *Let φ be given in (A)–(i). Then the function $\varphi(s)$ satisfies*

$$\varphi(s) - s\varphi'(s) \geq 0 \tag{2.6}$$

for any $s \in [\theta_1, \theta_2]$

Proof. Putting $f(s) = \varphi(s) - s\varphi'(s)$, we have $f'(s) = -s\varphi''(s) \geq 0$ and $f(\theta_1) = 0$. Then $f(s) = \varphi(s) - s\varphi'(s) \geq 0$ in $[\theta_1, \theta_2]$. □

To show Theorem 1.1 we apply the Leray-Schauder fixed point principle. We recall it here in one of its equivalent formulations for the reader's convenience .

Theorem 2.4 (Leray-Schauder Fixed Point Principle [8]). *Let X be a Banach space. Assume that $\Phi : [0,1] \times X \to X$ is a map with the following properties.*

(L1) For any fixed $\tau \in [0,1]$ the map $\Phi(\tau,\cdot) : X \to X$ is compact.

(L2) For every bounded subset $\mathcal{B}$ of X, the family of maps $\Phi(\cdot,\xi) : [0,1] \to X$, $\xi \in \mathcal{B}$, is uniformly equicontinuous.

(L3) $\Phi(0,\cdot)$ has precisely one fixed point in X.

(L4) There is a bounded subset $\mathcal{B}$ of X such that any fixed point in X of $\Phi(\tau,\cdot)$ is contained in $\mathcal{B}$ for every $0 \le \tau \le 1$.

Then $\Phi(1,\cdot)$ has at least one fixed point in X.

3 Truncated Problem

To prove the existence theorem we first consider the following truncated problem $(TE)_3^L$:

$$u_{tt} + \kappa QQu - \nu Qu_t = \Gamma_L\Big(\nabla \cdot [G(\theta)H_{,\varepsilon}(\varepsilon) + \overline{H}_{,\varepsilon}(\varepsilon)]\Big), \tag{3.1}$$

$$c_v\theta_t - k\Delta\theta = \theta G''(\theta)\theta_t H(\varepsilon) + \theta G'(\theta)\partial_t H(\varepsilon) + \nu(A\varepsilon_t) : \varepsilon_t \qquad \text{in } \Omega_T, \tag{3.2}$$

$$u = Qu = \nabla\theta \cdot n = 0 \qquad \text{on } S_T,$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad \theta(0,x) = \theta_0(x) \ge 0 \qquad \text{in } \Omega,$$

where

$$\Gamma_L(x) = \begin{cases} x & \text{if } |x| \le L, \\ L\frac{x}{|x|} & \text{if } |x| \ge L. \end{cases}$$

Theorem 3.1. *Fix L and $5 < p \le q < \infty$. Assume that $\theta_0 \ge 0$, (1.6) holds and $F(\varepsilon,\theta) \in C^4(\mathbb{S}^2 \times \mathbb{R}^+, \mathbb{R})$. Then for any $T > 0$ and $(u_0, u_1, \theta_0) \in U(p,q)$, there exists a unique solution (u_L, θ_L) to $(TE)_3^L$ satisfying $(u_L, \theta_L) \in V_T(p,q)$.*

Proof of Theorem 3.1. We apply Theorem 2.4 to the map Φ_τ^L from $V_T(p,q)$ into $V_T(p,q)$,

$$\Phi_\tau^L : (\bar{u}, \bar{\theta}) \mapsto (u, \theta), \qquad \tau \in [0,1],$$

defined by means of the following initial-boundary value problems:

$$u_{tt} + \kappa QQu - \nu Qu_t = \tau\Gamma_L\Big(\nabla\cdot[G(\bar\theta)H_{,\varepsilon}(\bar\varepsilon) + \overline{H}_{,\varepsilon}(\bar\varepsilon)]\Big),$$

$$c_v\theta_t - k\Delta\theta = \tau\left\{\bar\theta G''(\bar\theta)\theta_t H(\varepsilon) + \bar\theta G'(\bar\theta)\partial_t H(\varepsilon) + \nu(A\varepsilon_t):\varepsilon_t\right\} \qquad \text{in } \Omega_T,$$

$$u = Qu = \nabla\theta\cdot n = 0 \qquad \text{on } S_T,$$

$$u(0,x) = \tau u_0(x), \quad u_t(0,x) = \tau u_1(x), \quad \theta(0,x) = \tau\theta_0(x) \qquad \text{in } \Omega,$$

where $\bar\varepsilon = \varepsilon(\bar u)$. A fixed point of $\Phi_\tau^L(1,\cdot)$ in $V_T(p,q)$ is the desired solution of the system $(TE)_3^L$. Therefore to prove the existence statement it is sufficient to check that the map Φ_τ^L satisfies assumptions $(L1)$–$(L4)$ of Theorem 2.4. Noting that Γ_L is Lipschitz continuous, we can check assumptions $(L1)$, $(L2)$ and $(L3)$ in the same way as that in [21, Section 3]. Then it is sufficient to check the assumption $(L4)$, namely, to derive a priori bounds for a fixed point of the solution map Φ_τ^L. Without loss of generality we may set $\tau = 1$. Hence, from now on our purpose is to obtain a priori bounds for $(TE)_3^L$. To this end we prepare several lemmas. If there is no danger of confusion we write for simplicity (u,θ) instead of (u_L,θ_L).

Lemma 3.2 (Maximum Principle). *Let $(u_0,u_1,\theta_0) \in B_{p,p}^{4-2/p}\times B_{p,p}^{2-2/p}\times L^2$ for $p>5$. Assume that $\min_\Omega\theta_0 \ge 0$. Then the solution θ to $(TE)_3^L$ is non-negative almost everywhere in Ω_T.*

Proof. It follows from the maximal regularity (2.1) for (3.1) that

$$\begin{aligned}\|u\|_{W_p^{4,2}(\Omega_T)} &\le C\left(\|u_0\|_{B_{p,p}^{4-2/p}} + \|u_1\|_{B_{p,p}^{2-2/p}} + \|\Gamma_L\left\{\nabla\cdot[G(\bar\theta)H_{,\varepsilon}(\bar\varepsilon) + \overline{H}_{,\varepsilon}(\bar\varepsilon)]\right\}\|_{L^p(\Omega_T)}\right)\\ &\le C(\|u_0\|_{B_{p,p}^{4-2/p}} + \|u_1\|_{B_{p,p}^{2-2/p}} + L|\Omega_T|^{\frac1p})\\ &\le \Lambda(L).\end{aligned} \tag{3.3}$$

Then taking $p>5$, by Lemma 2.2 we have

$$\|\varepsilon\|_{L^\infty(\Omega_T)} + \|\varepsilon_t\|_{L^\infty(\Omega_T)} \le \Lambda(L) < \infty. \tag{3.4}$$

Therefore it holds that

$$\|\partial_t H(\varepsilon)\|_{L^\infty(\Omega_T)} \le \|\varepsilon_t\|_{L^\infty(\Omega_T)}\|\varepsilon\|_{L^\infty(\Omega_T)}^{K_1-1} \le \Lambda(L)$$

for $K_1>1$. Since $\sup_{\varepsilon\in S}|H_{,\varepsilon}(\varepsilon)| \le M$ for $K_1\le 1$, we conclude that

$$\|\partial_t H(\varepsilon)\|_{L^\infty(\Omega_T)} \le \Lambda(L) \tag{3.5}$$

for every $K_1 \geq 0$. From now on throughout this section we shall write $\Lambda = \Lambda(L)$.

Multiplying (3.2) by $\theta_- := \min\{\theta, 0\}$ and integrating over Ω, we have

$$\frac{c_v}{2}\frac{d}{dt}\int_\Omega \theta_-^2 dx + k\int_\Omega |\nabla\theta_-|^2 dx = \int_\Omega [\theta_-\theta G''(\theta)\theta_t H(\varepsilon) + \theta_-\theta G'(\theta)\partial_t H(\varepsilon) + \nu\theta_- A\varepsilon_t : \varepsilon_t]\, dx$$
$$= \frac{d}{dt}\int_\Omega H(\varepsilon)G_2(\theta_-)dx + \int_\Omega \overline{G}_2(\theta_-)\partial_t H(\varepsilon)dx + \int_\Omega \nu\theta_- A\varepsilon_t : \varepsilon_t dx,$$

where $G_2(\theta) = \theta^2 G'(\theta) - \overline{G}_2(\theta)$ and $\overline{G}_2(\theta) = 2\int_0^\theta sG'(s)ds$. We have $G_2(0) = 0$ and $G_2'(y) = y^2 G''(y) \geq 0$ for $y \leq 0$, because G'' is the odd function such that $G''(y) \leq 0$ for $y \geq 0$. Then $G_2(y) \leq 0$ for $y \geq 0$. Hence we have

$$-\int_\Omega H(\varepsilon)G_2(\theta_-)dx \geq 0.$$

It follows from (1.1) that

$$\int_\Omega \nu\theta_- A\varepsilon_t : \varepsilon_t dx \leq \nu\alpha_* \int_\Omega \theta_- |\varepsilon_t|^2 dx \leq 0.$$

Noting that $\overline{G}_2(\theta) = \frac{1}{2}C_1\theta^2$ for $\theta \in [-\theta_1, \theta_1]$, we have $\sup_{s\in\mathbb{R}} \frac{|\overline{G}_2(s)|}{s^2} \leq C$. Therefore we conclude that

$$\int_\Omega \overline{G}_2(\theta_-)\partial_t H(\varepsilon)dx \leq \int_\Omega |\theta_-|^2 \frac{|\overline{G}_2(\theta_-)|}{|\theta_-|^2}|\partial_t H(\varepsilon)|dx$$
$$\leq \Lambda\|\theta_-\|_{L^2}^2.$$

Consequently, we have

$$\frac{d}{dt}\left(c_v\|\theta_-(t)\|_{L^2}^2 - \int_\Omega H(\varepsilon)G_2(\theta_-)dx\right) \leq \Lambda\left(c_v\|\theta_-(t)\|_{L^2}^2 - \int_\Omega H(\varepsilon)G_2(\theta_-)dx\right).$$

Using the Gronwall inequality we obtain

$$\|\theta_-(t)\|_{L^2}^2 \leq \|\theta_-(t)\|_{L^2}^2 - \int_\Omega H(\varepsilon)G_2(\theta_-)dx$$
$$\leq \Lambda e^{\Lambda t}\left(\|\theta_-(0)\|_{L^2}^2 - \int_\Omega H(\varepsilon(0))G_2(\theta_-(0))dx\right)$$
$$= 0,$$

which completes the proof. □

Lemma 3.3. *Let $l > 2$ be arbitrary integer. Assume that $r \leq 1$. Then for any $(u_0, u_1, \theta_0) \in B_{p,p}^{4-2/p} \times B_{p,p}^{2-2/p} \times L^l =: U_1(l)$, the solution (u, θ) to $(TE)_3^L$ satisfies*

$$\|\theta\|_{L_T^\infty L^l} \leq \Lambda.$$

where $\Lambda = \Lambda(T, \|(u_1, u_2, \theta_0)\|_{U_1(l)})$. *Moreover, if* $(u_0, u_1, \theta_0) \in U_1(\infty)$, *then we have*

$$\|\theta\|_{L^\infty(\Omega_T)} \le \Lambda,$$

where $\Lambda = \Lambda(T, \|(u_1, u_2, \theta_0)\|_{U_1(\infty)})$.

Proof. Multiplying (3.2) by θ^{l-1} and integrating over Ω, we have

$$\begin{aligned} \frac{c_v}{l}\frac{d}{dt}\|\theta\|_{L^l}^l + k(l-1)\int_\Omega \theta^{l-2}|\nabla\theta|^2 dx &= \int_\Omega \left(\theta^l G''(\theta)\theta_t H(\varepsilon) + \theta^l G'(\theta)\partial_t H(\varepsilon)\right) dx \\ &\quad + \int_\Omega \nu\theta^{l-1} A\varepsilon_t : \varepsilon_t dx \\ &= \frac{d}{dt}\int_\Omega G_l(\theta)H(\varepsilon)dx + \int_\Omega \overline{G}_l(\theta)\partial_t H(\varepsilon)dx \\ &\quad + \nu\int_\Omega \theta^{l-1} A\varepsilon_t : \varepsilon_t dx, \end{aligned} \tag{3.6}$$

where $G_l(\theta) = \theta^l G'(\theta) - \overline{G}_l(\theta)$ and $\overline{G}_l(\theta) = l\int_0^\theta s^{l-1}G'(s)ds$. Since

$$\theta^l G''(\theta) = \begin{cases} C_2 r(r-1)\theta^{l+r-2} \le 0 & \text{for } \theta \ge \theta_2, \\ \theta^l \varphi''(\theta) \le 0 & \text{for } \theta_1 \le \theta \le \theta_2, \\ 0 & \text{for } \theta \le \theta_1, \end{cases} \tag{3.7}$$

we have $G_l'(\theta) = \theta^l G''(\theta) \le 0$ for $\theta \ge 0$ and $G_l'(0) = 0$. Thereby, we obtain

$$G_l(\theta) \le 0 \quad \text{for} \quad \theta \ge 0. \tag{3.8}$$

We put

$$\hat\theta = \theta\left(1 - \frac{lG_l(\theta)H(\varepsilon)}{c_v\theta^l}\right)^{1/l}.$$

We note that $\hat\theta \ge \theta$ due to (3.8). Since $\sup_{s\in[0,\infty)}|G'(s)| =: M < \infty$, we have

$$|\overline{G}_l(\theta)| = \left| l\int_0^\theta s^{l-1}G'(s)ds \right| \le C\theta^l$$

and

$$|G_l(\theta)| \le M\theta^l + |\overline{G}_l(\theta)| \le C\theta^l.$$

In view of (3.4) and (3.5) we obtain

$$\left|\int_\Omega \overline{G}_l(\theta)\partial_t H(\varepsilon)dx\right| \le C\|\theta^l\|_{L^1(\Omega)}\|\partial_t H(\varepsilon)\|_{L^\infty(\Omega)} \le \Lambda\|\theta\|_{L^l(\Omega)}^l$$

and

$$\int_\Omega \theta^{l-1} A\varepsilon_t : \varepsilon_t \le C\|\varepsilon_t\|^2_{L^\infty(\Omega)}\|\theta\|^{l-1}_{L^{l-1}(\Omega)} \le \Lambda\|\theta\|^{l-1}_{L^l(\Omega)}.$$

Since $\frac{1}{l}\partial_t\|\hat\theta\|^l_{L^l} = \|\hat\theta\|^{l-1}_{L^l}\partial_t\|\hat\theta\|_{L^l}$, it follows from (3.6) that

$$\begin{aligned}\frac{d}{dt}\|\hat\theta\|_{L^l(\Omega)} &\le \Lambda\|\theta\|_{L^l(\Omega)} + \Lambda \\ &\le \Lambda\|\hat\theta\|_{L^l(\Omega)} + \Lambda.\end{aligned}$$

Thus by the Gronwall inequality we have

$$\|\hat\theta\|_{L^\infty_T L^l} \le \Lambda\|\hat\theta_0\|_{L^l} + \Lambda. \tag{3.9}$$

Since

$$\begin{aligned}\hat\theta_0 &= \theta_0\left(1 - \frac{lG_l(\theta_0)H(\varepsilon_0)}{c_v\theta_0^l}\right)^{1/l} \\ &\le \theta_0\left(1 + lM\Lambda/c_v\right)^{1/l},\end{aligned}$$

we can obtain the first assertion. Here we note that the constant Λ in (3.9) is independent of l. Therefore taking a limit as $l \to \infty$ we can obtain the second assertion. This completes the proof. □

Lemma 3.4. *Let T be arbitrarily fixed. Assume that $r \le 1$. Then for any $(u_0, u_1, \theta_0) \in B^{4-2/p}_{p,p} \times B^{2-2/p}_{p,p} \times H^1 =: U_2$, the solution (u, θ) to $(TE)^L_3$ satisfies*

$$\|\theta\|_{W^{2,1}_2(\Omega_T)} \le \Lambda,$$

where Λ depends on T and $\|(u_0, u_1, \theta_0)\|_{U_2}$.

Proof. By using Lemma 3.3 thanks to $\theta_0 \in H^1 \hookrightarrow L^2$, we have

$$\|\theta\|_{L^\infty_T L^2} \le \Lambda. \tag{3.10}$$

Since $\theta G''(\theta) \le 0$ from (3.7) for $l = 1$, the following estimate holds true

$$\iint_{\Omega_T} \theta_t^2 \theta G''(\theta) H(\varepsilon)\,dx dt \le 0. \tag{3.11}$$

Multiplying (3.2) by θ_t and integrating over Ω_T, we have

$$c_v\|\theta_t\|^2_{L^2(\Omega_T)} + \frac{k}{2}\|\nabla\theta\|^2_{L^\infty_T L^2} \le \frac{k}{2}\|\theta_0\|^2_{H^1} + \iint_{\Omega_T} \theta_t^2\theta G''(\theta)H(\varepsilon)dxdt + \iint_{\Omega_T} \theta_t\theta G'(\theta)\partial_t H(\varepsilon)dxdt$$

$$+ \iint_{\Omega_T} \nu\theta_t A\varepsilon_t : \varepsilon_t dxdt$$

$$\le \|\theta_0\|^2_{H^1} + \Lambda\|\theta_t\|_{L^2(\Omega_T)}\|\theta\|^2_{L^\infty_T L^2}\|\partial_t H(\varepsilon)\|_{L^\infty(\Omega_T)} + \Lambda\|\theta_t\|_{L^\infty_T L^2}\|\varepsilon_t\|^2_{L^\infty(\Omega_T)}$$

$$\le \|\theta_0\|^2_{H^1} + \frac{c_v}{2}\|\theta_t\|^2_{L^2(\Omega_T)} + \Lambda,$$

thanks to (3.4), (3.5), (3.10) and (3.11). Therefore we arrive at

$$\|\theta_t\|_{L^2(\Omega_T)} + \|\nabla\theta\|_{L^\infty_T L^2} \le \Lambda(\|(u_0, u_1, \theta_0)\|_{U_2}).$$

Next multiplying (3.2) by $\frac{-\Delta\theta}{c_v - \theta G''(\theta)H(\varepsilon)}$ and integrating over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\|\nabla\theta(t)\|^2_{L^2} + \int_\Omega \frac{k(\Delta\theta)^2}{c_v - \theta G''(\theta)H(\varepsilon)}dx = -\int_\Omega \frac{\Delta\theta}{c_v - \theta G''(\theta)H(\varepsilon)}(\theta G'(\theta)\partial_t H(\varepsilon) + \nu A\varepsilon_t : \varepsilon_t)\, dx.$$

Here we remark that

$$c_v \le c_v - \theta G''(\theta)H(\varepsilon) \le c_v + M\Lambda,$$

where $0 \le \sup_{\theta\ge 0}(-\theta G''(\theta)) =: M < \infty$. Then integrating over $[0, t]$ for $t \le T$, we conclude the estimate

$$\|\nabla\theta(t)\|^2_{L^2} + \frac{2k}{c_v + \Lambda M}\|\Delta\theta\|^2_{L^2(\Omega_T)} \le \|\nabla\theta_0\|^2_{L^2} + 2\|\Delta\theta\|_{L^2(\Omega_T)}\|\theta G'(\theta)\partial_t H(\varepsilon) + \nu A\varepsilon_t : \varepsilon_t\|_{L^2(\Omega_T)}$$

$$\le \|\nabla\theta_0\|^2_{L^2} + \frac{k}{(c_v + \Lambda M)}\|\Delta\theta\|^2_{L^2(\Omega_T)}$$

$$+ \frac{c_v + \Lambda M}{k}\left(\Lambda\|\theta\|_{L^\infty_T L^2}\|\partial_t H(\varepsilon)\|_{L^\infty(\Omega_T)} + \Lambda\|\varepsilon_t\|_{L^\infty(\Omega_T)}\right)^2$$

$$\le \frac{k}{(c_v + \Lambda M)}\|\Delta\theta\|^2_{L^2(\Omega_T)} + \Lambda.$$

Consequently we arrive at the desired result. □

The same procedure as in [21, Lemma 6.1] allows to conclude that $\theta \in C^{\alpha,\alpha/2}(\Omega_T)$ for some Hölder exponent $0 < \alpha < 1$ depending on T, $\sup_\Omega \theta_0$ and $\|\theta\|_{L^\infty(\Omega_T)}$. The proof relies on the classical parabolic De Giorgi method. For more precise information of this method we refer to [14, Chapter II, §7] and [15, Chapter VI, §12]. Here we note that ε is Hölder continuous due to Lemma 2.2.

Lemma 3.5 ([21, Lemma 6.1]). *Assume that $k = \sup_\Omega \theta_0 < \infty$. Suppose that*

$$\|\varepsilon\|_{W^{2,1}_2(\Omega_T)} + \|\theta\|_{W^{2,1}_2(\Omega_T)} + \|\theta\|_{L^\infty(\Omega_T)} \le \Lambda \tag{3.12}$$

holds for any $s \in (1,\infty)$. Then $\theta \in C^{\alpha,\alpha/2}(\Omega_T)$ with Hölder exponent $\alpha \in (0,1)$ depending on Λ and k.

Lemma 3.6. *Assume that (3.12) holds. Then for any $(u_0, u_1, \theta_0) \in U(p,q)$ and $5 < p, q < \infty$ we have*

$$\|(u,\theta)\|_{V_T(p,q)} = \|u\|_{W_p^{4,2}(\Omega_T)} + \|\theta\|_{W_q^{2,1}(\Omega_T)} \leq \Lambda,$$

where Λ depends on $\|(u_0, u_1, \theta_0)\|_{U(p,q)}$ and T.

Proof. We can construct a unique time local solution $(u,\theta) \in W_p^{4,2}(\Omega_{\widetilde{T}}) \times W_q^{2,1}(\Omega_{\widetilde{T}})$ of $(TE)_3^L$ for sufficiently small $\widetilde{T} < T$, using the result of Clément and Li [7] (see also [27, Lemma 3.3.7]). Then from the embedding we have $\theta \in C([0,\widetilde{T}] \times \Omega)$. By combining this regularity result with Lemma 3.5, we obtain $\theta \in C^{\alpha,\alpha/2}([0,T] \times \Omega)$.

For brevity of notation we denote $c_v - \theta G''(\theta)H(\varepsilon)$ by $c_0(\varepsilon,\theta)$, and $\theta G'(\theta)\partial_t H(\varepsilon) + \nu(A\varepsilon_t) : \varepsilon_t$ by $R(\varepsilon,\theta)$. Then the equation (1.3) can be rewritten as

$$c_0(\varepsilon_0,\theta_0)\theta_t - \Delta\theta = (c_0(\varepsilon_0,\theta_0) - c_0(\varepsilon,\theta))\theta_t + R(\varepsilon,\theta).$$

By the assumptions we have

$$\begin{aligned} \|R(\varepsilon,\theta)\|_{L^q(\Omega_T)} &\leq C\|\theta\|^r_{L^\infty(\Omega_T)}\|H_{,\varepsilon}(\varepsilon)\|_{L^\infty(\Omega_T)}\|\varepsilon_t\|_{L^q(\Omega_T)} + C\|\varepsilon_t\|^2_{L^{2q}(\Omega_T)} \\ &\leq \Lambda. \end{aligned}$$

From the Hölder continuity it follows that

$$\|c_0(\varepsilon_0,\theta_0) - c_0(\varepsilon,\theta)\|_{L^\infty(\Omega_{T_1})} \leq K T_1^{\frac{\alpha}{2}},$$

where K is the Hölder constant independent of T_1. Here $T_1 \ll T$ will be determined later.

Next, we show that $1/c_0(\varepsilon,\theta)(x,T_2)$ is Hölder continuous with respect to the space variable for T_2 fixed in $[0,T]$. We remark that

$$\mathcal{G}(y) := yG''(y) \leq M$$

and $\mathcal{G} \in C^1$ is Lipschitz continuous. Then we have

$$\begin{aligned}
\left|\frac{1}{c_0}(x,T_2) - \frac{1}{c_0}(x',T_2)\right| &= \left|\frac{\mathcal{G}(\theta(x',T_2))H(\varepsilon(x',T_2)) - \mathcal{G}(\theta(x,T_2))H(\varepsilon(x,T_2))}{\{c_v - \mathcal{G}(\theta(x,T_2))H(\varepsilon(x,T_2))\}\{c_v - \mathcal{G}(\theta(x',T_2))H(\varepsilon(x',T_2))\}}\right| \\
&\le \frac{1}{c_v^2}|\{\mathcal{G}(\theta(x',T_2))H(\varepsilon(x',T_2)) - \mathcal{G}(\theta(x,T_2))H(\varepsilon(x',T_2))\} \\
&\qquad + \{\mathcal{G}(\theta(x,T_2))H(\varepsilon(x',T_2)) - \mathcal{G}(\theta(x,T_2))H(\varepsilon(x,T_2))\}| \\
&\le \frac{1}{c_v^2}|H(\varepsilon(x',T_2))|\,|\mathcal{G}(\theta(x',T_2)) - \mathcal{G}(\theta(x,T_2))| \\
&\qquad + \frac{1}{c_v^2}|\mathcal{G}(\theta(x,T_2))|\,|H(\varepsilon(x',T_2)) - H(\varepsilon(x,T_2))| \\
&\le \Lambda K|x-x'|^\alpha + CM|x-x'|^\alpha \\
&\le \Lambda|x-x'|^\alpha,
\end{aligned}$$

where Λ is independent of T_2. Therefore $[1/c_0(\varepsilon,\theta)](x,T_2)$ is Hölder continuous for any $T_2 \in [0,T]$. Moreover, we have $\sup_{\Omega_T}[1/c_0(\varepsilon,\theta)] \ge 1/(c_v + M\Lambda)$. These conditions assure that $\frac{1}{c_0(\varepsilon(T_2),\theta(T_2))}\Delta$ has the maximal regularity property according to (2.3). Hence, taking $T_1 = \left(\frac{1}{2\Lambda(K,M,T)K}\right)^{\frac{1}{\alpha}}$, we have

$$\begin{aligned}
\|\theta\|_{W_q^{2,1}(\Omega_{T_1})} &\le \Lambda(K,M,T)\left(\|c_0(\varepsilon_0,\theta_0) - c_0(\varepsilon,\theta)\|_{L^\infty(\Omega_{T_1})}\|\theta_t\|_{L^q(\Omega_{T_1})} + \|R(\varepsilon,\theta)\|_{L^q(\Omega_{T_1})} + \|\theta_0\|_{B^{2-2/q}_{q,q}(\Omega)}\right) \\
&\le \frac{1}{2}\|\theta_t\|_{L^q(\Omega_{T_1})} + \Lambda + \Lambda\|\theta_0\|_{B^{2-2/q}_{q,q}(\Omega)},
\end{aligned}$$

which yields

$$\|\theta\|_{W_q^{2,1}(\Omega_{T_1})} \le \Lambda + \Lambda\|\theta_0\|_{B^{2-2/q}_{q,q}(\Omega)}.$$

Here we remark that

$$\|\theta(T_1)\|_{B^{2-2/q}_{q,q}} \le C(T_1)\|\theta\|_{W_q^{2,1}(\Omega_{T_1})} \le C(T_1)(\Lambda + \Lambda\|u_0\|_{B^{2-2/q}_{q,q}})$$

thanks to the embedding $W_q^{2,1}(\Omega_{T_1}) \hookrightarrow BUC([0,T_1], B^{2-\frac{2}{q}}_{q,q})$ (see [4], [18]). Then similarly for the interval $[T_1, 2T_1]$ we have

$$\|\theta\|_{W_q^{2,1}(\Omega_{[T_1,2T_1]})} \le \Lambda + \Lambda\|u(T_1)\|_{B^{2-2/q}_{q,q}} \le \Lambda + \Lambda\|u_0\|_{B^{2-2/q}_{q,q}} \le \Lambda.$$

Repeating the same operation yields

$$\|\theta\|_{W_q^{2,1}(\Omega_{[kT_1,(k+1)T_1]})} \le \Lambda.$$

Summing the inequalities from $k = 0$ to $k = m$ satisfying $(m+1)T_1 > T$ and $mT_1 \le T$, we conclude that

$$\|\theta\|_{W_q^{2,1}(\Omega_T)} \le \Lambda.$$

Next we estimate the norm $\|u\|_{W_p^{4,2}(\Omega_T)}$. From Lemma 2.2 it follows that

$$\|\nabla\theta\|_{L^\infty(\Omega_T)} + \|\nabla\varepsilon\|_{L^\infty(\Omega_T)} \le \Lambda$$

for $q > 5$. Therefore, by virtue of the maximal regularity (2.1), we have

$$\begin{aligned}
\|u\|_{W_p^{4,2}(\Omega_T)} &\le C\|(u_0, u_1, 0)\|_{U(p,q)} + \|\nabla\cdot(G(\theta)H_{,\varepsilon}(\varepsilon))\|_{L^p(\Omega_T)} + \|\nabla\cdot\overline{H}_{,\varepsilon}(\varepsilon)\|_{L^p(\Omega_T)} \\
&\le C\|(u_0, u_1, 0)\|_{U(p,q)} + \Lambda\|\nabla\theta\|_{L^\infty(\Omega_T)}\|G'(\theta)\|_{L^\infty(\Omega_T)}\|H_{,\varepsilon}(\varepsilon)\|_{L^\infty(\Omega_T)} \\
&\quad + \Lambda\|\theta\|^r_{L^\infty(\Omega_T)}\|\nabla\varepsilon\|_{L^\infty(\Omega_T)}\|H_{,\varepsilon\varepsilon}(\varepsilon)\|_{L^\infty(\Omega_T)} + \Lambda\|\nabla\varepsilon\|_{L^\infty(\Omega_T)}\|\overline{H}_{,\varepsilon\varepsilon}(\varepsilon)\|_{L^\infty(\Omega_T)} \\
&\le \Lambda(\|(u_0, u_1, 0)\|_{U(p,q)}),
\end{aligned}$$

which completes the proof.

□

Proof of Theorem 3.1 (continuation). The assumption (L4) is satisfied thanks to Lemma 3.6 and the estimate (3.3). Then the existence of a solution to problem $(TE)_3^L$ results from Theorem 2.4. Noting that Γ_L is Lipschitz continuous, we can obtain the uniqueness result by repeating the arguments of [22, Section 6]. We remark also that the assumption $p \le q$ is required to show $(L1)$, see [21]. Thereby the proof of Theorem 3.1 is completed.

□

4 Proof of Theorem 1.1 (Existence)

The idea of the proof consists in showing that the solution (u_L, θ_L) to $(TE)_3^L$ constructed in Section 3 satisfies also the original system (1.2)–(1.5) for sufficiently large truncation size L. To this purpose, assuming that there exists a sufficiently smooth solution of problem (1.2)–(1.5) such that $\theta \ge 0$, we derive for it a sequence of a priori estimates which are independent of L.

Lemma 4.1 (Energy Conservation Law). *Assume that $\theta \geq 0$ a.e. in Ω_T, $K_2 \leq 6$ and $6r + K_1 \leq 6$. Then for any $t \in [0,T]$ a smooth solution of (1.2)–(1.5) satisfies*

$$\|\theta(t)\|_{L^1(\Omega)} + \|u_t(t)\|_{L^2(\Omega)} + \|Qu(t)\|_{L^2(\Omega)} \leq C(\|(u_0, u_1, \theta_0)\|_{H^2 \times L^2 \times L^1}). \tag{4.1}$$

Proof. Multiplying (1.2) by u_t and integrating the resulting equation with respect to the space variable, we have

$$\frac{d}{dt}\left(\frac{1}{2}\|u_t\|^2_{L^2} + \frac{\kappa}{2}\|Qu\|^2_{L^2} + \int_\Omega \overline{H}(\varepsilon)dx\right) + \nu \int_\Omega (A\varepsilon_t) : \varepsilon_t dx = -\int_\Omega G(\theta)\frac{\partial}{\partial t}H(\varepsilon)dx.$$

Integrating (1.3) over Ω, we obtain

$$c_v \frac{d}{dt}\int_\Omega \theta dx = \nu \int_\Omega (A\varepsilon_t) : \varepsilon_t dx + \int_\Omega \theta G'(\theta)\frac{\partial}{\partial t}H(\varepsilon)dx + \int_\Omega \theta G''(\theta)\theta_t H(\varepsilon)dx.$$

Combining these equalities, we deduce

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\|u_t\|^2_{L^2} + \frac{\kappa}{2}\|Qu\|^2_{L^2} + c_v\int_\Omega \theta dx + \int_\Omega \overline{H}(\varepsilon)dx\right) \\
&= \int_\Omega \left(\theta G'(\theta)\frac{\partial}{\partial t}H(\varepsilon) + \theta G''(\theta)\theta_t H(\varepsilon) - G(\theta)\frac{\partial}{\partial t}H(\varepsilon)\right)dx \\
&= -\frac{d}{dt}\int_\Omega \overline{G}(\theta)H(\varepsilon)dx,
\end{aligned}$$

where $\overline{G}(\theta) = G(\theta) - \theta G'(\theta)$. Consequently,

$$\frac{d}{dt}\left(\frac{1}{2}\|u_t\|^2_{L^2} + \frac{\kappa}{2}\|Qu\|^2_{L^2} + c_v\int_\Omega \theta dx + \int_\Omega \overline{H}(\varepsilon)dx + \int_\Omega \overline{G}(\theta)H(\varepsilon)dx\right) = 0.$$

Here we recall that $\theta \geq 0$ and $H(\varepsilon) \geq 0$. By the structure of $G(\theta)$ the function $\overline{G}(\theta)$ is as follows:

$$\overline{G}(r) = \begin{cases} 0 & \text{if } \theta \in [0, \theta_1], \\ \varphi(\theta) - \theta\varphi'(\theta) & \text{if } \theta \in [\theta_1, \theta_2], \\ C_2(1-r)\theta^r & \text{if } \theta \in [\theta_2, \infty). \end{cases}$$

According to Lemma 2.3 we have $\overline{G}(\theta) \geq 0$. Consequently, it follows from (A)–(iii) that

$$\begin{aligned}
\frac{1}{2}\|u_t(t)\|^2_{L^2} + \frac{\kappa}{2}\|u(t)\|^2_{H^2} + c_v\|\theta(t)\|_{L^1} &\leq \frac{1}{2}\|u_0\|^2_{H^2} + \frac{\kappa}{2}\|u_1\|^2_{L^2} + c_v\|\theta_0\|_{L^1} + \int_\Omega |\overline{H}(\varepsilon_0)|dx \\
&+ C_3|\Omega| + \int_{\{\theta_2 \geq \theta_0 \geq \theta_1\}\cap\Omega} [\varphi(\theta_0) - \theta_0\varphi'(\theta_0)]H(\varepsilon_0)dx + C_2(1-r)\int_{\{\theta_0 > \theta_2\}\cap\Omega} \theta_0^r H(\varepsilon_0)dx,
\end{aligned}$$

where $\varepsilon_0 = \varepsilon(u_0)$. Since the smooth function $\varphi(s) - s\varphi'(s)$ is bounded for $s \in [\theta_1, \theta_2]$, it follows that

$$\int_{\{\theta_2 \geq \theta_0 \geq \theta_1\}\cap\Omega} [\varphi(\theta_0) - \theta_0\varphi'(\theta_0)]H(\varepsilon_0)dx \leq C\int_\Omega |\varepsilon_0|^{K_1} dx$$
$$\leq C\|u_0\|_{H^2}^{K_1}$$

for $K_1 \leq 6$,

$$\int_{\{\theta_0 > \theta_2\}\cap\Omega} \theta_0^r H(\varepsilon_0)dx \leq C\|\theta_0\|_{L^1}^r \|\varepsilon_0\|_{L^{\frac{K_1}{1-r}}}^{K_1}$$
$$\leq C\|\theta_0\|_{L^1}^r \|u_0\|_{H^2}^{K_1}$$

for $6r + K_1 \leq 6$ and

$$\int_\Omega |\overline{H}(\varepsilon_0)|dx \leq \|u_0\|_{H^2}^{K_2}$$

for $K_2 \leq 6$. Hence we conclude the assertion. □

Lemma 4.2. *Let T be fixed. Assume that $\theta \geq 0$ a.e. in Ω_T and (1.6) holds. Then for any $(u_0, u_1, \theta_0) \in B^{19/8}_{16/5,16/5} \times B^{3/8}_{16/5,16/5} \times L^2 =: U_3$, the solution (u, θ) to (1.2)–(1.5) satisfies*

$$\|\varepsilon\|_{W^{2,1}_{16/5}(\Omega_T)} + \|\nabla\theta\|_{L^2(\Omega_T)} + \|\theta\|_{L^\infty_T L^2} \leq \Lambda, \tag{4.2}$$

where Λ depends on T and $\|(u_0, u_1, \theta_0)\|_{U_3}$. Moreover,

$$\|\varepsilon\|_{L^\infty(\Omega_T)} + \|\theta\|_{L^{10/3}(\Omega_T)} \leq \Lambda. \tag{4.3}$$

Proof. Remark that $\|(u_0, u_1, \theta_0)\|_{H^2\times L^2\times L^1} \leq C\|(u_0, u_1, \theta_0)\|_{U_3}$ (see [1]).

From the Gagliardo-Nirenberg inequality and Lemma 4.1 it follows that

$$\begin{aligned}\|\varepsilon\|_{L^{5p}(\Omega_T)} &\leq C\left\|\|\varepsilon\|_{L^6(\Omega)}^{\frac{4}{5}}\|\varepsilon\|_{W^2_p(\Omega)}^{\frac{1}{5}}\right\|_{L^{5p}_T}\\ &\leq C\|\varepsilon\|_{L^\infty_T L^6}^{\frac{4}{5}}\|\varepsilon\|_{W^{2,1}_p(\Omega_T)}^{\frac{1}{5}}\\ &\leq C\|u\|_{L^\infty_T H^2}^{\frac{4}{5}}\|\varepsilon\|_{W^{2,1}_p(\Omega_T)}^{\frac{1}{5}}\\ &\leq C\|\varepsilon\|_{W^{2,1}_p(\Omega_T)}^{\frac{1}{5}}\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}\|\theta\|_{L^{8/3}(\Omega_T)} &\leq C\left\|\|\theta\|_{L^1(\Omega)}^{\frac{1}{4}}\|\theta\|_{H^1(\Omega)}^{\frac{3}{4}}\right\|_{L^\infty_T}\\ &\leq C\|\theta\|_{L^\infty_T L^1}^{\frac{1}{4}}\|\theta\|_{L^2_T H^1}^{\frac{3}{4}}\\ &\leq \Lambda(\|\nabla\theta\|_{L^2(\Omega_T)} + \|\theta\|_{L^\infty_T L^2})^{\frac{3}{4}}.\end{aligned} \tag{4.5}$$

It follows from (4.4) that

$$\|\overline{H}_{,\varepsilon}(\varepsilon)\|_{L^{10/6}(\Omega_T)} \leq C\|\varepsilon\|^{K_2-1}_{L^{10}(\Omega_T)} \leq \Lambda\|\varepsilon\|^{\frac{K_2-1}{5}}_{W^{2,1}_{10/6}(\Omega_T)} \leq \frac{1}{4}\|\varepsilon\|_{W^{2,1}_{10/6}(\Omega_T)} + \Lambda$$

for $K_2 \in [1,6)$, and

$$\|\overline{H}_{,\varepsilon}(\varepsilon)\|_{L^{10/6}(\Omega_T)} \leq M|\Omega_T|^{\frac{6}{10}} \leq \Lambda$$

for $K_2 \in [0,1)$.

We first consider the case of $K_1 \geq 1$. Applying the growth condition and the Young inequality, we have

$$\begin{aligned}\|G(\theta)H_{,\varepsilon}(\varepsilon)\|_{L^{\frac{10}{6}}(\Omega_T)} &\leq \|\theta\|^r_{L^{\frac{8}{3}}(\Omega_T)}\|\varepsilon\|^{K_1-1}_{L^{\frac{10(K_1-1)}{6-6r}}(\Omega_T)} + \sup_{\theta\in[0,\theta_2]}|G(\theta)|\|\varepsilon\|^{K_1-1}_{L^{\frac{10(K_1-1)}{6}}(\Omega_T)}\\ &\leq \Lambda\|\theta\|^r_{L^{\frac{8}{3}}(\Omega_T)}\|\varepsilon\|^{K_1-1}_{L^{10}(\Omega_T)} + \Lambda\|\varepsilon\|^{K_1-1}_{L^{10}(\Omega_T)}\end{aligned}$$

for $6r + K_1 \leq 6$ (and $K_1 \leq 6$). Then

$$\begin{aligned}\|\theta\|^r_{L^{8/3}(\Omega_T)}\|\varepsilon\|^{K_1-1}_{L^{10}(\Omega_T)} + \|\varepsilon\|^{K_1-1}_{L^{10}(\Omega_T)} &\leq \Lambda(\|\nabla\theta\|_{L^2(\Omega_T)} + \|\theta\|_{L^\infty_T L_2})^{3r/4}\|\varepsilon\|^{(K_1-1)/5}_{W^{2,1}_{10/6}(\Omega_T)} + \Lambda\|\varepsilon\|^{(K_1-1)/5}_{W^{2,1}_{10/6}(\Omega_T)}\\ &\leq \frac{1}{4}\|\varepsilon\|_{W^{2,1}_{10/6}(\Omega_T)} + \Lambda(\|\nabla\theta\|_{L^2(\Omega_T)} + \|\theta\|_{L^\infty_T L_2})^{\frac{16r}{3(6-K_1)}} + \Lambda\end{aligned}$$

for $6r + K_1 < 6$ (and $K_1 < 6$). From the maximal regularity (2.2) it follows that

$$\begin{aligned}\|\varepsilon\|_{W^{2,1}_{10/6}(\Omega_T)} &\leq C\|(u_0,u_1,\theta_0)\|_{U_3} + \|G(\theta)H_{,\varepsilon}(\varepsilon)\|_{L^{10/6}(\Omega_T)} + \|\overline{H}_{,\varepsilon}(\varepsilon)\|_{L^{10/6}(\Omega_T)}\\ &\leq C\|(u_0,u_1,\theta_0)\|_{U_3} + \Lambda + \Lambda(\|\nabla\theta\|_{L^2(\Omega_T)} + \|\theta\|_{L^\infty_T L_2})^{\frac{16r}{3(6-K_1)}}.\end{aligned} \tag{4.6}$$

Next, multiplying (1.3) by θ and integrating over Ω, we have

$$\begin{aligned}\frac{c_v}{2}\frac{d}{dt}\|\theta(t)\|^2_{L^2} + k\|\nabla\theta\|^2_{L^2} &= \int_\Omega \theta^2 G''(\theta)\theta_t H(\varepsilon)dx + \int_\Omega \theta^2 G'(\theta)\partial_t H(\varepsilon)dx + \nu\int_\Omega \theta A\varepsilon_t : \varepsilon_t dx\\ &= \int_\Omega G_2'(\theta)\theta_t H(\varepsilon)dx + \int_\Omega G_2(\theta)\partial_t H(\varepsilon)dx + 2\int_\Omega \overline{G}_2(\theta)\partial_t H(\varepsilon)dx\\ &\quad + \nu\int_\Omega \theta A\varepsilon_t : \varepsilon_t dx\\ &= \frac{d}{dt}\int_\Omega G_2(\theta)H(\varepsilon)dx + 2\int_\Omega \overline{G}_2(\theta)\partial_t H(\varepsilon)dx + \nu\int_\Omega \theta A\varepsilon_t : \varepsilon_t dx,\end{aligned} \tag{4.7}$$

where $G_2(\theta)$ and $\overline{G}_2(\theta)$ are given in the proof of Lemma 3.2. Recall that

$$G_2(\theta) = \frac{C_2 r(r-1)}{r+1}\theta^{r+1} \leq 0 \qquad \text{and} \qquad \overline{G}_2(\theta) = \frac{2C_2 r}{r+1}\theta^{r+1} \qquad \text{for } \theta \geq \theta_2,$$

and

$$\sup_{\theta\in[0,\theta_2]} |G_2(\theta)| + \sup_{\theta\in[0,\theta_2]} |\overline{G}_2(\theta)| =: M < \infty.$$

Then we have

$$-\int_\Omega G_2(\theta)H(\varepsilon)dx = -\int_{\Omega\cap\{\theta\geq\theta_2\}} G_2(\theta)H(\varepsilon)dx - \int_{\Omega\cap\{\theta_1\leq\theta\leq\theta_2\}} G_2(\theta)H(\varepsilon)dx$$
$$\geq -M\int_\Omega |H(\varepsilon)|dx.$$

Hence integrating (4.7) with respect to time variable, we obtain

$$\frac{c_v}{2}\|\theta\|^2_{L^\infty_T L^2} + k\|\nabla\theta\|^2_{L^2(\Omega_T)} \leq \frac{c_v}{2}\|\theta_0\|^2_{L^2} + \|\overline{G}_2(\theta)\partial_t H(\varepsilon)\|_{L^1(\Omega_T)} + \nu\|\theta A\varepsilon_t : \varepsilon_t\|_{L^1(\Omega_T)}$$
$$+ M\sup_{t\in[0,T]}\int_\Omega |H(\varepsilon(t))|dx + \int_\Omega |G_2(\theta_0)H(\varepsilon_0)|dx.$$

By (4.4), (4.5) and the assumptions we infer that

$$\|\theta^{r+1}\partial_t H(\varepsilon)\|_{L^1(\Omega_T)} \leq \Lambda\|\theta\|^{r+1}_{L^{8/3}(\Omega_T)}\|u\|_{W^{2,1}_{10/5}(\Omega_T)}\|\varepsilon\|^{K_1-1}_{L^{10}(\Omega_T)}$$
$$\leq \Lambda(\|\nabla\theta\|_{L^2(\Omega_T)} + \|\theta\|_{L^\infty_T L_2})^{\frac{3(r+1)}{4}}\|u\|^{1+\frac{K_1-1}{5}}_{W^{2,1}_{10/5}(\Omega_T)}.$$

$$\|\theta A\varepsilon_t : \varepsilon_t\|_{L^1(\Omega_T)} \leq C\|\theta\|_{L^{\frac{5}{3}}(\Omega_T)}\|\varepsilon_t\|^2_{L^{\frac{10}{3}}(\Omega_T)} \leq \Lambda(\|\nabla\theta\|_{L^2(\Omega_T)} + \|\theta\|_{L^\infty_T L_2})^{\frac{3}{4}}\|\varepsilon_t\|^2_{\frac{10}{3}(\Omega_T)},$$

$$\int_\Omega |H(\varepsilon(t))|dx \leq C\|u(t)\|^{K_1}_{H^2} \leq \Lambda$$

and

$$\|\theta_0^{r+1}H(\varepsilon_0)\|_{L^1(\Omega)} \leq C\|\theta_0\|^{r+1}_{L^2(\Omega)}\|\varepsilon_0\|^{K_1}_{L^{\frac{2K_1}{1-r}}(\Omega)}$$
$$\leq C\|\theta_0\|^{r+1}_{L^2(\Omega)}\|u_0\|^{K_1}_{H^2(\Omega)}.$$

Consequently we arrive at

$$\|\theta\|^2_{L^\infty_T L^2} + \|\nabla\theta\|^2_{L^2(\Omega_T)} \leq \Lambda(\|(u_0, u_1, \theta_0)\|_{U_3}) + \Lambda(\|\nabla\theta\|_{L^2(\Omega_T)} + \|\theta\|_{L^\infty_T L_2})^{\frac{3(r+1)}{4}}\|\varepsilon\|^{\frac{4}{5}+\frac{K_1}{5}}_{W^{2,1}_{10/5}(\Omega_T)}$$
$$+ \Lambda(\|\nabla\theta\|_{L^2(\Omega_T)} + \|\theta\|_{L^\infty_T L_2})^{\frac{3}{4}}\|\varepsilon_t\|^2_{L^{\frac{10}{3}}(\Omega_T)} \tag{4.8}$$

Substituting (4.6) into (4.8) yields

$$\|\theta\|^2_{L^\infty_T L^2} + \|\nabla\theta\|^2_{L^2(\Omega_T)} \leq \Lambda(\|(u_0, u_1, \theta_0)\|_{U_3})$$
$$+ \Lambda(\|\nabla\theta\|_{L^2(\Omega_T)} + \|\theta\|_{L^\infty_T L_2})^{\frac{3(r+1)}{4}}\left(\|(u_0, u_1, \theta_0)\|_{U_3} + \|\nabla\theta\|^{\frac{16r}{4(6-K_1)}}_{L^2(\Omega_T)}\right)^{\frac{4}{5}+\frac{K_1}{5}}$$
$$+ \Lambda(\|\nabla\theta\|_{L^2(\Omega_T)} + \|\theta\|_{L^\infty_T L_2})^{\frac{3}{4}}\left(\|(u_0, u_1, \theta_0)\|_{U_3} + \|\nabla\theta\|^{\frac{16r}{4(6-K_1)}}_{L^2(\Omega_T)}\right)^2.$$

Here from the assumption $6r + K_1 < 6$ it follows that

$$\frac{3(r+1)}{4} + \frac{15r}{4(6-K_1)}\left(\frac{4}{5} + \frac{K_1}{5}\right) = \frac{30r + 3(6-K_1)}{4(6-K_1)} < \frac{5(6-K_1) + 3(6-K_1)}{4(6-K_1)} = 2,$$

$$\frac{3}{4} + \frac{30r}{4(6-K_1)} < \frac{3}{4} + \frac{5}{4} = 2.$$

Thus we conclude that

$$\|\theta\|_{L_T^\infty L^2} + \|\nabla\theta\|_{L^2(\Omega_T)} \leq \Lambda(\|(u_0, u_1, \theta_0)\|_{U_3}) + \Lambda\|\nabla\theta\|_{L^2(\Omega_T)}^{1-}.$$

Here we use $p-$ to denote a number less than p. Hence, by the Young inequality, we have

$$\|\theta\|_{L_T^\infty L^2} + \frac{1}{2}\|\theta\|_{L^2(\Omega_T)} \leq \Lambda(\|(u_0, u_1, \theta_0)\|_{U_3}).$$

Substituting the above inequality into (4.6), we deduce also the following

$$\|\varepsilon\|_{W_{10/5}^{2,1}(\Omega_T)} \leq \Lambda(\|(u_0, u_1, \theta_0)\|_{U_3}).$$

Next, we consider the case of $0 \leq K_1 \leq 1$ and $0 \leq r < 5/6$. In this case it follows that

$$|H_{,\varepsilon}(\varepsilon)| \leq C < \infty.$$

By an argument similar to the presented above we have

$$\begin{aligned}
\|\varepsilon\|_{W_{10/5}^{2,1}(\Omega_T)} &\leq \|(u_0, u_1, 0)\|_{U_3} + \|G(\theta)H_{,\varepsilon}(\varepsilon)\|_{L^{10/5}(\Omega_T)} \\
&\leq \|(u_0, u_1, 0)\|_{U_3} + C\|\theta\|_{L^{\frac{10r}{5}}(\Omega_T)}^r + C \sup_{\theta\in[0,\theta_2]} G(\theta) \qquad (4.9) \\
&\leq \|(u_0, u_1, 0)\|_{U_3} + \Lambda\|\theta\|_{L^{\frac{8}{3}}(\Omega_T)}^r + C.
\end{aligned}$$

Noting that

$$\|\theta^{r+1}\partial_t H(\varepsilon)\|_{L^1(\Omega_T)} \leq \Lambda\|\theta\|_{L^{8/3}(\Omega_T)}^{r+1}\|u\|_{W_{10/5}^{2,1}(\Omega_T)},$$

we obtain

$$\begin{aligned}
\|\theta\|_{L_T^\infty L^2}^2 + \|\nabla\theta\|_{L^2(\Omega_T)}^2 &\leq \|\theta_0\|_{L^2}^2 + \|\theta^{r+1}\partial_t H(\varepsilon)\|_{L^1(\Omega_T)} + \|\theta A\varepsilon_t : \varepsilon_t\|_{L^1(\Omega_T)} \\
&\quad + M \sup_{t\in[0,T]} \int_\Omega |H(\varepsilon(t))| dx + \int_\Omega |G_2(\theta_0)H(\varepsilon_0)| dx \\
&\leq \Lambda(\|(u_0, u_1, \theta_0)\|_{U_3}) + \Lambda\|\theta\|_{L^{8/3}(\Omega_T)}^{r+1}\|u\|_{W_{10/5}^{2,1}(\Omega_T)} + C\|\theta\|_{L^{8/3}(\Omega_T)}\|u\|_{W_{10/5}^{2,1}(\Omega_T)}^2 \\
&\leq \Lambda(\|(u_0, u_1, \theta_0)\|_{U_3}) + \Lambda(\|\nabla\theta\|_{L^2(\Omega_T)} + \|\theta\|_{L_T^\infty L^2})^{3(2r+1)/4}.
\end{aligned}$$

Since $3(2r+1)/4 < 2$, we arrive at the desired estimate (4.2).

The estimate (4.3) follows with the help of the embeddings

$$\|\varepsilon\|_{L^\infty(\Omega_T)} \le \Lambda \|\varepsilon\|_{W^{2,1}_{10/5}(\Omega_T)}$$

and of the inequality

$$\|\theta\|_{L^{10/3}(\Omega_T)} \le C \left\| \|\theta\|^{2/5}_{L^2(\Omega)} \|\theta\|^{3/5}_{H^1(\Omega)} \right\|_{L^{10/3}_T} \le C \|\theta\|^{2/5}_{L^\infty_T L^2} \|\theta\|^{3/5}_{L^2 H^1}.$$

This completes the proof. □

Lemma 4.3. *Let T be any fixed. Assume that $\theta \ge 0$ a.e. in Ω_T and (1.6) holds. Then for any $(u_0, u_1, \theta_0) \in B^{5/2}_{4,4} \times B^{1/2}_{4,4} \times H^1 = U_4$ the following estimate holds*

$$\|\varepsilon\|_{W^{2,1}_4(\Omega_T)} + \|\nabla\theta\|_{L^\infty_T L^2} + \|\theta\|_{W^{2,1}_2(\Omega_T)} \le \Lambda,$$

where constant Λ depends on T and $\|(u_0, u_1, \theta_0)\|_{U_4}$. Moreover, we have

$$\|\nabla\theta\|_{L^{10/3}(\Omega_T)} + \|\theta\|_{L^{10}(\Omega_T)} + \|\nabla\varepsilon\|_{L^{20}(\Omega_T)} \le \Lambda.$$

Proof. Remark that $U_4 \hookrightarrow U_3$. Using (4.3) we have

$$\|G(\theta)H_{,\varepsilon}(\varepsilon)\|_{L^4(\Omega_T)} \le \begin{cases} \Lambda \|\theta\|^r_{L^{10/3}(\Omega_T)} \|\varepsilon\|^{K_1-1}_{L^\infty(\Omega_T)} \le \Lambda & \text{if } K_1 \ge 1, \\ \Lambda \sup |H_{,\varepsilon}| \|\theta\|^r_{L^{10/3}(\Omega_T)} \le \Lambda & \text{if } K_1 \le 1, \end{cases} \tag{4.10}$$

for $r \le 5/6$. Then from the maximal regularity (2.2) it follows that

$$\|\varepsilon\|_{W^{2,1}_4} \le \|(u_0, u_1, \theta_0)\|_{U_4} + \|G(\theta)H_{,\varepsilon}(\varepsilon)\|_{L^4} \le \Lambda. \tag{4.11}$$

Multiplying (1.3) by θ_t and integrating over Ω_T, we get

$$\begin{aligned} c_v \|\theta_t\|^2_{L^2(\Omega_T)} + \frac{k}{2}\|\nabla\theta\|^2_{L^\infty_T L^2} &\le \frac{k}{2}\|\theta_0\|^2_{H^1} + \iint_{\Omega_T} \theta_t^2 \theta G''(\theta) H(\varepsilon)\,dxdt + \iint_{\Omega_T} \theta_t \theta G'(\theta) \partial_t H(\varepsilon)\,dxdt \\ &\quad + \iint_{\Omega_T} \theta_t A\varepsilon_t : \varepsilon_t\,dxdt \\ &\le \frac{k}{2}\|\theta_0\|^2_{H^1} + C\|\theta_t\|_{L^2(\Omega_T)} \|\theta^r H_{,\varepsilon}(\varepsilon)\|_{L^4} \|\varepsilon_t\|_{L^4} + C\|\theta_t\|_{L^2}\|\varepsilon_t\|^2_{L^4} \\ &\le \frac{k}{2}\|\theta_0\|^2_{H^1} + \Lambda(\|(u_0, u_1, \theta_0)\|_{U_4})\|\theta_t\|_{L^2(\Omega_T)} \\ &\le \Lambda(\|(u_0, u_1, \theta_0)\|_{U_4}) + \frac{1}{2}\|\theta_t\|^2_{L^2(\Omega_T)}, \end{aligned}$$

where we applied (3.11), (4.10) and (4.11). Therefore we arrive at

$$\|\varepsilon\|_{W_4^{2,1}(\Omega_T)} + \|\theta_t\|_{L^2(\Omega_T)} + \|\nabla\theta\|_{L_T^\infty L^2} \le \Lambda(\|(u_0,u_1,\theta_0)\|_{U_4}). \tag{4.12}$$

Next multiplying (1.3) by $\frac{-\Delta\theta}{c_v-\theta G''(\theta)H(\varepsilon)}$ and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla\theta(t)\|_{L^2}^2 + \int_\Omega \frac{k|\Delta\theta|^2}{c_v-\theta G''(\theta)H(\varepsilon)}dx \le \int_\Omega \frac{\Delta\theta}{c_v-\theta G''(\theta)H(\varepsilon)}(\theta G'(\theta)\partial_t H(\varepsilon) + \nu A\varepsilon_t : \varepsilon_t)\,dx.$$

Here we recall that

$$c_v \le c_v - \theta G''(\theta)H(\varepsilon) \le c_v + M\Lambda,$$

where $0 \le \sup_{\theta\ge0}(-\theta G''(\theta)) =: M < \infty$. Then integrating the above inequality with respect to time variable, we conclude that

$$\begin{aligned}
\|\nabla\theta(t)\|_{L^2}^2 + \frac{2k}{c_v+\Lambda M}\|\Delta\theta\|_{L^2(\Omega_T)}^2 &\le \|\nabla\theta_0\|_{L^2}^2 + \frac{k}{c_v+\Lambda M}\|\Delta\theta\|_{L^2(\Omega_T)}^2 \\
&\quad + \frac{c_v+\Lambda M}{k}\|\theta G'(\theta)\partial_t H(\varepsilon) + A\varepsilon_t : \varepsilon_t\|_{L^2(\Omega_T)}^2 \\
&\le \Lambda + \frac{k}{c_v+\Lambda M}\|\Delta\theta\|_{L^2(\Omega_T)}^2 + \Lambda(M)\|\theta^r H_{,\varepsilon}(\varepsilon)\|_{L^4(\Omega_T)}\|\varepsilon_t\|_{L^4(\Omega_T)} \\
&\quad + \Lambda(M)\|\varepsilon_t\|_{L^4(\Omega_T)}^2 \\
&\le \Lambda + \frac{k}{2(1+\Lambda M)}\|\Delta\theta\|_{L^2(\Omega_T)}^2
\end{aligned}$$

due to (4.10) and (4.11). Consequently we obtain the first assertion.

With the help of Lemma 2.2, we also obtain estimate

$$\|\nabla\theta\|_{L^{10/3}(\Omega_T)} + \|\theta\|_{L^{10}(\Omega_T)} + \|\nabla\varepsilon\|_{L^{20}(\Omega_T)} \le \Lambda(\|\theta\|_{W_2^{2,1}(\Omega_T)} + \|\varepsilon\|_{W_4^{2,1}(\Omega_T)}) \le \Lambda,$$

which completes the proof. □

Lemma 4.4. *Let T be arbitrary fixed and $p \in [20/9, 10/3]$ fixed. Assume that $\theta \ge 0$ a.e. in Ω_T and (1.6) holds. Then for any $(u_0,u_1,\theta_0) \in B_{p,p}^{4-2/p} \times B_{p,p}^{2-2/p} \times H^1 =: U_5(p)$, the solution (u,θ) to (1.2)-(1.5) satisfies*

$$\|u\|_{W_p^{4,2}(\Omega_T)} \le \Lambda,$$

where Λ depends on T and $\|(u_0,u_1,\theta_0)\|_{U_5(p)}$.

Proof. Since the embedding $B^{4-\frac{2}{p}}_{p,p} \hookrightarrow B^{\frac{5}{2}}_{4,4}$ holds for any $\frac{20}{9} \leq p$, by the Lemma 4.3 we find that

$$\|\epsilon\|_{W^{2,1}_4(\Omega_T)} + \|\theta\|_{W^{2,1}_2(\Omega_T)} \leq \Lambda(\|(u_0, u_1, \theta_0)\|_{B^{5/2}_{4,4} \times B^{1/2}_{4,4} \times H^1})$$
$$\leq \Lambda(\|(u_0, u_1, \theta_0)\|_{B^{4-2/p}_{p,p} \times B^{2-2/p}_{p,p} \times H^1}).$$

For any $p \leq \frac{10}{3}$ we have

$$\|\nabla \cdot (G(\theta)H_{,\epsilon}(\epsilon))\|_{L^p(\Omega_T)} \leq \Lambda\|\nabla\theta\|_{L^{10/3}(\Omega_T)}\|G'(\theta)\|_{L^\infty(\Omega_T)}\|H_{,\epsilon}(\epsilon)\|_{L^\infty(\Omega_T)}$$
$$+ \Lambda\|\theta\|^r_{L^{10}(\Omega_T)}\|\nabla\epsilon\|_{L^{20}(\Omega_T)}\|H_{,\epsilon\epsilon}(\epsilon)\|_{L^\infty(\Omega_T)}$$
$$\leq \Lambda$$

and

$$\|\nabla \cdot \overline{H}_{,\epsilon}(\epsilon)\|_{L^p(\Omega_T)} \leq \Lambda\|\nabla\epsilon\|_{L^{20}(\Omega_T)}\|\overline{H}_{,\epsilon\epsilon}(\epsilon)\|_{L^\infty(\Omega_T)} \leq \Lambda,$$

thanks to Lemmas 4.2 and 4.3. Then from the maximal regularity (2.1) it follows that

$$\|u\|_{W^{4,2}_p(\Omega_T)} \leq C\|(u_0, u_1, 0)\|_{U_5(p)} + C(\|\nabla \cdot (G(\theta)H_{,\epsilon}(\epsilon))\|_{L^p(\Omega_T)} + \|\nabla \cdot \overline{H}_{,\epsilon}(\epsilon)\|_{L^p(\Omega_T)})$$
$$\leq \Lambda.$$

This completes the proof. □

Lemma 4.5. *Let T be arbitrary fixed, $l > 2$ integer and $p \in (1, \infty)$. Assume that $\theta \geq 0$ a.e. in Ω_T and (1.6) holds. Then for any $(u_0, u_1, \theta_0) \in B^{17/5}_{10/3,10/3} \times B^{7/5}_{10/3,10/3} \times (L^l \cap H^1) =: U_6(l)$, the solution (u, θ) to (1.2)–(1.5) satisfies*

$$\|\theta\|_{L^\infty_T L^l_x} \leq \Lambda,$$

where $\Lambda = \Lambda(T, \|(u_0, u_1, \theta_0)\|_{U_6(l)})$. Moreover, if $(u_0, u_1, \theta_0) \in U_6(\infty)$ then

$$\|\theta\|_{L^\infty(\Omega_T)} \leq \Lambda,$$

where $\Lambda = \Lambda(T, \|(u_0, u_1, \theta_0)\|_{U_6(\infty)})$, and for $(u_0, u_1, \theta_0) \in (B^{3-2/p}_{p,p} \cap B^{17/5}_{10/3,10/3}) \times (B^{1-2/p}_{p,p} \cap B^{7/5}_{10/3,10/3}) \times (L^\infty \cap H^1) =: U_7(p)$ it holds

$$\|\epsilon\|_{W^{2,1}_p(\Omega_T)} \leq \Lambda,$$

where $\Lambda = \Lambda(T, \|(u_0, u_1, \theta_0)\|_{U_7(p)})$.

Proof. The same operation as in the proof of Lemma 3.3 yields

$$\frac{c_v}{l}\frac{d}{dt}\|\hat{\theta}\|_{L^l}^l + k(l-1)\int_\Omega \theta^{l-2}|\nabla\theta|^2 dx = \int_\Omega \overline{G}_l(\theta)\partial_t H(\varepsilon)dx + \nu\int \theta^{l-1}A\varepsilon_t : \varepsilon_t dx. \tag{4.13}$$

Here we recall that $G_l(\theta) = \theta^l G'(\theta) - \overline{G}_l(\theta)$, $\overline{G}_l(t) = l\int_0^\theta s^{l-1}G'(s)ds$ and

$$\hat{\theta} = \theta\left(1 - \frac{lG_l(\theta)H(\varepsilon)}{c_v\theta^l}\right)^{1/l} \geq \theta. \tag{4.14}$$

Since $\|H_{,\varepsilon}(\varepsilon)\|_{L^\infty(\Omega_T)} = \Lambda < \infty$ from (4.3), we have

$$\begin{aligned}\left|\int_\Omega \overline{G}_l(\theta)\partial_t H(\varepsilon)dx\right| &\leq C\|\theta^{l-1}\|_{L^1(\Omega)}\|\theta\|_{L^\infty(\Omega)}\|\varepsilon_t\|_{L^\infty(\Omega)}\|H_{,\varepsilon}(\varepsilon)\|_{L^\infty(\Omega)} \\ &\leq \Lambda\|\theta\|_{L^l(\Omega)}^{l-1}\|\theta\|_{H^2(\Omega)}\|\varepsilon_t\|_{L^\infty(\Omega)}.\end{aligned}$$

Therefore, we conclude from (4.13) that

$$\frac{c_v}{l}\frac{d}{dt}\|\hat{\theta}\|_{L^l(\Omega)}^l \leq \Lambda\|\varepsilon_t\|_{L^\infty(\Omega)}\|\theta\|_{H^2(\Omega)}\|\theta\|_{L^l(\Omega)}^{l-1} + C\|\varepsilon_t\|_{L^\infty(\Omega)}^2\|\theta\|_{L^l(\Omega)}^{l-1}. \tag{4.15}$$

Here note that the equality $\partial_t\|\hat{\theta}\|_{L^l(\Omega)}^l = l\|\hat{\theta}\|_{L^l(\Omega)}^{l-1}\partial_t\|\hat{\theta}\|_{L^l(\Omega)}$, and the Sobolev embedding and Lemma 4.4 yield estimates

$$\begin{aligned}\|\varepsilon_t\|_{L_T^2L^\infty} &\leq \Lambda\|\varepsilon_t\|_{L_T^2W^1_{10/3}} \leq \Lambda\|u\|_{W^{4,2}_{10/3}(\Omega_T)} \leq \Lambda, \\ \|\theta\|_{L_T^2H^2} &\leq \|\theta\|_{W_2^{2,1}(\Omega_T)} \leq \Lambda,\end{aligned}$$

where Λ is independent of l. Thus, integrating (4.15) with respect to time variable gives

$$\begin{aligned}\|\hat{\theta}\|_{L_T^\infty L^l} &\leq \|\hat{\theta}_0\|_{L^l} + \Lambda\|\varepsilon_t\|_{L_T^2L^\infty}\|\theta\|_{L_T^2H^2} + \Lambda\|\varepsilon_t\|_{L_T^2L^\infty}^2 \\ &\leq \Lambda + \|\hat{\theta}_0\|_{L^l}\end{aligned}$$

In view of the inequality $\hat{\theta}_0 \leq \theta_0(1 + lM\Lambda/c_v)^{1/l}$, the desired result can be obtained. For the $W_p^{2,1}$-norm of ε, we find that

$$\|\varepsilon\|_{W_p^{2,1}} \leq C\|(u_0, u_1, 0)\|_{U_T(p)} + \Lambda\|\theta\|_{L^\infty(\Omega_T)}^r\|H_{,\varepsilon}(\varepsilon)\|_{L^\infty(\Omega_T)} + \Lambda\|\overline{H}_\varepsilon(\varepsilon)\|_{L^\infty(\Omega_T)} \leq \Lambda$$

for $p \in (1, \infty)$, by virtue of the maximal regularity (2.2). This completes the proof. □

Using again Lemma 3.4, we can also prove the Hölder continuity of θ. The Hölder continuity of ε is assured on account of Lemma 2.2. Hence from Lemma 3.6 we can obtain the bounds in higher Sobolev norms, i.e. for $5 < p, q < \infty$

$$\|(u,\theta)\|_{V_T(p,q)} = \|u\|_{W_p^{4,2}(\Omega_T)} + \|\theta\|_{W_q^{2,1}(\Omega_T)} \leq \Lambda =: \widehat{\Lambda}, \tag{4.16}$$

where $\widehat{\Lambda}$ is independent of L.

This a priori estimate says that if there exists a solution to problem $(TE)_3$ such that $\theta \geq 0$ then this solution satisfies estimate (4.16). Let us consider now problem $(TE)_3^L$ from Section 3 assuming that the truncation size L is sufficiently large such that

$$\left|\nabla \cdot [G(\theta)H_{,\varepsilon}(\varepsilon) + \overline{H}_{,\varepsilon}(\varepsilon)]\right| \leq \widehat{\Lambda}^{K_1+r-1} + \widehat{\Lambda}^{K_2-1} \ll L.$$

In this case we may regard Γ_L as the identity operator because the internal part of Γ_L in (3.1) is smaller than L. Therefore the unique solution (u_L, θ_L) to $(TE)_3^L$ satisfies (4.16) for large L. In other words, $V_T(p,q)$-norm bound for (u_L, θ_L) does not depend on L. Hence (u_L, θ_L) satisfies also the original system $(TE)_3$.

The positivity of θ follows by the same argument as the proof of Lemma 3.1 in [22]. This completes the proof of Theorem 1.1.

5 Two-Dimensional Case

In this section we consider the solvability of 2-D system $(TE)_2$. We prove the following theorem.

Theorem 5.1. *Fix $4 < p \leq q < \infty$. Assume that $\min_\Omega \theta_0 \geq 0$, $\nu > 0$ and (A) with (1.7). Then for any $T > 0$ and $(u_0, u_1, \theta_0) \in U(p,q)$, there exists at least one solution (u,θ) to $(TE)_2$ satisfying $(u,\theta) \in V_T(p,q)$.*

Moreover, if we assume $\min_\Omega \theta_0 = \theta_* > 0$ then there exists a positive constant ω such that

$$\theta \geq \theta_* \exp(-\omega t) \quad \text{in } \Omega_T.$$

Theorem 5.2. *In addition to assumptions of Theorem 1.1, suppose that $F(\varepsilon,\theta) \in C^4(\mathbf{S}^2 \times \mathbf{R}^+, \mathbf{R})$. Then the solution $(u,\theta) \in V_T(p,q)$ to $(TE)_2$ constructed above is unique.*

Proof of Theorem 5.1. With the exception of a priori bounds the result follows by the same procedure as in the proof of 3-D case. Thus, it remains to check the bounds corresponding to Lemmas 4.1, 4.2 and 4.3 under the assumption (A) with (1.7).

Lemma 5.3 (Energy Conservation Law). *Assume that $\theta \geq 0$ a.e. in Ω_T and (1.7) holds. Then for any $t \in [0, T]$ the smooth solution of $(TE)_2$ satisfies*

$$\|\theta(t)\|_{L^1(\Omega)} + \|u_t(t)\|_{L^2(\Omega)} + \|Qu(t)\|_{L^2(\Omega)} \leq C(\|(u_0, u_1, \theta_0)\|_{H^2 \times L^2 \times L^1}).$$

Proof. The same operation as in the proof of Lemma 4.1 yields

$$\frac{d}{dt}\left(\frac{1}{2}\|u_t\|_{L^2}^2 + \frac{\kappa}{2}\|Qu\|_{L^2}^2 + c_v\int_\Omega \theta dx + \int_\Omega \overline{H}(\varepsilon)dx + \int_\Omega \overline{G}(\theta)H(\varepsilon)dx\right) = 0,$$

where $\overline{G}(\theta) = G(\theta) - \theta G'(\theta)$. Here we recall that $\theta \geq 0$, $H(\varepsilon) \geq 0$ and $\overline{G}(\theta) \geq 0$. Consequently, it follows from (A)–(iii) that

$$\frac{1}{2}\|u_t\|_{L_T^\infty L^2}^2 + \frac{\kappa}{2}\|u\|_{L_T^\infty H^2}^2 + c_v\|\theta\|_{L_T^\infty L^1} \leq \frac{\kappa}{2}\|u_0\|_{H^2}^2 + \frac{1}{2}\|u_1\|_{L^2}^2 + c_v\|\theta_0\|_{L^1}$$
$$+ \int_\Omega \{|\overline{H}(\varepsilon_0)| + |\overline{G}(\theta_0)H(\varepsilon_0)|\}\, dx + C_3|\Omega|,$$

where $\varepsilon_0 = \varepsilon(u_0)$. From the Sobolev embedding it holds that

$$\|\varepsilon_0\|_{L^s(\Omega)} \leq C\|u_0\|_{H^2(\Omega)} \tag{5.1}$$

for any $s \in [1, \infty)$. Then we have

$$\int_\Omega |\overline{G}(\theta_0)H(\varepsilon_0)|dx \leq C\|\theta_0\|_{L^1(\Omega)}^r\|\varepsilon_0\|_{L^{\frac{K_1}{1-r}}(\Omega)}^{K_1}$$
$$\leq C\|\theta_0\|_{L^1(\Omega)}^r\|u_0\|_{H^2}^{K_1}$$

for $r < 1$ and $K_1 < \infty$, and

$$\int_\Omega \overline{H}(\varepsilon_0)dx \leq \|\varepsilon_0\|_{L^{K_2}}^{K_2}$$
$$\leq C\|u_0\|_{H^2}^{K_2}$$

for $K_2 < \infty$. This completes the proof. □

Lemma 5.4. *Let T and $p \in [2,4)$ be fixed. Assume that (1.7) holds. Then for any $(u_0, u_1, \theta_0) \in B^{3-2/p}_{p,p} \times B^{1-2/p}_{p,p} \times L^2 =: U_3'(p)$, the solution (u, θ) to $(TE)_2$ satisfies*

$$\|\varepsilon\|_{W^{2,1}_p(\Omega_T)} + \|\nabla\theta\|_{L^2(\Omega_T)} + \|\theta\|_{L^\infty_T L^2} \leq \Lambda, \tag{5.2}$$

where Λ depends on T and $\|(u_0, u_1, \theta_0)\|_{U_3'(p)}$. Moreover, we have

$$\|\varepsilon\|_{L^\infty(\Omega_T)} + \|\theta\|_{L^p(\Omega_T)} \lesssim \Lambda. \tag{5.3}$$

Proof. We first show (5.2) for p such that $p < 3$. From the Sobolev inequality (5.1) and Lemma 5.3, it follows that

$$\|\varepsilon\|_{L^s(\Omega_T)} \leq \Lambda \|u\|_{L^\infty_T H^2} \leq \Lambda$$

for every $s < \infty$, and hence we obtain

$$\|H_{,\varepsilon}\|_{L^{K_1}(\Omega_T)} + \|\overline{H}_{,\varepsilon}\|_{L^{K_2}(\Omega_T)} \leq \Lambda \tag{5.4}$$

for any K_1, $K_2 < \infty$. Moreover, by using the Hölder inequality, we have

$$\|\theta\|_{L^p(\Omega_T)} \leq C \left\| \|\theta\|_{L^1}^{1-2/p} \|\theta\|_{L^{2/(3-p)}}^{2/p} \right\|_{L^p_T} \leq C \|\theta\|_{L^\infty_T L^1}^{1-2/p} \|\theta\|_{L^2_T H^1}^{2/p} \leq \Lambda \|\theta\|_{L^2_T H^1}^{2/p} \tag{5.5}$$

for $p \in [2, 3)$.

We fix $\bar{p}$ such that $r + 2 < \bar{p} < 3$. From (5.4), (5.5) and the maximal regularity (2.2) it follows that

$$\begin{aligned}
\|\varepsilon\|_{W^{2,1}_{\bar{p}}(\Omega_T)} &\leq C\|(u_0, u_1, \theta_0)\|_{U_3'(\bar{p})} + C\|G(\theta) H_{,\varepsilon}(\varepsilon)\|_{L^{\bar{p}}(\Omega_T)} + C\|\overline{H}_{,\varepsilon}(\varepsilon)\|_{L^{\bar{p}}(\Omega_T)} \\
&\leq \Lambda + C\|\theta\|^r_{L^{\bar{p}}(\Omega_T)} \|H_{,\varepsilon}(\varepsilon)\|_{L^{\frac{\bar{p}}{(1-r)}}(\Omega_T)} + C\|\overline{H}_{,\varepsilon}(\varepsilon)\|_{L^{\bar{p}}(\Omega_T)} \qquad (5.6) \\
&\leq \Lambda + \Lambda \|\theta\|^{2r/\bar{p}}_{L^2_T H^1}.
\end{aligned}$$

Next, the same operation as in the proof of Lemma 4.2 yields

$$\begin{aligned}
\frac{c_v}{2}\|\theta\|^2_{L^\infty_T L^2} + k\|\nabla\theta\|^2_{L^2(\Omega_T)} \leq{}& \frac{c_v}{2}\|\theta_0\|^2_{L^2} + \|\overline{G}_2(\theta)\partial_t H(\varepsilon)\|_{L^1(\Omega_T)} + \nu\|\theta A\varepsilon_t : \varepsilon_t\|_{L^1(\Omega_T)} \\
&+ M \sup_{t\in[0,T]} \int_{\Omega_T} |H(\varepsilon(t))| dx + \int_\Omega |G_2(\theta_0) H(\varepsilon_0)| dx.
\end{aligned}$$

By (5.4), (5.5) and (5.6) we have

$$\begin{aligned}
\|\theta^{r+1}\partial_t H(\varepsilon)\|_{L^1(\Omega_T)} &\leq \Lambda \|\theta\|^{r+1}_{L^{\bar{p}}(\Omega_T)} \|\varepsilon\|_{W^{2,1}_{\bar{p}}(\Omega_T)} \|H_{,\varepsilon}(\varepsilon)\|_{L^{\frac{\bar{p}}{\bar{p}-(r+2)}}(\Omega_T)} \\
&\leq \Lambda \|\theta\|^{\frac{2(r+1)}{\bar{p}}}_{L^2_T H^1} (\Lambda + \|\theta\|^{\frac{2r}{\bar{p}}}_{L^2_T H^1})
\end{aligned}$$

for $\bar{p} > r + 2$,

$$\|\theta A\varepsilon_t : \varepsilon_t\|_{L^1(\Omega_T)} \leq \Lambda\|\theta\|_{L^{\bar{p}}(\Omega_T)}\|\varepsilon_t\|^2_{L^{\frac{2\bar{p}}{\bar{p}-1}}(\Omega_T)} \leq \Lambda\|\theta\|^{\frac{2}{\bar{p}}}_{L^2_T H^1}(\Lambda + \|\theta\|^{\frac{4r}{\bar{p}}}_{L^2_T H^1}),$$

$$\int_\Omega |H(\varepsilon(t))|dx \leq C\|u(t)\|^{K_1}_{H^2} \leq \Lambda$$

and

$$\begin{aligned}\|\theta_0^{r+1}H(\varepsilon_0)\|_{L^1(\Omega)} &\leq C\|\theta_0\|^{r+1}_{L^2(\Omega)}\|\varepsilon_0\|^{K_1}_{L^{\frac{2K_1}{1-r}}(\Omega)} \\ &\leq C\|\theta_0\|^{r+1}_{L^2(\Omega)}\|u_0\|^{K_1}_{H^2(\Omega)}.\end{aligned}$$

Consequently we arrive at

$$\|\theta\|^2_{L^\infty_T L^2} + \|\nabla\theta\|^2_{L^2(\Omega_T)} \leq \Lambda(\|(u_0, u_1, \theta_0)\|_{U_3'(p)}) + \Lambda\|\theta\|^{\frac{2(2r+1)}{\bar{p}}}_{L^2_T H^1}$$

Since $2r + 1 < r + 2 < \bar{p}$, by using the Young inequality we have

$$\|\theta\|_{L^\infty_T L^2} + \|\nabla\theta\|_{L^2(\Omega_T)} \leq \Lambda(\|(u_0, u_1, \theta_0)\|_{U_3'(p)}). \tag{5.7}$$

Substituting (5.7) into (5.6), we obtain (5.2) for $p < 3$.

We shall show the rest of proof. Taking $p \in [2, 4)$, by the same operation as (5.5) we have

$$\|\theta\|_{L^p(\Omega_T)} \leq C\|\theta\|^{1-2/p}_{L^\infty_T L^2}\|\theta\|^{2/p}_{L^2_T H^1} \leq \Lambda$$

for $p < 4$ thanks to (5.7). Then from (2.2) we conclude that

$$\begin{aligned}\|\varepsilon\|_{W^{2,1}_p} &\leq \Lambda + \|\theta\|^r_{L^p}\|H_{,\varepsilon}(\varepsilon)\|_{L^{\frac{p}{(1-r)}}} + \|\overline{H}_{,\varepsilon}(\varepsilon)\|_{L^p} \\ &\leq \Lambda.\end{aligned}$$

This completes the proof. □

Lemma 5.5. *Let T be any fixed. Assume that (1.7) holds. Then for any $(u_0, u_1, \theta_0) \in B^{5/2}_{4,4} \times B^{1/2}_{4,4} \times H^1 = U_4'$ the following estimate holds*

$$\|\varepsilon\|_{W^{2,1}_4(\Omega_T)} + \|\nabla\theta\|_{L^\infty_T L^2} + \|\theta\|_{W^{2,1}_2(\Omega_T)} \leq \Lambda,$$

where constant Λ depends on T and $\|(u_0, u_1, \theta_0)\|_{U_4'}$. Moreover, we have

$$\|\nabla\theta\|_{L^4(\Omega_T)} + \|\theta\|_{L^s(\Omega_T)} + \|\nabla\varepsilon\|_{L^s(\Omega_T)} \leq \Lambda$$

for any $s < \infty$.

Proof. It follows from Lemma 5.4 and (2.2) that

$$\|\varepsilon\|_{W_4^{2,1}(\Omega_T)} \le C\|(u_0,u_1,\theta_0)\|_{U_4'} + C\|\theta\|_{L^{4r}}^r\|H_{,\varepsilon}(\varepsilon)\|_{L^\infty(\Omega_T)} + C\|\overline{H}_{,\varepsilon}(\varepsilon)\|_{L^\infty(\Omega_T)} \le \Lambda \tag{5.8}$$

thanks to $r<1$. The same operation as in the proof of Lemma 4.3 yields

$$\begin{aligned} c_v\|\theta_t\|_{L^2(\Omega_T)}^2 + \frac{k}{2}\|\nabla\theta\|_{L_T^\infty L^2}^2 &\le \frac{k}{2}\|\theta_0\|_{H^1}^2 + C\|\theta_t\|_{L^2(\Omega_T)}\|\theta^r H_{,\varepsilon}(\varepsilon)\|_{L^4(\Omega_T)}\|\varepsilon_t\|_{L^4(\Omega_T)} \\ &\quad + C\|\theta_t\|_{L^2(\Omega_T)}\|\varepsilon_t\|_{L^4(\Omega_T)}^2 \\ &\le \Lambda + \frac{c_v}{2}\|\theta_t\|_{L^2(\Omega_T)}^2 \end{aligned}$$

on account of (5.3) and (5.8). Therefore, we arrive at the estimate

$$\|\varepsilon\|_{W_4^{2,1}(\Omega_T)} + \|\theta_t\|_{L^2(\Omega_T)} + \|\nabla\theta\|_{L_T^\infty L^2} \le \Lambda(\|(u_0,u_1,\theta_0)\|_{U_4'}).$$

Moreover, applying the same argument as in the proof of Lemma 4.3, we get

$$\|\Delta\theta\|_{L^2(\Omega_T)} \le \Lambda.$$

This completes the proof of the first assertion. With the help of Lemma 2.2 we obtain the second assertion. This completes the proof of Lemma 5.5. □

From a modification similar to that presented in Section 4 we can derive the estimate

$$\|(u,\theta)\|_{V_T(p,q)} = \|u\|_{W_p^{4,2}(\Omega_T)} + \|\theta\|_{W_q^{2,1}(\Omega_T)} \le \Lambda.$$

Hence the proof of Theorem 5.1 are completed. □

Acknowledgment. The first author is partly supported by the Research Fellowships of the Japan Society of Promotion of Science (JSPS) for Young Scientists. The authors would like to express deep gratitude to Professor Yoshio Tsutsumi for valuable advices.

References

[1] R. A. Adams and J. J. F. Fournier, *Sobolev Spaces (2nd ed.)*, Academic Press, Amsterdam, 2003.

[2] T. Aiki, Weak solutions for Falk's model of shape memory alloys, Math. Meth. Appl. Sci., **23** (2000), 299-319.

[3] T. Aiki, A. Kadoya and S. Yoshikawa, One-dimensional shape memory alloy problem with small viscosity. submitted to GAKUTO International Series Math. Sci. Appl.

[4] H. Amann, *Linear and quasilinear parabolic problems, vol. I, abstract linear theory*, Monographs in Mathematics, Birkhäuser, Basel, vol. 89, 1995.

[5] M. Brokate and J. Sprekels, *Hysteresis and phase transitions*, Appl. Math. Sci., Springer, Berlin, Vol. 121, 1996.

[6] N. Bubner and J. Sprekels, Optimal control of martensitic phase transitions in a deformation-driven experiment on shape memory alloys, Adv. Math. Sci. Appl., **8** (1998), 299-325.

[7] P. Clément and S. Li, Abstract parabolic quasilinear equations and application to a groundwater flow problem, Adv. Math. Sci. Appl., **3** (1993/94), 17-32.

[8] C. M. Dafermos and L. Hsiao, Global smooth thermomechanical processes in one-dimensional nonlinear thermoviscoelasticity, Nonlinear Anal., **6** (1982), 435-454.

[9] R. Denk, M. Hieber and J. Prüss, *$\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type,* Mem. Amer. Math. Soc., **166** (2003), Number788.

[10] F. Falk, Elastic phase transitions and nonconvex energy functions, Free boundary problems: Theory and applications I (K.-H. Hoffmann, J. Sprekels, eds.), Longman, London, (1990), 45-59.

[11] F. Falk and P. Konopka, Three-dimensional Landau theory describing the martensitic phase transformation of shape memory alloys, J. Phys.: Condensed Matter, **2** (1990), 61-77.

[12] K.-H. Hoffmann and A. Żochowski, Existence of solutions to some non-linear thermoelastic systems with viscosity, Math. Meth. Appl. Sci., **15** (1992), 187-204.

[13] M. Hieber and J. Prüss, Heat kernels and maximal L^p-L^q estimates for parabolic evolution equations, Comm. Part. Diff. Eq. **22** (1997), 1647-1669.

[14] O. A. Ladyzenskaja, V. A. Solonnikov and N. N. Ural'ceva, *Linear and quasi-linear equations of parabolic type,* Trans. Math. Monographs., American Mathmatical Society, Providence, Rhode Island, vol. 23, 1968.

[15] G. M. Lieberman, *Second order parabolic differential equations,* World Scientific Publishing, Singapore, 1996.

[16] J. Sprekels and S. Zheng, Global solutions to the equations of a Ginzburg-Landau theory for structural phase transitions in shape memory alloys, Physica D, Nonlinear Phenomena, 39 (1989), 59-76.

[17] J. Sprekels, S. Zheng and P. Zhu, Asymptotic behavior of the solutions to a Landau-Ginzburg system with viscosity for martensitic phase transitions in shape memory alloys, SIAM Jour. Math. Anal., 29 (1998), 69-84.

[18] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators,* North-Holland, Amsterdam, 1978.

[19] I. Pawłow, Three-dimensional model of thermomechanical evolution of shape memory materials, Control and Cybernetics, 29 (2000), 341-365.

[20] I. Pawłow and A. Żochowski, Existence and uniqueness of solutions for a three-dimensional thermoelastic system, Dissert. Math., 406 (2002), 1-46.

[21] I. Pawłow and W. M. Zajączkowski Global existence to a three-dimensional nonlinear thermoelasticity system arising in shape memory materials, Math. Meth. Appl. Sci., 28 (2005), 407-442.

[22] I. Pawłow and W. M. Zajączkowski Unique global solvability in two-dimensional non-linear thermoelasticity, Math. Meth. Appl. Sci., 28 (2005), 551-592.

[23] I. Pawłow and W. M. Zajączkowski New existence result for 3-D shape memory model, in Dissipative Phase Transitions, P. Colli, N. Kenmochi; J. Sprekels (Eds.). Series on Advances in Mathematical and Applied Sciences, World Sci. Publishing (to appear).

[24] S. Yoshikawa, Weak solutions for the Falk model system of shape memory alloys in energy class, Math. Meth. Appl. Sci., 28 (2005), 1423-1443.

[25] S. Yoshikawa, Unique global existence for a three-dimensional thermoelastic system of shape memory alloys, Adv. Math. Sci. Appl., **15** (2005), 603-627.

[26] S. Yoshikawa, Small energy global existence for a two-dimensional thermoelastic system of shape memory materials, submitted to GAKUTO International Series Math. Sci. Appl.

[27] S. Yoshikawa, *Global solutions for shape memory alloy systems*, Doctor Thesis, Tohoku University, (2006).